

Hopf bifurcation and optimal control in a diffusive predator-prey system with time delay and prey harvesting*

Xiaoyuan Chang^{a,b}, Junjie Wei^a

^aDepartment of Mathematics, Harbin Institute of Technology
Harbin, Heilongjiang 150001, China
weijj@hit.edu.cn

^bSchool of Applied Sciences, Harbin University of Science and Technology
52 Xuefu Street, Harbin, Heilongjiang 150080, China

Received: 17 December 2011 / **Revised:** 15 May 2012 / **Published online:** 22 November 2012

Abstract. In this paper, we investigated the dynamics of a diffusive delayed predator-prey system with Holling type II functional response and nonzero constant prey harvesting on no-flux boundary condition. At first, we obtain the existence and the stability of the equilibria by analyzing the distribution of the roots of associated characteristic equation. Using the time delay as the bifurcation parameter and the harvesting term as the control parameter, we get the existence and the stability of Hopf bifurcation at the positive constant steady state. Applying the normal form theory and the center manifold argument for partial functional differential equations, we derive an explicit formula for determining the direction and the stability of Hopf bifurcation. Finally, an optimal control problem has been considered.

Keywords: Hopf bifurcation, predator-prey system, harvesting, delay, optimal control.

1 Introduction

Predator-prey systems model some biological phenomena and relationship between predator and prey in the real world, which play a crucial role in mathematics and have been extensively considered in many ways by many researchers (see e.g. [1–6] and [7–12]).

Recently, as the continuous exploitation of biological resources, the harvest of population in fishery, forestry, and wildlife management become more crucial. Concerning the conservation for the long-term benefits of humanity, there is a wide-range of interest in the use of bioeconomic modeling to gain insight in the scientific management of renewable resources like fisheries and forestry [13]. In biological, both harvesting in prey and in predator could lead some dangers in real-life harvesting such as no equilibrium exists and

*This research is supported by National Natural Science Foundation of China (grants No. 11031002 and No. 11126269).

either prey or predator goes to extinction for some values of harvesting rate. It would be interesting that the nonzero constant harvesting in both models can prevent mutual extinction of prey and predator, and remove the singularity of the origin, which was regarded as “pathological behavior” by some researchers (see [14,15]). And in mathematics, the study of predator-prey models with harvesting have attracted the attention of many researchers which have rich bifurcating phenomena (see e.g. [13,14,16–18] and references therein).

Xiao and Jennings [13] considered the Michaelis–Menten type (ratio-dependent) predator-prey model with a constant rate prey harvesting. It has been showed that system has four equilibria and undergoes two saddle-node bifurcations, the separatrix connecting a saddle and a saddle-node bifurcation and heteroclinic bifurcation. When the spatial dispersal is considered in ratio-dependent predator-prey model, system has been investigated by Zhang [19]. They derived the conditions for Hopf and Turing bifurcation on the spatial domain. Their results have shown that modeling by reaction-diffusion equations is an appropriate tool for investigating fundamental mechanisms of complex spatiotemporal dynamics.

Many researchers focus on constant-yield harvesting of a predator-prey system (1) with Holling type II functional response and its variants (see [20–24]),

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \frac{m}{\delta} \frac{xy}{x+A} - h_1, \\ \frac{dy}{dt} &= y\left(-B + \frac{mx}{x+A}\right) - h_2,\end{aligned}\tag{1}$$

where $h_1 > 0$ and $h_2 > 0$ are harvesting or removal rate for the prey and the predator, respectively. They observed very rich and interesting dynamical behaviors such as the existence of multiple equilibria, homoclinic loop, Hopf bifurcation etc. Brauer and Soudack [25] developed the theory of global behavior of system (1) and shown by examples which of the theoretically possible transitions can actually occur for a class of biologically motivated models. Brauer and Soudack [26] carried out the phase portrait analysis and considered the general form of system (1). They have shown how to approximate the region of asymptotic stability in biological terms the initial states which lead to coexistence of the two species by efficient computer simulations and how to identify values of the harvest rates for which the region of asymptotic stability disappears corresponding to collapse of the biological system for every initial state.

If a time delay in the predator response term is considered in system (1) with $h_1 = H$ and $h_2 = 0$, then it becomes

$$\begin{aligned}\frac{dx}{dt} &= rx(t)\left(1 - \frac{x(t)}{K}\right) - \frac{x(t)y(t)}{x(t)+A} - H, \\ \frac{dy}{dt} &= y(t)\left(-B + \frac{x(t-\tau)}{x(t-\tau)+A}\right).\end{aligned}\tag{2}$$

The time delay represents a gestation time of the predators. The reproduction of predating the prey is not instantaneous, but will be mediated by some discrete time lag required for

gestation of the predators (see also [27]). Xia et al. in [27] have investigated system (2), applying the normal form theory of retarded functional differential equations developed by Faria and Magalhães [28]. Also, Martin and Ruan in [16] considered dynamics of the general model of system above.

If system (2) is further considered a spatial diffusion with no-flux boundary conditions, then it can be described by the following system:

$$\begin{aligned} \frac{\partial u(t, x)}{\partial t} &= D_1 \Delta u(t, x) + ru(t, x) \left(1 - \frac{u(t, x)}{K} \right) - \frac{u(t, x)v(t, x)}{A + u(t, x)} - H, \\ x &\in \Omega, \quad t > 0, \\ \frac{\partial v(t, x)}{\partial t} &= D_2 \Delta v(t, x) + v(t, x) \left(-B + \frac{u(t - \tau, x)}{A + u(t - \tau, x)} \right), \quad x \in \Omega, \quad t > 0, \\ \frac{\partial u(t, x)}{\partial \nu} &= \frac{\partial v(t, x)}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ u(\theta, x) &= \phi(\theta, x), \quad v(\theta, x) = \psi(\theta, x), \quad x \in \bar{\Omega}, \quad \theta \in [-\tau, 0]. \end{aligned} \quad (3)$$

Here the habitat Ω is a bounded domain in \mathbb{R}^n ($n \geq 1$), ν is the outer normal direction for $x \in \partial\Omega$ and the imposed no-flux boundary condition means the system is a closed one. Also, r is the intrinsic growth rate of the prey; K is the carry capacity of the prey; A is the half saturation constant for the predators which is the prey density at which the functional response is half maximal; B is the death rate of predators. System (3) with $H = 0$ has been investigated by many researchers (see e.g. [1, 29–33] and references therein). System (3) with $H \neq 0$ have no results as far as we know. We assume that the predator in system (3) is not of commercial importance. The prey is continuously being harvested at a constant rate by a harvesting agency. The harvesting activity does not affect the predator population directly. It is obvious that the harvesting activity does reduce the predator population indirectly by reducing the availability of the prey to the predator.

The rest of the paper is organized as follows. In Section 2, we study the distribution of the roots of the characteristic equation and the stability of a positive constant steady state and show the conditions under which Hopf bifurcation occurs. In Section 3, we derived an algorithm for determining the direction and stability of the Hopf bifurcation by applying the normal form theory and the center manifold theory of partial functional differential equations ([34] and [35]). Some numerical analysis is given which support the analysis of Section 2 and Section 3 in Section 4. In Section 5, an optimal harvesting policy is considered by using Pontryagin's maximal principle.

2 Analysis of the characteristic equations

In this section, we consider the stability of the equilibria for system (3) on $\Omega = (0, l\pi)$ with $l \in \mathbb{R}^+$ by analyzing the characteristic roots.

Denote $X = C([0, l\pi], \mathbb{R}^2)$, $\mathcal{C}_\tau = C([-\tau, 0], X)$. Assume $(\phi, \psi) \in \mathcal{C}_\tau$, $u(0, x) = \phi(0, x) > 0$, $v(0, x) = \psi(0, x) > 0$, $\phi(\theta, x) \geq 0$, and $\psi(\theta, x) \geq 0$ for $\theta \in [-\tau, 0]$.

For convenience, some useful results of system (2) from [27] are rewritten as follows. Denote

$$\begin{aligned}\tau_0 &= \frac{1}{\sigma_+} \arccos\left(\frac{\sigma_+^2}{\alpha_2\beta_1}\right), \quad \sigma_+ = \sqrt{\frac{(-\alpha_1^2 + \sqrt{\alpha_1^4 + 4\alpha_2^2\beta^2})}{2}}, \\ \beta_1 &= \frac{Av^*}{(A+u^*)^2}, \quad \alpha_1 = r\left(1 - \frac{2u^*}{K}\right) - \frac{Av^*}{(A+u^*)^2}, \quad \alpha_2 = B = \frac{u^*}{A+u^*}, \\ u^* &= \frac{AB}{(1-B)}, \quad v^* = (A+u^*)\left(r - \frac{ru^*}{K} - \frac{H}{u^*}\right).\end{aligned}$$

Assume hypotheses (A1) and (A2) hold, where

- (A1) $rAB(K - KB - AB) - HK(1 - B)^2 > 0$ and $0 < B < 1$;
 (A2) $rKAB^2 + (1 - B)^3HK - rA^2B^2 - rKAB^3 - rA^2B^3 < 0$.

The following statements are true:

- (i) System (2) has a unique positive equilibrium $E = (u^*, v^*)$;
 (ii) Hopf bifurcation occurs in system (2);
 (iii) The Hopf bifurcation values for system (2) are $\tau_k = \tau_0 + 2k\pi/\sigma_+$, $k = 0, 1, 2, \dots$

Note that hypotheses (A1) and (A2) imply:

$$\begin{aligned}H < H_0 &:= ru^*\left(1 - \frac{u^*}{K}\right) = \frac{rAB}{K(1-B)^2}(K(1-B) - AB) \quad \text{and} \quad 0 < B < 1; \\ H < H_2 &:= \frac{ru^{*2}}{AK}(2u^* + A - K) = \frac{rAB^2}{K(1-B)^3}(A(1+B) - K(1-B)).\end{aligned}$$

The constant equilibrium points of system (2) are also the ones of system (3). Denote

$$u_+ = \frac{K}{2} + \sqrt{\frac{K(rK - 4H)}{4r}}, \quad u_- = \frac{K}{2} - \sqrt{\frac{K(rK - 4H)}{4r}}, \quad H_1 := \frac{rK}{4}.$$

And it is not difficult to show that the following lemma holds.

Theorem 1.

- (i) If $0 < H < H_0$, then system (3) has one interior equilibrium point $E^* = (u^*, v^*)$ and two boundary equilibrium points $E_1 = (u_+, 0)$ and $E_2 = (u_-, 0)$;
 (ii) If $H_0 \leq H < H_1$, then system (3) has two boundary equilibrium points E_1 and E_2 ;
 (iii) If $H = H_1$, then system (3) has only one equilibrium point $(K/2, 0)$;
 (iv) If $H > H_1$, then system (3) has no equilibrium points.

Remark 1. The existence and multiplicity of equilibria in system (3) are determined by the value of H from Lemma 1 above.

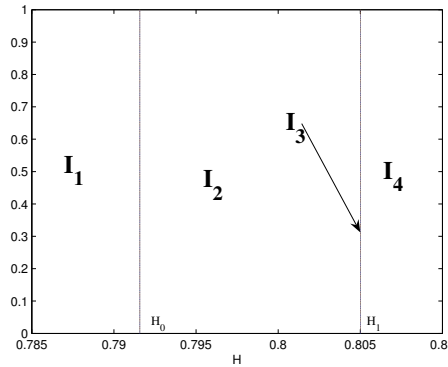


Fig. 1. The existence and multiplicity of equilibria with parameter H . If $H \in I_1 := (0, H_0)$, then system (3) has three equilibria; if $H \in I_2 := [H_0, H_1)$, then system (3) has two equilibria; if $H \in I_3 := \{H = H_1\}$, then system (3) has only one equilibrium; if $H \in I_4 := (H_1, \infty)$, then system (3) has no equilibria. Here $D_1 = 2, D_2 = 0.5, r = 0.7, K = 4.6, A = 1, B = 0.667, l = 5$.

2.1 The boundary equilibria

Now, we consider the stability of two bounded equilibria in $0 < H < H_1$.

2.1.1 Consider $E_1 = (u_+, 0)$

Linearizing system (3) around E_1 , we get the eigenvalues are

$$\lambda_{1n} = -D_1 \frac{n^2}{l^2} + r \left(1 - \frac{2u_+}{K} \right), \quad \lambda_{2n} = -D_2 \frac{n^2}{l^2} + \left(B - \frac{u_+}{A + u_+} \right).$$

Obviously, the eigenvalues are independent of the time delay, and the characteristic equation is coincide with system (3) with $\tau = 0$. $u_+ > K/2$ implies $\lambda_{1n} < 0$. From

$$B - \frac{u_+}{A + u_+} = \frac{1 - B}{A + u_+} \left(u^* - u_+ \right) = \frac{1 - B}{A + u_+} \left(u^* - \frac{K}{2} - \sqrt{\frac{K(Kr - 4H)}{4r}} \right),$$

we have, if $u^* < K/2$ or $u^* > K/2$ but $H < H_0$, then $u^* < u_+$ and $\lambda_{2n} < 0$; if $u^* > K/2$ and $H_0 < H < H_1$, then $u^* > u_+$ and $\lambda_{20} > 0$; if $u^* = K/2$, then $u_+ = K/2, H_0 = H_1$ and $\lambda_{10} = \lambda_{20} = 0, \lambda_{1n} < 0, \lambda_{2n} < 0$ as $n > 0$. If $u^* > K/2$ and $H = H_0$, then $u_+ = u^*, \lambda_{1n} < 0, \lambda_{20} = 0$, and $\lambda_{2n} < 0$, whose stability is determined by center manifold theory and normal form theory, and which is postponed to Appendix. Hence, we get the following lemma.

Theorem 2. For the boundary equilibrium E_1 of system (3), we have:

- (i) If $u^* < K/2$ but $0 < H < H_1$ or $u^* > K/2$ but $H \leq H_0$, then E_1 is local asymptotically stable;

- (ii) If $u^* > K/2$ and $H_0 < H < H_1$, then E_1 is unstable;
- (iii) If $u^* \equiv K/2$ and $0 < H < H_1$, then codimension two bifurcation occurs.

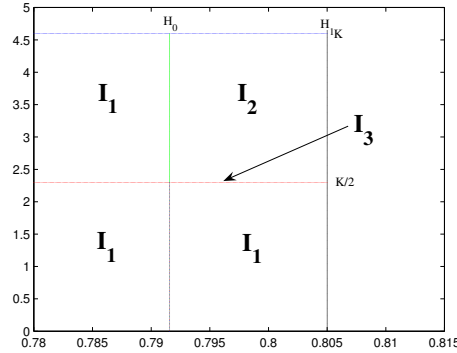


Fig. 2. Stability diagram of u_+ with H and u^* . u_+ is locally asymptotically stable, when $(H, u^*) \in I_1 := \{(H, u^*): u^* < K/2 \text{ but } 0 < H < H_1 \text{ or } u^* > K/2 \text{ but } H \leq H_0\}$; u_+ is unstable, when $(H, u^*) \in I_2 := \{(H, u^*): u^* > K/2 \text{ and } H_0 < H < H_1\}$; codimension two bifurcation occurs, when $(H, u^*) \in I_3 := \{(H, K/2): 0 < H < H_1\}$. Here $D_1 = 2, D_2 = 0.5, r = 0.7, K = 4.6, A = 1, B = 0.667, l = 5$.

2.1.2 Consider $E_2 = (u_-, 0)$

Using the same method above, we linearize the system around E_2 , and the eigenvalues are

$$\lambda_{1n} = r \left(1 - \frac{2u_-}{K} \right) - D_1 \frac{n^2}{l^2}, \quad \lambda_{2n} = B - \frac{u_-}{A + u_-} - D_2 \frac{n^2}{l^2}.$$

This implies that E_2 is unstable.

2.2 The interior equilibrium

Now, we focus on considering the dynamics behaviors of the interior equilibrium $E^* = (u^*, v^*)$. Let $\hat{u} = u - u^*, \hat{v} = v - v^*$, also let u and v denote \hat{u} and \hat{v} , respectively. System (3) is transformed into the following equivalent system:

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= D_1 \Delta u + ru \left(1 - \frac{2u^*}{K} - \frac{u}{K} \right) - \frac{(u + u^*)v}{A + u + u^*} - \frac{Auv^*}{(A + u + u^*)(A + u^*)}, \\ x \in \Omega, \quad t > 0, \\ \frac{\partial v(t)}{\partial t} &= D_2 \Delta v + (v + v^*) \frac{Au(t - \tau)}{(A + u^*)(A + u(t - \tau) + u^*)}, \quad x \in \Omega, \quad t > 0, \end{aligned} \tag{4}$$

$$\begin{aligned} u'(t, 0) = u'(t, l\pi) = 0, \quad v'(t, 0) = v'(t, l\pi) = 0, \quad t \geq 0, \\ u(\theta, x) = \phi(\theta, x) - u^*, \quad v(\theta, x) = \psi(\theta, x) - v^*, \quad x \in \bar{\Omega}, \quad \theta \in [-\tau, 0]. \end{aligned}$$

Denote $u_1(t) = u(t, \cdot)$, $u_2(t) = v(t, \cdot)$, and $U = (u_1, u_2)^T$. Then system (4) can be rewritten as an abstract differential equation in the phase space \mathcal{C}_τ ,

$$\dot{U}(t) = d\Delta U(t) + L(U_t) + F(U_t), \tag{5}$$

where $\text{dom}(d\Delta) = \{(u, v)^T \mid u, v \in C^2([0, l\pi], \mathbb{R}), \partial_\nu u = \partial_\nu v = 0, v = 0, l\pi\}$, $L : \mathcal{C}_\tau \mapsto X$, $F : \mathcal{C}_\tau \mapsto X$ are defined as

$$d = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad L(\phi) = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta_1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix},$$

$$F(\phi) = \begin{pmatrix} -\frac{r}{K}\phi_1^2(0) + \frac{Av^*\phi_1^2(0) - A(A+u^*)\phi_1(0)\phi_2(0)}{(A+u^*)^2(A+u^*+\phi_1(0))} \\ \frac{A\phi_1(-\tau)\phi_2(0)}{(A+u^*)(A+u^*+\phi_1(-\tau))} - \frac{Av^*\phi_1^2(-\tau)}{(A+u^*)^2(A+u^*+\phi_1(-\tau))} \end{pmatrix},$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in \mathcal{C}_\tau$.

The linearized equation of system (4) at the origin has the form:

$$\dot{U}(t) = d\Delta U(t) + L(U_t), \tag{6}$$

and its characteristic equation from [35] is

$$\lambda y - d\Delta y - L(e^{\lambda \cdot} y) = 0, \quad y \in \text{dom}(d\Delta) \setminus \{0\}, \tag{7}$$

where \cdot represents a time delay variable. It is well known that the operator $u \mapsto \Delta u$ with $\partial_\nu u = 0$ at 0 and $l\pi$ has eigenvalues $-n^2/l^2$, $n = 0, 1, 2, \dots$, with corresponding eigenfunctions $\cos(n\pi x/l)$. Let

$$\phi = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos\left(\frac{n\pi x}{l}\right)$$

be an eigenfunction for $\Delta + L$ with eigenvalue λ , see also [36]. Hence, Eq. (7) can be written as

$$\Delta_n(\lambda, \tau) = \lambda^2 + A_n\lambda + B_n + Ce^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0, \tag{5_n}$$

where

$$A_n = (D_1 + D_2)\frac{n^2}{l^2} - \alpha_1, \quad B_n = \frac{D_2 n^2}{l^2} \left(D_1 \frac{n^2}{l^2} - \alpha_1 \right), \quad C = \frac{ABv^*}{(A+u^*)^2} = \alpha_2\beta_1.$$

$$(A3) \ H < H_2^* := ABD_1 / ((1-B)^2 l^2) + H_2.$$

Lemma 1. Assume that hypotheses (A1) and (A3) hold. Then $\lambda = 0$ is not a root of Eq. (5_n) for any $n \in \mathbb{N}_0$.

Proof. From Eq. (5_n), we have

$$\Delta_n(0, \tau) = B_n + C = \frac{D_2 n^2}{l^2} \left(D_1 \frac{n^2}{l^2} - \alpha_1 \right) + \alpha_2\beta_1.$$

Sending n to either 0 or 1 in $\Delta_n(0, \tau)$, we obtain that

$$\Delta_0(0, \tau) = \alpha_2\beta_1 > 0, \quad \Delta_1(0, \tau) = \left(\frac{D_1}{l^2} - \alpha_1\right)\frac{D_2}{l^2} + \alpha_2\beta_1,$$

and $\Delta_{n+1}(0, \tau) > \Delta_n(0, \tau)$. By hypothesis (A3), we get $\alpha_1 < D_1/l^2$. That is, $\Delta_n(0, \tau) > 0$ for all $n \in \mathbb{N}_0$, which implies $\lambda = 0$ is not a root of Eq. (5_n) for any $n \in \mathbb{N}_0$. \square

Lemma 2. *Assume that hypotheses (A1) and (A3) hold. Then Eq. (5_n) with $\tau = 0$ has two negative real part roots for all $n \in \mathbb{N}_0$. Furthermore, equilibrium E^* of system (3) with $\tau = 0$ is local asymptotically stable.*

Proof. Eq. (5_n) with $\tau = 0$ is equivalent to the following quadratic equation,

$$\Delta_n(\lambda, 0) = \lambda^2 + A_n\lambda + B_n + C, \quad n \in \mathbb{N}_0. \quad (6_n)$$

Let λ_{1n} and λ_{2n} be two roots of Eq. (6_n). Then for any $n \in \mathbb{N}_0$,

$$\begin{aligned} \lambda_{1n} + \lambda_{2n} &= -A_n = \alpha_1 - (D_1 + D_2)\frac{n^2}{l^2}, \\ \lambda_{1n}\lambda_{2n} &= B_n + C = D_1D_2\frac{n^4}{l^4} - D_2\alpha_1\frac{n^2}{l^2} + \alpha_2\beta_1. \end{aligned}$$

From hypothesis (A3), $\lambda_{1n} + \lambda_{2n} < 0$, $\lambda_{1n}\lambda_{2n} > 0$, which implies Eq. (5_n) with $\tau = 0$ has two negative real part roots by Viète theorem. \square

From the result of Ruan and Wei [37], we know that the sum of the orders of the zeros of Eq. (5_n) in the open right half plane can change only if a zero appears on or crosses the imaginary axis, as τ varies. Now, we focus on the conditions under which the cases above occur. Denote

$$\tilde{N} = l\sqrt{\frac{\alpha_1 D_2 + \sqrt{4D_1 D_2 \alpha_2 \beta_1 + \alpha_1^2 D_2^2}}{2D_1 D_2}} \quad \text{and} \quad N_1 = \begin{cases} \tilde{N} - 1, & \tilde{N} \in \mathbb{N}, \\ [\tilde{N}], & \tilde{N} \notin \mathbb{N}. \end{cases}$$

Lemma 3. *Eq. (5_n) has a pair of purely imaginary roots $\pm i\omega_n$ ($0 \leq n \leq N_1$) at*

$$\tau = \tau_n^j = \tau_n^0 + \frac{2j\pi}{\omega_n}, \quad j = 0, 1, 2, \dots, \quad (8)$$

where

$$\begin{aligned} \tau_n^0 &= \frac{1}{\omega_n} \arccos \frac{w_n^2 - B_n}{C}, \\ \omega_n &= \sqrt{\left(\frac{2B_n - A_n^2 + \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C^2)}}{2}\right)}. \end{aligned} \quad (9)$$

Proof. Fix $n \in \mathbb{N}_0$. $i\omega$ ($\omega > 0$) is a root of Eq. (5_n) if and only if ω satisfies

$$-\omega^2 + A_n\omega i + B_n + C(\cos \omega\tau - i \sin \omega\tau) = 0.$$

Separating the real and imaginary parts, we have

$$-\omega^2 + B_n + C \cos \omega\tau = 0, \quad A_n\omega - C \sin \omega\tau = 0,$$

which is equivalent to

$$w^4 + (A_n^2 - 2B_n)w^2 + B_n^2 - C^2 = 0. \tag{10}$$

Setting $z = \omega^2$, Eq. (10) is transformed into

$$z^2 + (A_n^2 - 2B_n)z + B_n^2 - C^2 = 0, \tag{11}$$

where

$$\begin{aligned} A_n^2 - 2B_n &= \left[(D_1 + D_2) \frac{n^2}{l^2} - \alpha_1^2 \right]^2 - 2D_2 \left(D_1 \frac{n^4}{l^4} - \frac{n^2}{l^2} \alpha_1 \right) \\ &= D_2 \frac{n^4}{l^4} + \left(D_1 \frac{n^2}{l^2} - \alpha_1 \right)^2, \\ B_n^2 - C^2 &= (B_n + C)(B_n - C) \\ &= \left[D_2 \frac{n^2}{l^2} \left(D_1 \frac{n^2}{l^2} - \alpha_1 \right) + \alpha_2\beta_1 \right] \left[D_2 \frac{n^2}{l^2} \left(D_1 \frac{n^2}{l^2} - \alpha_1 \right) - \alpha_2\beta_1 \right]. \end{aligned}$$

From hypothesis (A3), $B_n + C > 0$,

$$B_n - C = D_2 \frac{n^2}{l^2} \left(D_1 \frac{n^2}{l^2} - \alpha_1 \right) - \alpha_2\beta_1 = D_1 D_2 \left(\frac{n^2}{l^2} - \frac{\alpha_1}{2D_1} \right)^2 - \alpha_2\beta_1 - \frac{D_2}{4D_1} \alpha_1^2.$$

For any $u^* \in (0, K]$, Eq. (11) has no positive roots for $n > N_1$, and $0 \leq n \leq N_1$ is the necessary condition of Eq. (11) having positive roots. For $0 \leq n \leq N_1$, a unique positive root z_n of Eq. (11) is

$$z_n = \frac{2B_n - A_n^2 + \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C^2)}}{2},$$

and

$$\omega_n = \sqrt{z_n} = \sqrt{\frac{2B_n - A_n^2 + \sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C^2)}}{2}},$$

is the imaginary part of the purely imaginary root. At

$$\tau = \tau_n^j = \tau_n^0 + \frac{2j\pi}{\omega_n} = \frac{1}{\omega_n} \arccos \frac{\omega_n^2 - B_n}{C} + \frac{2j\pi}{\omega_n}, \quad j = 0, 1, 2, \dots$$

Eq. (5_n) has a pair of purely imaginary roots $\pm i\omega_n$ ($0 \leq n \leq N_1$). Completing the proof. \square

Remark 2. Notice that Eq. (5₀) is the characteristic equation of the linearization of the system (2) at the positive equilibrium, which has been considered in [27]. From the results in [27] listed above, we know when $\tau = \tau_k$, system (2) has simple imaginary roots $\pm i\sigma_+$, which are just coincide with τ_0^k and $\pm i\omega_0$. That is, when $n = 0$, the conclusions of Lemma 2.3 are also the results in [27].

From Eq. (8) and Eq. (9), we have $\tau_n^j < \tau_n^{j+1}$ for any $0 \leq n \leq N_1$, $j \in \mathbb{N}_0$ and the following lemma point out the increasing proposition of τ_n^j in n .

Lemma 4. Let τ_n^j be defined as Eq. (8) and Eq. (9). Then for any $0 \leq n \leq N_1$, $j \in \mathbb{N}_0$, $\tau_n^j < \tau_{n+1}^j$.

Proof. From Eq. (8) and Eq. (9), we have

$$\begin{aligned}\omega_n^2 &= \frac{\sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C^2)} - (A_n^2 - 2B_n)}{2} \\ &= \frac{2}{\sqrt{\left(\frac{A_n^2 - 2B_n}{C^2 - B_n^2}\right)^2 + \frac{4}{(c^2 - B_n^2)} + \frac{A_n^2 - 2B_n}{C^2 - B_n^2}}} = \frac{2}{\sqrt{Y_n^2 + 4 + Y_n}},\end{aligned}$$

where

$$\begin{aligned}Y_n &= \frac{A_n^2 - 2B_n}{C^2 - B_n^2} = \frac{D_2^2 \frac{n^4}{l^4} + (D_1 \frac{n^2}{l^2} - \alpha_1)^2}{(\alpha_2 \beta_1)^2 - (D_1 D_2 \frac{n^4}{l^4} - D_2 \alpha_1 \frac{n^2}{l^2})^2} \\ &= \frac{D_2^2 y^2 + (D_1 y - \alpha_1)^2}{(\alpha_2 \beta_1)^2 - (D_1 D_2 y^2 - D_2 \alpha_1 y)^2}, \\ y &= \frac{n^2}{l^2}.\end{aligned}$$

Simple computation shows that

$$\begin{aligned}\frac{d\omega_n}{dY_n} &= \frac{-(1 + Y_n/\sqrt{Y_n^2 + 4})}{\sqrt{2}(\sqrt{Y_n^2 + 4 + Y_n})^{\frac{3}{2}}} < 0, \\ \frac{dY_n}{dy} &= \frac{2(D_2^2 y^2 + (D_1 y - \alpha_1)^2)(D_1 D_2 y^2 - \alpha_1 D_2 y)(2D_1 D_2 y - \alpha_1 D_2)}{((\alpha_2 \beta_1)^2 - (D_1 D_2 y^2 - \alpha_1 D_2 y)^2)^2} \\ &\quad + \frac{2D_2^2 y + 2D_1(D_1 y - \alpha_1)}{(\alpha_2 \beta_1)^2 - (D_1 D_2 y^2 - \alpha_1 D_2 y)^2} \\ &= \frac{2D_2^2 y(D_1 y - \alpha_1)(D_1 D_2^2 y^3 + (D_1 y - \alpha_1)^3)}{((\alpha_2 \beta_1)^2 - (D_1 D_2 y^2 - \alpha_1 D_2 y)^2)^2} \\ &\quad + \frac{(\alpha_2 \beta_1)^2 (2yD_2^2 + 2D_1(D_1 y - \alpha_1))}{((\alpha_2 \beta_1)^2 - (D_1 D_2 y^2 - \alpha_1 D_2 y)^2)^2} > 0.\end{aligned}$$

That is, ω_n is strictly decreasing in n for $0 \leq n \leq N_1$. Notice that B_n is strictly increasing in n for $0 \leq n \leq N_1$. Then we obtain $(\omega_n^2 - B_n)/C$ is strictly decreasing in n for $0 \leq n \leq N_1$. And $\arccos((\omega_n^2 - B_n)/C)$ is strictly increasing in n . Hence, τ_n^j is strictly increasing in n . \square

From the above proposition, we have $\tau_n^0 < \tau_n^1 < \tau_n^2 < \dots < \tau_n^j < \dots$ for any $0 \leq n \leq N_1$ and $\tau_0^j < \tau_1^j < \tau_2^j < \dots < \tau_n^j < \dots < \tau_{N_1}^j, j \in \mathbb{N}_0$. Notice that what possibility will occur is $\tau_m^j = \tau_n^k$ for some $m > n, j < k$. In this paper, we do not consider this case. In other words, we consider

$$\tau \in \mathcal{D} := \{\tau_n^j: \tau_m^j \neq \tau_n^k, m \neq n, j \neq k, 0 \leq n, m \leq N_1, j, k \in \mathbb{N}_0\}.$$

Let $\lambda_n(\tau) = \alpha_n(\tau) + i\omega_n(\tau)$ denote a root of Eq. (5_n) satisfying $\alpha_n(\tau_n^j) = 0$ and $\omega_n(\tau_n^j) = \omega_n$, when τ is close to τ_n^j . Then we have the following transversality condition.

Lemma 5. $\alpha'(\tau_n^j) = \text{Re}(d\lambda/d\tau)_{\tau=\tau_n^j} > 0$.

Proof. From Eq. (5_n), we have

$$\frac{d\lambda}{d\tau}(2\lambda + A_n - C\tau e^{-\lambda\tau}) = C\lambda e^{-\lambda\tau}.$$

Hence,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A_n - C\tau e^{-\lambda\tau}}{C\lambda e^{-\lambda\tau}} = \frac{2}{C}e^{\lambda\tau} + \frac{A_n}{C\lambda}e^{\lambda\tau} - \frac{\tau}{\lambda}.$$

Substituting τ_n^j into the above equation, we obtain

$$\begin{aligned} (\lambda'(\tau_n^j))^{-1} &= \text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_n^j} = \frac{2 \cos(\omega_n \tau_n^j)}{C} + \frac{A_n \sin(\omega_n \tau_n^j)}{C\omega_n} \\ &= \frac{\sqrt{(A_n^2 - 2B_n)^2 - 4(B_n^2 - C^2)}}{C^2} > 0, \end{aligned}$$

where $\cos(\omega_n \tau_n^j) = (\omega_n^2 - B_n)/C, \sin(\omega_n \tau_n^j) = (A_n \omega_n)/C$. □

From the above analysis, we have the following theorem.

Theorem 3. *Suppose that hypotheses (A1) and (A3) hold. For system (3), the following statements are true:*

- (i) *If $\tau \in [0, \tau_0^0)$, then the equilibrium point $E^*(u^*, v^*)$ is local asymptotically stable.*
- (ii) *If $\tau > \tau_0^0$, then the equilibrium point $E^*(u^*, v^*)$ is unstable.*
- (iii) *$\tau = \tau_0^j (j \in \mathbb{N}_0)$ are Hopf bifurcation values of system (3) and the bifurcating periodic solutions are all spatially homogeneous, which coincides with the periodic solution of the corresponding ODE system. And when $\tau \in \mathcal{D} \setminus \{\tau_0^k: k \in \mathbb{N}_0\}$, system (3) also undergoes a Hopf bifurcation of spatially inhomogeneous periodic orbits.*

3 Direction and stability of spatial Hopf bifurcation

In this section, we shall study the directions, stability and the period of bifurcating periodic solutions by applying the normal form theory and the center manifold theorem of partial functional differential equations presented in [34, 35]. For fixed $j \in \mathbb{N}_0$, $0 \leq n \leq N_1$, we denote $\tilde{\tau} = \tau_n^j$. Let $\tilde{u}(t, x)$ and $\tilde{v}(t, x)$ denote $u(\tau t, x)$ and $v(\tau t, x)$, respectively. And drop the tilde as a matter of convenience. Then system (4) can be transformed into

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= \tau \left(d\Delta u + ru \left(1 - \frac{2u^*}{K} - \frac{u}{K} \right) - \frac{(u+u^*)v}{A+u+u^*} - \frac{Auv^*}{(A+u+u^*)(A+u^*)} \right), \\ \frac{\partial v(t)}{\partial t} &= \tau \left(\Delta v + \frac{A(v+v^*)u(t-1)}{(A+u^*)(A+u(t-1)+u^*)} \right) \end{aligned} \quad (12)$$

for $x \in \Omega$, $t > 0$. Let $\tau = \tilde{\tau} + \mu$ ($\mu \in \mathbb{R}$), $u_1(t) = u(t, \cdot)$, $u_2(t) = v(t, \cdot)$, $U = (u_1, u_2)^T$, $\gamma = r/K$ and $G := A + u^* = A/(1 - B)$. Then Eq. (12) can be rewritten in an abstract form in the phase space $\mathcal{C}_1 := C([-1, 0], X)$,

$$\frac{dU(t)}{dt} = \tilde{\tau} d\Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t, \mu), \quad (13)$$

where $L_{\tilde{\tau}} : \mathcal{C}_1 \mapsto X$, $F : \mathcal{C}_1 \mapsto X$ are defined by

$$L_{\mu}(\phi) = \mu \begin{pmatrix} \alpha_1 \phi_1(0) - \alpha_2 \phi_2(0) \\ \beta_1 \phi_1(-1) \end{pmatrix},$$

and

$$\begin{aligned} F(\phi, \mu) &= \mu D\Delta \phi + L_{\mu}(\phi) + f(\phi, \mu), \\ f(\phi, \mu) &= (\tilde{\tau} + \mu) \begin{pmatrix} -\gamma \phi_1^2(0) + \frac{Av^* \phi_1^2(0) - AG\phi_1(0)\phi_2(0)}{G^2(G+\phi_1(0))} \\ \frac{A\phi_1(-1)\phi_2(0)}{G(G+\phi_1(-1))} - \frac{Av^* \phi_1^2(-1)}{G^2(G+\phi_1(-1))} \end{pmatrix}, \end{aligned}$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in \mathcal{C}_1$.

Consider the linear equation

$$\frac{dU(t)}{dt} = \tilde{\tau} d\Delta U(t) + L_{\tilde{\tau}}(U_t). \quad (14)$$

According to results in Section 2, we know the origin $(0, 0)$ is an equilibrium of system (12), and $\Lambda_n := \{i\omega_n \tilde{\tau}, -i\omega_n \tilde{\tau}\}$ are characteristic values of the system (12) of ordinary functional differential equation

$$\frac{dz(t)}{dt} = -\tilde{\tau} d \frac{n^2}{l^2} z(t) + L_{\tilde{\tau}}(z_t). \quad (15)$$

By the Riesz representation theorem, there exists a 2×2 matrix function $\eta^n(\theta, \tilde{\tau})$, $-1 \leq \theta \leq 0$, whose entries are of bounded variation such that

$$-\tilde{\tau}d \frac{n^2}{l^2} \phi(0) + L_{\tilde{\tau}}(\phi) = \int_{-1}^0 d\eta^n(\theta, \tau) \phi(\theta)$$

for $\phi \in C([-1, 0], \mathbb{R}^2)$.

In fact, we can choose

$$\eta^n(\theta, \tau) = \begin{cases} \tau \begin{pmatrix} \alpha_1 & -\alpha_2 \\ 0 & 0 \end{pmatrix}, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ \tau \begin{pmatrix} 0 & 0 \\ \beta_1 & 0 \end{pmatrix}, & \theta = -1. \end{cases}$$

Let $A(\tilde{\tau})$ denote the infinitesimal generators of the semigroup induced by the solutions of Eq. (15) and A^* be the formal adjoint of $A(\tilde{\tau})$ under the bilinear pairing

$$\begin{aligned} (\psi, \phi) &= \psi(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \psi(\xi - \theta) d\eta^n(\theta, \tilde{\tau}) \phi(\xi) d\xi \\ &= \psi(0)\phi(0) + \tilde{\tau} \int_{-1}^0 \psi(\xi + 1) \begin{pmatrix} 0 & 0 \\ \beta_1 & 0 \end{pmatrix} \phi(\xi) d\xi \end{aligned} \tag{16}$$

for $\phi \in C([-1, 0], \mathbb{R}^2)$, $\psi \in C([0, 1], (\mathbb{R}^2)^*)$ (see [36, 39]). $A(\tilde{\tau})$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_n \tilde{\tau}$, and they are also eigenvalues of A^* . Let P and P^* be the center subspace, that is, the generalized eigenspace of $A(\tilde{\tau})$ and A^* associated with Λ_n , respectively. Then P^* is the adjoint space of P and $\dim P = \dim P^* = 2$.

It can be verified that $p_1(\theta) = (1, \xi)^T e^{i\omega_n \tilde{\tau} \theta}$ ($\theta \in [-1, 0]$), $p_2(\theta) = \overline{p_1(\theta)}$ is a basis of $A(\tilde{\tau})$ with Λ_n and $q_1(s) = (1, \eta)e^{-i\omega_n \tilde{\tau} s}$ ($s \in [0, 1]$), $q_2(s) = q_1(s)$ is a basis of A^* with Λ_n , where

$$\xi = \frac{\alpha_1 - i\omega_n - D_1 \frac{n^2}{l^2}}{\alpha_2} = \frac{\beta_1 e^{-i\omega_n \tilde{\tau}}}{i\omega_n + D_2 \frac{n^2}{l^2}}, \quad \eta = \frac{\alpha_2}{i\omega_n - D_2 \frac{n^2}{l^2}} = \frac{D_1 \frac{n^2}{l^2} - i\omega_n - \alpha_1}{\beta_1 e^{i\omega_n \tilde{\tau}}}.$$

Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)^T$ with

$$\begin{aligned} \Phi_1(\theta) &= \frac{p_1(\theta) + p_2(\theta)}{2} = \begin{pmatrix} \operatorname{Re}(e^{i\omega_n \tilde{\tau} \theta}) \\ \operatorname{Re}(\xi e^{i\omega_n \tilde{\tau} \theta}) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega_n \tilde{\tau} \theta) \\ \frac{1}{\alpha_2} (\alpha_1 \cos(\omega_n \tilde{\tau} \theta) + \omega_n \sin(\omega_n \tilde{\tau} \theta)) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}\Phi_2(\theta) &= \frac{p_1(\theta) - p_2(\theta)}{2i} = \begin{pmatrix} \operatorname{Im}(e^{i\omega_n \tilde{\tau}\theta}) \\ \operatorname{Im}(\xi e^{i\omega_n \tilde{\tau}\theta}) \end{pmatrix} \\ &= \begin{pmatrix} \sin(\omega_n \tilde{\tau}\theta) \\ \frac{1}{\alpha_2}(\alpha_1 \sin(\omega_n \tilde{\tau}\theta) - \omega_n \cos(\omega_n \tilde{\tau}\theta)) \end{pmatrix}\end{aligned}$$

for $\theta \in [-1, 0]$ and

$$\begin{aligned}\Psi_1^*(s) &= \frac{q_1(s) + q_2(s)}{2} = (\operatorname{Re}(e^{-i\omega_n \tilde{\tau}s}), \operatorname{Re}(\eta e^{-i\omega_n \tilde{\tau}s})) \\ &= \left(\cos(\omega_n \tilde{\tau}s), -\frac{\alpha_2}{\omega_n} \sin(\omega_n \tilde{\tau}s) \right), \\ \Psi_2^*(s) &= \frac{q_1(s) - q_2(s)}{2i} = (\operatorname{Im}(e^{-i\omega_n \tilde{\tau}s}), \operatorname{Im}(\eta e^{-i\omega_n \tilde{\tau}s})) \\ &= \left(-\sin(\omega_n \tilde{\tau}s), -\frac{\alpha_2}{\omega_n} \cos(\omega_n \tilde{\tau}s) \right)\end{aligned}$$

for $s \in [0, 1]$. Then we have by (16)

$$\begin{aligned}D_1^* &:= (\Psi_1^*, \Phi_1) = 1 - \frac{\tilde{\tau}}{2\omega_n} \alpha_2 \beta_1 \sin(\omega_n \tilde{\tau}), \\ D_2^* &:= (\Psi_1^*, \Phi_2) = \frac{\tilde{\tau}}{2\omega_n} \alpha_2 \beta_1 \left(\frac{\sin(\omega_n \tilde{\tau})}{\omega_n \tilde{\tau}} - \cos(\omega_n \tilde{\tau}) \right), \\ D_3^* &:= (\Psi_2^*, \Phi_1) = -\frac{\alpha_1}{\omega_n} - \frac{\tilde{\tau}}{2\omega_n} \alpha_2 \beta_1 \left(\cos(\omega_n \tilde{\tau}) + \frac{\sin(\omega_n \tilde{\tau})}{\omega_n \tilde{\tau}} \right), \\ D_4^* &:= (\Psi_2^*, \Phi_2) = 1 + \frac{\tilde{\tau}}{2\omega_n} \alpha_2 \beta_1 \sin(\omega_n \tilde{\tau}).\end{aligned}$$

Define

$$(\Psi^*, \Phi) = (\Psi_j^*, \Phi_k) = \begin{pmatrix} (\Psi_1^*, \Phi_1) & (\Psi_1^*, \Phi_2) \\ (\Psi_2^*, \Phi_1) & (\Psi_2^*, \Phi_2) \end{pmatrix}$$

and construct a new basis Ψ for P^* by $\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1} \Psi^*$. Then $(\Psi, \Phi) = I_2$ and

$$\begin{aligned}\Psi_1(0) &= \frac{1}{D_1^* D_4^* - D_2^* D_3^*} \left(D_4^*, \frac{\alpha_2}{\omega_n} D_2^* \right), \\ \Psi_2(0) &= \frac{1}{D_1^* D_4^* - D_2^* D_3^*} \left(-D_3^*, -\frac{\alpha_2}{\omega_n} D_1^* \right).\end{aligned}$$

In addition, $f_n := (\beta_n^1, \beta_n^2)$, where

$$\beta_n^1 = \begin{pmatrix} \cos(\frac{n}{l}x) \\ 0 \end{pmatrix}, \quad \beta_n^2 = \begin{pmatrix} 0 \\ \cos(\frac{n}{l}x) \end{pmatrix}.$$

Let $c \cdot f_n$ be defined by $c \cdot f_n = c_1 \beta_n^1 + c_2 \beta_n^2$ for $c = (c_1, c_2)^T \in \mathcal{C}_1$.

Thus the center subspace of linear equation (14) is given by $P_{CN}\mathcal{C}_1$, where for $\phi \in \mathcal{C}_1$, $P_{CN}\phi := \Phi(\Psi, \langle \phi, f_n \rangle) \cdot f_n$. $\mathcal{C}_1 = P_{CN}\mathcal{C}_1 \oplus P_S\mathcal{C}_1$ and $P_S\mathcal{C}_1$ denotes the complement subspace of $P_{CN}\mathcal{C}_1$ in \mathcal{C}_1 ,

$$\langle u, v \rangle := \frac{1}{l\pi} \int_0^{l\pi} u_1 \bar{v}_1 \, dx + \frac{1}{l\pi} \int_0^{l\pi} u_2 \bar{v}_2 \, dx$$

for $u = (u_1, u_2), v = (v_1, v_2), u, v \in X$ and $\langle \phi, f_0 \rangle = (\langle \phi, f_0^1 \rangle, \langle \phi, f_0^2 \rangle)^T$.

Let $A_{\bar{\tau}}$ denote the infinitesimal generator of an analytic semigroup induced by the linear system(14), and Eq. (12) can be rewritten as the following abstract form

$$\frac{dU(t)}{dt} = A_{\bar{\tau}}U_t + R(U_t, \mu),$$

where

$$R(U_t, \mu) = \begin{cases} 0, & \theta \in [-1, 0), \\ F(U_t, \mu), & \theta = 0. \end{cases}$$

By the decomposition of \mathcal{C}_1 , the solution above can be written as

$$U_t = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot f_n + h(x_1, x_2, \mu), \tag{17}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\Psi, \langle U_t, f_n \rangle),$$

and $h(x_1, x_2, \mu) \in P_S\mathcal{C}_1, h(0, 0, 0) = 0, Dh(0, 0, 0) = 0$. In particular, the solution of Eq. (13) on the center manifold is given by

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_n + h(x_1, x_2, 0).$$

Let $z = x_1 - ix_2$, and notice that $p_1 = \Phi_1 + i\Phi_2$. Then we have

$$\Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_n = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{z+\bar{z}}{2} \\ \frac{i(z-\bar{z})}{2} \end{pmatrix} \cdot f_n = \frac{1}{2}(p_1z + \bar{p}_1\bar{z}) \cdot f_n,$$

and

$$h(x_1, x_2, 0) = h\left(\frac{z + \bar{z}}{2}, \frac{(z - \bar{z})i}{2}, 0\right).$$

Hence, Eq. (17) can be transformed into

$$\begin{aligned} U_t &= \frac{1}{2}(p_1z + \bar{p}_1\bar{z}) \cdot f_n + h\left(\frac{z + \bar{z}}{2}, \frac{(z - \bar{z})i}{2}, 0\right) \\ &= \frac{1}{2}(p_1z + \bar{p}_1\bar{z}) \cdot f_n + W(z, \bar{z}), \end{aligned} \tag{18}$$

where

$$W(z, \bar{z}) = h\left(\frac{z + \bar{z}}{2}, \frac{(z - \bar{z})i}{2}, 0\right).$$

From [35], z satisfies

$$\dot{z} = i\omega_n \tilde{\tau} z + g(z, \bar{z}),$$

where

$$g(z, \bar{z}) = (\Psi_1(0) - i\Psi_2(0)) \langle F(U_t, 0), f_n \rangle. \quad (19)$$

Let

$$W(z, \bar{z}) = W_{20} \frac{z^2}{2} + W_{11} z\bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots, \quad (20)$$

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} \dots, \quad (21)$$

from Eq. (18) and Eq. (20), we have

$$u_t(0) = \frac{1}{2}(z + \bar{z}) \cos\left(\frac{n}{l}x\right) + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots,$$

$$v_t(0) = \frac{1}{2}(\xi z + \bar{\xi} \bar{z}) \cos\left(\frac{n}{l}x\right) + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots,$$

$$u_t(-1) = \frac{1}{2}(ze^{-i\omega_n \tilde{\tau}} + \bar{z}e^{i\omega_n \tilde{\tau}}) \cos\left(\frac{n}{l}x\right) + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} \\ + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \dots,$$

$$\frac{u}{A + u^* + u} = \frac{u}{G + u} = \frac{u}{G} - \frac{u^2}{G^2} + \dots,$$

and

$$\bar{F}_1(U_t, 0) = \frac{1}{\tilde{\tau}} F_1(U_t, 0) = -\gamma u_t^2(0) + \frac{Av^* u_t^2(0) - AGu_t(0)v_t(0)}{G^2(G + u_t(0))}$$

$$= -\gamma u_t^2(0) + \frac{Au_t(0)}{G^4} (G - u_t(0)) (v^* u_t(0) - Gv_t(0)),$$

$$\bar{F}_2(U_t, 0) = \frac{1}{\tilde{\tau}} F_2(U_t, 0) = \frac{Au_t(-1)v_t(0)}{G(G + u_t(-1))} - \frac{Av^* u_t^2(-1)}{G^2(G + u_t(-1))}$$

$$= \frac{Au_t(-1)}{G^4} (G - u_t(-1)) (Gv_t(0) - v^* u_t(-1)).$$

Hence,

$$\begin{aligned} \bar{F}_1(U_t, 0) &= \frac{z^2}{2} \left(\frac{\cos^2(\frac{n}{l}x)}{2G^3} (-\gamma G^3 + Av^* - A\xi G) \right) \\ &\quad + z\bar{z} \left(\frac{\cos^2(\frac{n}{l}x)}{4G^3} (-2\gamma G^3 + 2Av^* - AG(\xi + \bar{\xi})) \right) \\ &\quad + \frac{\bar{z}^2}{2} \left(\frac{\cos^2(\frac{n}{l}x)}{2G^3} (-\gamma G^3 + Av^* - AG\bar{\xi}) \right) \\ &\quad + \frac{z^2\bar{z}}{2} \left(\cos\left(\frac{n}{l}x\right) \left(-\gamma + \frac{Av^*}{G^3} - \frac{A\xi}{G^2} \right) W_{11}^{(1)}(0) \right. \\ &\quad \quad \left. + \cos\left(\frac{n}{l}x\right) \left(-\gamma + \frac{Av^*}{2G^3} - \frac{A\bar{\xi}}{2G^2} \right) W_{20}^{(1)}(0) \right. \\ &\quad \quad \left. - \frac{A}{G^2} \left(W_{11}^{(2)}(0) + \frac{W_{20}^{(2)}(0)}{2} \right) \cos\left(\frac{n}{l}x\right) \right. \\ &\quad \quad \left. + \frac{A}{4G^4} ((2\xi + \bar{\xi})G - 3v^*) \cos^3\left(\frac{n}{l}x\right) \right) + \dots, \end{aligned}$$

$$\begin{aligned} \bar{F}_2(U_t, 0) &= \frac{z^2}{2} \left(\frac{Ae^{-i\omega_n\bar{\tau}}}{2G^3} (\xi G - v^*e^{-i\omega_n\bar{\tau}}) \cos^2\left(\frac{n}{l}x\right) \right) \\ &\quad + z\bar{z} \left(\frac{A}{4G^3} (\bar{\xi}Ge^{-i\omega_n\bar{\tau}} - 2v^* + \xi Ge^{i\omega_n\bar{\tau}}) \cos^2\left(\frac{n}{l}x\right) \right) \\ &\quad + \frac{\bar{z}^2}{2} \left(\frac{Ae^{i\omega_n\bar{\tau}}}{2G^3} (\bar{\xi}G - v^*e^{i\omega_n\bar{\tau}}) \cos^2\left(\frac{n}{l}x\right) \right) \\ &\quad + \frac{z^2\bar{z}}{2} \left(\cos\left(\frac{n}{l}x\right) \frac{A}{G^3} (e^{-i\omega_n\bar{\tau}}GW_{11}^{(2)}(0) + (\xi G - 2v^*e^{-i\omega_n\bar{\tau}})W_{11}^{(1)}(-1)) \right. \\ &\quad \quad \left. + \cos\left(\frac{n}{l}x\right) \frac{A}{2G^3} (GW_{20}^{(2)}(0)e^{i\omega_n\bar{\tau}} + (G\bar{\xi} - 2v^*e^{i\omega_n\bar{\tau}})W_{20}^{(1)}(-1)) \right. \\ &\quad \quad \left. + \cos^3\left(\frac{n}{l}x\right) \frac{A}{4G^4} (3v^*e^{-i\omega_n\bar{\tau}} - (2\xi + \bar{\xi}e^{-2i\omega_n\bar{\tau}})G) \right) + \dots, \end{aligned}$$

and

$$\begin{aligned} \langle F(U_t, 0), f_n \rangle &= \tilde{\tau} (\bar{F}_1(U_t, 0)f_n^1 + \bar{F}_2(U_t, 0)f_n^2) \\ &= \frac{\tilde{\tau}}{2G^3} \frac{z^2}{2} \left(\frac{-\gamma G^3 + Av^* - A\xi G}{Ae^{-i\omega_n\bar{\tau}}(\xi G - v^*e^{-i\omega_n\bar{\tau}})} \right) \frac{1}{l\pi} \int_0^{l\pi} \cos^3\left(\frac{n}{l}x\right) dx \end{aligned}$$

$$\begin{aligned}
 & + \frac{\tilde{\tau}}{4G^3} z\bar{z} \left(-2\gamma G^3 + 2Av^* - A(\xi + \bar{\xi})G \right) \frac{1}{l\pi} \int_0^{l\pi} \cos^3 \left(\frac{n}{l}x \right) dx \\
 & + \frac{\tilde{\tau}}{2G^3} \frac{z^2}{2} \left(A e^{i\omega_n \tilde{\tau}} (\bar{\xi}G - v^* e^{i\omega_n \tilde{\tau}}) \right) \frac{1}{l\pi} \int_0^{l\pi} \cos^3 \left(\frac{n}{l}x \right) dx \\
 & + \frac{\tilde{\tau}}{G^3} \frac{z^2 \bar{z}}{2} \left(\begin{aligned} & (-2\gamma G^3 + Av^* - A\xi G) \langle W_{11}^{(1)}(0) \cos \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & + \frac{1}{2} (-2\gamma G^3 + Av^* - A\bar{\xi}G) \langle W_{20}^{(1)}(0) \cos \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & - AG \langle (W_{11}^{(2)}(0) + \frac{W_{20}^{(2)}(0)}{2}) \cos \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & + \frac{A}{4} ((2\xi + \bar{\xi}) - \frac{3v^*}{4G}) \langle \cos^3 \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & + AGe^{-i\omega_n \tilde{\tau}} \langle W_{11}^{(2)}(0) \cos \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & + A(\xi G - 2v^* e^{-i\omega_n \tilde{\tau}}) \langle W_{11}^{(1)}(-1) \cos \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & + \frac{AGe^{i\omega_n \tilde{\tau}}}{2} \langle W_{20}^{(2)}(0) \cos \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & + \frac{A}{2} (\bar{\xi}G - 2v^* e^{i\omega_n \tilde{\tau}}) \langle W_{20}^{(1)}(-1) \cos \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & + \frac{3Av^* e^{-i\omega_n \tilde{\tau}}}{4G} \langle \cos^3 \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \\ & - \frac{A}{4} (2\xi + \bar{\xi} e^{-2i\omega_n \tilde{\tau}}) \langle \cos^3 \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \rangle \end{aligned} \right) + \dots
 \end{aligned}$$

Denote $\Gamma = \int_0^{l\pi} \cos^3(\frac{n}{l}x) dx$, and

$$\Psi_1(0) - i\Psi_2(0) = \frac{1}{D_1^* D_4^* - D_2^* D_3^*} \left(D_4^* + D_3^* i, \frac{\alpha_2}{\omega_n} (D_2^* + D_1^* 1) \right) := (\Upsilon_1, \Upsilon_2).$$

Notice that $\int_0^{l\pi} \cos^3((n/l)x) dx = 0$ for $n = 1, 2, 3, \dots$, and we have

$$\begin{aligned}
 & (\Psi_1(0) - i\Psi_2(0)) \langle F(U_t, 0), f_n \rangle \\
 & = \tilde{\tau} \left(\frac{\Gamma}{4G^3} \left(\frac{z^2}{2} \left(2\Upsilon_1 (-\gamma G^3 + Av^* - AG\xi) + 2\Upsilon_2 A e^{-i\omega_n \tilde{\tau}} (G\xi - v^* e^{-i\omega_n \tilde{\tau}}) \right) \right. \right. \\
 & \quad \left. \left. + z\bar{z} \left(\Upsilon_1 (-2\gamma G^3 + 2Av^* - AG(\xi + \bar{\xi})) + \Upsilon_2 A (G\bar{\xi} e^{-i\omega_n \tilde{\tau}} - 2v^* + G\xi e^{i\omega_n \tilde{\tau}}) \right) \right) \right. \\
 & \quad \left. + \frac{\bar{z}^2}{2} \left(2\Upsilon_1 (-\gamma G^3 + Av^* - AG\bar{\xi}) + 2\Upsilon_2 A e^{i\omega_n \tilde{\tau}} (G\bar{\xi} - v^* e^{i\omega_n \tilde{\tau}}) \right) \right) \\
 & \quad \left. + \kappa \frac{z^2 \bar{z}}{2} + \dots \right), \tag{22}
 \end{aligned}$$

where

$$\kappa = \Upsilon_1 \left(\left(-2\gamma + \frac{Av^*}{G^3} - \frac{A\xi}{G^2} \right) \left\langle W_{11}^{(1)}(0) \cos \left(\frac{n}{l}x \right), \cos \left(\frac{n}{l}x \right) \right\rangle \right)$$

$$\begin{aligned}
 & + \left(-\gamma + \frac{Av^*}{2G^3} - \frac{A\bar{\xi}}{2G^2} \right) \left\langle W_{20}^{(1)}(0) \cos\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \\
 & - \frac{A}{G^2} \left\langle \left(W_{11}^{(2)}(0) + \frac{W_{20}^{(2)}(0)}{2} \right) \cos\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \\
 & + \frac{A}{4G^3} \left((2\xi + \bar{\xi}) - \frac{3v^*}{4G} \right) \left\langle \cos^3\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \\
 & + \frac{A}{G^3} \Upsilon_2 \left(Ge^{-i\omega_n \tilde{\tau}} \left\langle W_{11}^{(2)}(0) \cos\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \right. \\
 & \quad + (G\xi - 2v^* e^{-i\omega_n \tilde{\tau}}) \left\langle W_{11}^{(1)}(-1) \cos\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \\
 & \quad + \frac{G}{2} e^{i\omega_n \tilde{\tau}} \left\langle W_{20}^{(2)}(0) \cos\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \\
 & \quad + \frac{1}{2} (G\bar{\xi} - 2v^* e^{i\omega_n \tilde{\tau}}) \left\langle W_{20}^{(1)}(-1) \cos\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \\
 & \quad + \frac{3}{4} \frac{3v^*}{G} e^{-i\omega_n \tilde{\tau}} \left\langle \cos^3\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \\
 & \quad \left. - \frac{1}{4} (2\xi + \bar{\xi} e^{-2i\omega_n \tilde{\tau}}) \left\langle \cos^3\left(\frac{n}{l}x\right), \cos\left(\frac{n}{l}x\right) \right\rangle \right).
 \end{aligned}$$

Then by (19), (21) and (22), we have $g_{20} = g_{11} = g_{02} = 0$, when $n = 1, 2, 3 \dots$. When $n = 0$, we obtain the following quantities:

$$\begin{aligned}
 g_{20} &= \frac{\tilde{\tau}}{2G^3} (\Upsilon_1 (-G^3\gamma + Av^* - AG\xi) + \Upsilon_2 Ae^{-i\omega_n \tilde{\tau}} (G\xi - v^* e^{-i\omega_n \tilde{\tau}})), \\
 g_{11} &= \frac{\tilde{\tau}}{4G^3} (\Upsilon_1 (-2\gamma G^3 + 2Av^* - AG(\xi + \bar{\xi})) + \Upsilon_2 A(\bar{\xi} Ge^{-i\omega_n \tilde{\tau}} - 2v^* + \xi Ge^{i\omega_n \tilde{\tau}})), \\
 g_{02} &= \frac{\tilde{\tau}}{2G^3} (\Upsilon_1 (-\gamma G^3 + Av^* - AG\bar{\xi}) + \Upsilon_2 Ae^{i\omega_n \tilde{\tau}} (G\bar{\xi} - v^* e^{i\omega_n \tilde{\tau}})).
 \end{aligned}$$

And for $n \in \mathbb{N}_0$, $g_{21} = \tilde{\tau}\kappa$.

Now, a complete description for g_{21} depends on the algorithm for $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-1, 0]$ which we will compute.

From (20), we have

$$\begin{aligned}
 \dot{W}(z, \bar{z}) &= W_{20} z \dot{z} + W_{11} \dot{z} \bar{z} + W_{11} z \dot{\bar{z}} + W_{02} \bar{z} \dot{\bar{z}} + \dots, \\
 A_{\tilde{\tau}} W(z, \bar{z}) &= A_{\tilde{\tau}} W_{20} \frac{z^2}{2} + A_{\tilde{\tau}} W_{11} z \bar{z} + A_{\tilde{\tau}} W_{02} \frac{\bar{z}^2}{2} + \dots.
 \end{aligned}$$

By [35], $W(z, \bar{z})$ satisfies

$$\dot{W} = A_{\tilde{\tau}} W + H(z, \bar{z}),$$

where

$$\begin{aligned} H(z, \bar{z}) &= H_{20} \frac{z^2}{2} + H_{11} z \bar{z} + H_{02} \frac{\bar{z}^2}{2} + \cdots \\ &= X_0 F(U_t, 0) - \Phi(\Psi, \langle X_0 F(U_t, 0), f_n \rangle) \cdot f_n. \end{aligned} \quad (23)$$

Hence,

$$(2i\omega_n \tilde{\tau} - A_{\tilde{\tau}})W_{20} = H_{20}, \quad -A_{\tilde{\tau}}W_{11} = H_{11}, \quad (-2i\omega_n \tilde{\tau} - A_{\tilde{\tau}})W_{02} = H_{02}. \quad (24)$$

That is,

$$W_{20} = (2i\omega_n \tilde{\tau} - A_{\tilde{\tau}})^{-1} H_{20}, \quad W_{11} = -A_{\tilde{\tau}}^{-1} H_{11}, \quad W_{02} = (-2i\omega_n \tilde{\tau} - A_{\tilde{\tau}})^{-1} H_{02}.$$

By Eq. (22), we know that for $\theta \in [-1, 0)$,

$$\begin{aligned} H(z, \bar{z}) &= -\Phi(\theta)\Psi(0)\langle F(U_t, 0), f_n \rangle \cdot f_n \\ &= -\left(\frac{p_1(\theta) + p_2(\theta)}{2}, \frac{p_1(\theta) - p_2(\theta)}{2i}\right) \begin{pmatrix} \Psi_1(0) \\ \Psi_2(0) \end{pmatrix} \langle F(U_t, 0), f_n \rangle \cdot f_n \\ &= -\frac{1}{2}(p_1(\theta)(\Psi_1(0) - i\Psi_2(0)) + p_2(\theta)(\Psi_1(0) + i\Psi_2(0))) \langle F(U_t, 0), f_n \rangle \cdot f_n \\ &= -\frac{1}{2} \left((p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}) \frac{z^2}{2} + (p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}) z\bar{z} \right) \\ &\quad - \frac{1}{2} \left((p_1(\theta)g_{02} + p_2(\theta)\bar{g}_{20}) \frac{\bar{z}^2}{2} \right) + \cdots. \end{aligned}$$

Therefore, by (23), for $-1 \leq \theta < 0$,

$$\begin{aligned} H_{20}(\theta) &= \begin{cases} 0, & n = 1, 2, 3, \dots, \\ -\frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}) \cdot f_0, & n = 0, \end{cases} \\ H_{11}(\theta) &= \begin{cases} 0, & n = 1, 2, 3, \dots, \\ -\frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}) \cdot f_0, & n = 0, \end{cases} \\ H_{02}(\theta) &= \begin{cases} 0, & n = 1, 2, 3, \dots, \\ -\frac{1}{2}(p_1(\theta)g_{02} + p_2(\theta)\bar{g}_{20}) \cdot f_0, & n = 0, \end{cases} \end{aligned}$$

and $H(z, \bar{z})(0) = F(U_t, 0) - \Phi(\Psi, \langle F(U_t, 0), f_n \rangle) \cdot f_n$, where

$$H_{20}(0) = \begin{cases} \frac{\tilde{\tau}}{2G^3} \begin{pmatrix} -\gamma G^3 + Av^* - AG\xi \\ Ae^{-i\omega_n \tilde{\tau}} (G\xi - v^* e^{-i\omega_n \tilde{\tau}}) \end{pmatrix} \cos^2(\frac{n}{7}x), & n = 1, 2, 3, \dots, \\ \frac{\tilde{\tau}}{2G^3} \begin{pmatrix} -\gamma G^3 + Av^* - AG\xi \\ Ae^{-i\omega_n \tilde{\tau}} (G\xi - v^* e^{-i\omega_n \tilde{\tau}}) \end{pmatrix} \\ \quad - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\bar{g}_{02}) \cdot f_0, & n = 0, \end{cases}$$

$$H_{11}(0) = \begin{cases} \frac{\tilde{\tau}}{4G^3} \left(\begin{matrix} -2\gamma G^3 + 2Av^* - AG(\xi + \bar{\xi}) \\ A(G\bar{\xi}e^{-i\omega_n\tilde{\tau}} - 2v^* + G\xi e^{i\omega_n\tilde{\tau}}) \end{matrix} \right) \cos^2\left(\frac{n}{l}x\right), & n = 1, 2, 3, \dots, \\ \frac{\tilde{\tau}}{4G^3} \left(\begin{matrix} -2\gamma G^3 + 2Av^* - AG(\xi + \bar{\xi}) \\ A(G\bar{\xi}e^{-i\omega_n\tilde{\tau}} - 2v^* + G\xi e^{i\omega_n\tilde{\tau}}) \end{matrix} \right) \\ -\frac{1}{2}(p_1(0)g_{11} + p_2(0)\bar{g}_{11}) \cdot f_0, & n = 0. \end{cases}$$

By the definition of $A_{\tilde{\tau}}$ and (24), we have

$$\dot{W}_{20} = A_{\tilde{\tau}}W_{20} = 2i\omega_n\tilde{\tau}W_{20} + \frac{1}{2}(p_1g_{20} + p_2\bar{g}_{02}) \cdot f_n, \quad -1 \leq \theta < 0.$$

That is,

$$W_{20}(\theta) = \frac{i}{2\omega_n\tilde{\tau}} \left(g_{20}p_1(\theta) + \frac{\bar{g}_{02}}{3}p_2(\theta) \right) \cdot f_n + E_1e^{2i\omega_n\tilde{\tau}\theta},$$

where

$$E_1 = \begin{cases} W_{20}(0), & n = 1, 2, 3, \dots, \\ W_{20}(0) - \frac{i}{2\omega_n\tilde{\tau}}(g_{20}p_1(0) + \frac{\bar{g}_{02}}{3}p_2(0)) \cdot f_0, & n = 0. \end{cases}$$

Using the definition of $A_{\tilde{\tau}}$ and (24), we get that for $-1 \leq \theta < 0$,

$$\begin{aligned} & -\left(g_{20}p_1(0) + \frac{\bar{g}_{02}}{3}p_2(0) \right) \cdot f_0 + 2i\omega_0\tilde{\tau}E_1 \\ & -A_{\tilde{\tau}} \left(\frac{i}{2\omega_0\tilde{\tau}} \left(g_{20}p_1(0) + \frac{\bar{g}_{02}}{3}p_2(0) \right) \cdot f_0 \right) \\ & -A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}} \left(\frac{i}{2\omega_0\tilde{\tau}} \left(g_{20}p_1(\theta) + \frac{\bar{g}_{02}}{3}p_2(\theta) \right) \cdot f_n + E_1e^{2i\omega_0\tilde{\tau}\theta} \right) \\ & = \frac{\tilde{\tau}}{2G^3} \left(\begin{matrix} -\gamma G^3 + Av^* - AG\xi \\ Ae^{-i\omega_0\tilde{\tau}}(G\xi - v^*e^{-i\omega_0\tilde{\tau}}) \end{matrix} \right) - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\bar{g}_{02}) \cdot f_0. \end{aligned}$$

As

$$\begin{aligned} A_{\tilde{\tau}}p_1(0) \cdot f_0 + L_{\tilde{\tau}}(p_1 \cdot f_0) &= i\omega_0p_1(0) \cdot f_0, \\ A_{\tilde{\tau}}p_2(0) \cdot f_0 + L_{\tilde{\tau}}(p_2 \cdot f_0) &= -i\omega_0p_2(0) \cdot f_0, \end{aligned}$$

we get for $n = 0, 1, 2, \dots$,

$$2i\omega_0E_1 - A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}}E_1e^{2i\omega_0\tilde{\tau}\theta} = \frac{\tilde{\tau}}{2G^3} \left(\begin{matrix} -\gamma G^3 + Av^* - AG\xi \\ Ae^{-i\omega_0\tilde{\tau}}(G\xi - v^*e^{-i\omega_0\tilde{\tau}}) \end{matrix} \right) \cos^2\left(\frac{n}{l}x\right),$$

where \cdot represents a time delay variable. That is

$$E_1 = \frac{1}{2G^3}E \left(\begin{matrix} -\gamma G^3 + Av^* - AG\xi \\ Ae^{-i\omega_0\tilde{\tau}}(G\xi - v^*e^{-i\omega_0\tilde{\tau}}) \end{matrix} \right) \cos^2\left(\frac{n}{l}x\right),$$

where

$$E = \begin{pmatrix} 2i\omega_n + D_1 \frac{n^2}{l^2} - \alpha_1 & \alpha_2 \\ \beta_1 e^{-2i\omega_n \tilde{\tau}} & 2i\omega_n \tilde{\tau} + D_2 \frac{n^2}{l^2} \end{pmatrix}^{-1}.$$

Similarly, from (24), we get

$$-\dot{W}_{11}(\theta) = -\frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}) \cdot f_n,$$

that is

$$W_{11}(\theta) = \frac{i}{2\omega_n \tilde{\tau}} (p_2(\theta)\bar{g}_{11} - p_1(\theta)g_{11}) + E_2.$$

Similar to the computing method of W_{20} , we have

$$E_2 = \frac{1}{4G^3} E^* \begin{pmatrix} -2\gamma G^3 + 2Av^* - AG(\xi + \bar{\xi}) \\ A(G\bar{\xi}e^{-i\omega_n \tilde{\tau}} - 2v^* + G\xi e^{i\omega_n \tilde{\tau}}) \end{pmatrix} \cos^2\left(\frac{n}{l}x\right),$$

where

$$E^* = \begin{pmatrix} D_1 \frac{n^2}{l^2} - \alpha_1 & \alpha_2 \\ -\beta_1 & D_2 \frac{n^2}{l^2} \end{pmatrix}^{-1}.$$

Thus we can compute the following quantities which determine the direction and stability of bifurcating periodic orbits:

$$c_1(0) = \frac{i}{2\omega_n \tilde{\tau}} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{1}{2}g_{21}, \quad \mu_2 = -\frac{\text{Re}(c_1(0))}{\text{Re}(\lambda'(\tau_n^j))},$$

$$T_2 = -\frac{1}{\omega_n \tilde{\tau}} (\text{Im}(c_1(0)) + \mu_2 \text{Im}(\lambda'(\tau_n^j))), \quad \beta_2 = 2 \text{Re}(c_1(0)),$$

and we have

Theorem 4. For any critical value τ_n^j , we have:

- (i) μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ (< 0), then the direction of the Hopf bifurcation is forward (backward), that is, the bifurcating periodic solutions exists for $\tau > \tau_n^j$ ($\tau < \tau_n^j$);
- (ii) β_2 determines the stability of the bifurcating periodic solutions on the center manifold: if $\beta_2 < 0$ (> 0), then the bifurcating periodic solutions are orbitally asymptotically stable (unstable);
- (iii) T_2 determines the period of the bifurcating periodic solutions: if $T_2 > 0$ (< 0), then the period increases (decreases).

4 Numerical simulations

In this section, we present some numerical simulations to illustrate the theoretical analysis and symbolic mathematical software Matlab is used to plot numerical graphs.

We consider the system (3) with $D_1 = 2$, $D_2 = 0.5$, $r = 0.6$, $K = 4$, $A = 1$, $H = 0.05$, $B = 0.667$, $l = 5$. By a direct computation, we have $E(u^*, v^*) = (2.003, 0.8246)$,

$\omega_0 = 0.2385$, $\tau_0 = 1.549$, $\alpha_1 = -0.0923 < 0$, $H_0 = 0.6 > H$, $H_2 = 1.5678 > H$. By Theorem 3 and Theorem 4, when τ crosses $\tau_0 = 1.549$, the positive solution E^* loses its stability and Hopf bifurcation occurs. $\mu_2 > 0$ means the direction of the bifurcation is supercritical, $\beta_2 < 0$ implies the bifurcating periodic solutions are orbitally asymptotically stable, and $T_2 > 0$ means the period increases. This is shown in Fig. 3 and Fig. 4, where the initial condition is taken at $(2.503, 0.8256)$. In Fig. 5, we simulate the value of τ_0^0 as H varies. And the result coincide with the one of lemmas in Section 2.

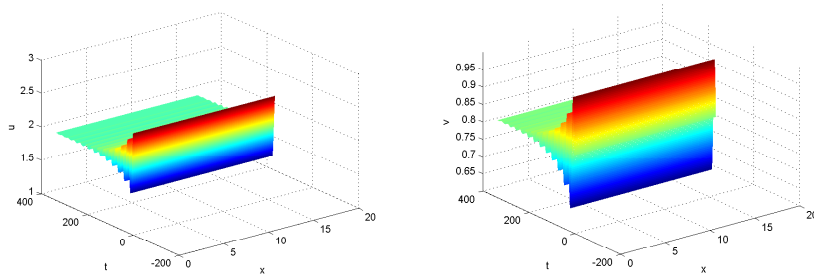


Fig. 3. The numerical simulations of system (4) with $\tau = 1 < \tau_0$. Left: component u (stable). Right: component v (stable).

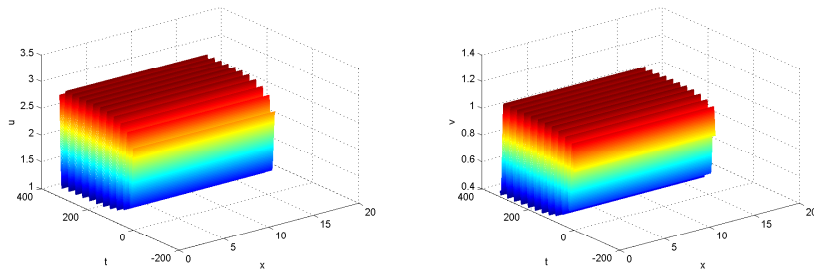


Fig. 4. The numerical simulations of a stable homogeneous equilibrium solution of system (4) with $\tau = 2 > \tau_0$. Left: component u (stable). Right: component v (stable).

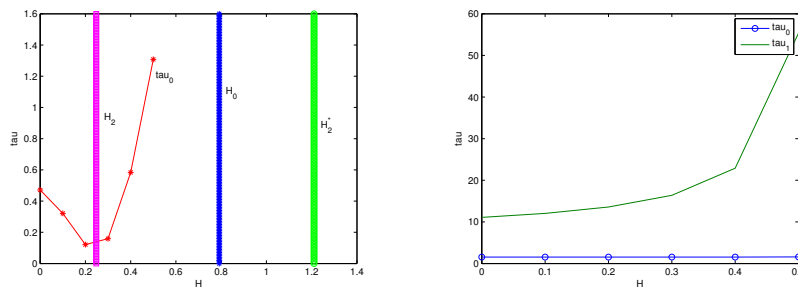


Fig. 5. Left: τ_0 exists if $H < \min\{H_0, H_2^*\}$. Right: $\tau_0^0 < \tau_0^1$.

5 Optimal control strategy

In this section, applying the method and notations of Fister [38], we consider optimal control strategy for the system (3) with $\tau = 0$. For the cause of prey conservation and economic benefit, we add predator harvesting term to the optimal control system and obtain the following system

$$\begin{aligned} L_1 u &:= u_t - D_1 \Delta u = ru \left(1 - \frac{u}{K}\right) - \frac{uv}{A+u} - E_1 u, \quad (x, t) \in Q := \Omega \times (0, T), \\ L_2 v &:= v_t - D_2 \Delta v = v \left(-B + \frac{u}{A+u}\right) - E_2 v, \quad (x, t) \in Q, \end{aligned} \quad (25)$$

$$\begin{aligned} \partial_\nu u &= \partial_\nu v = 0, \quad (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \end{aligned}$$

where Ω is smooth bounded domain in \mathbb{R}^n , the functions E_1 and E_2 are controls that represent harvesting a proportion of the population. Our class of admissible control is

$$\mathcal{E} = \{(E_1, E_2) \in L^2(Q): 0 \leq E_i \leq \Gamma_i \text{ a.e. for } i = 1, 2\}.$$

Our payoff function is

$$J(E_1, E_2) = \int_Q \{K_1 E_1 u + K_2 E_2 v - M_1 E_1^2 - M_2 E_2^2\} dx dt,$$

where $K_1 E_1 u$, $K_2 E_2 v$ represent the revenue of harvesting and $M_1 E_1^2$, $M_2 E_2^2$ denote the cost of the controls. We want to maximize the functional over the admissible class of controls, that is, there exists E_1^* and E_2^* such that $J(E_1^*, E_2^*) = \max_{(E_1, E_2) \in \mathcal{E}} J(E_1, E_2)$.

For any $(E_1, E_2) \in \mathcal{E}$, we assume that there exists a pair of solution (u, v) of the system (25). Applying the method of Fister [38] and the estimates:

$$\begin{aligned} & \frac{1}{2} \int_\Omega (|u^n(x, t)|^2 + |v^n(x, t)|^2) dx + \int_Q D_1 |\Delta u^n|^2 + D_2 |\Delta v^n|^2 dx dt \\ &= \frac{1}{2} \int_\Omega (|u_0^n(x)|^2 + |v_0^n(x)|^2) dx + \int_Q (r - E_1) |u^n|^2 - \frac{r}{K} \int_Q |u^n|^3 \\ & \quad - \int_Q \frac{v^n}{A + u^n} |u^n|^2 - \int_Q (B + E_2) |v^n|^2 + \int_Q \frac{u^n}{A + u^n} |v^n|^2 \\ &\leq \frac{1}{2} \int_\Omega (|u_0^n(x)|^2 + |v_0^n(x)|^2) dx + \int_Q (r - E_1) |u^n|^2 - \frac{r}{K} \int_Q |u^n|^3 + \int_Q |v^n|^2 \\ &\leq \frac{1}{2} \int_\Omega (|u_0^n(x)|^2 + |v_0^n(x)|^2) dx + C \int_Q \{|u^n|^2 + |v^n|^2\}, \end{aligned}$$

$$\int_{\Omega} (|u^n(x, t)|^2 + |v^n(x, t)|^2) dx \leq e^{2CT} \int_{\Omega} (|u_0^n(x)|^2 + |v_0^n(x)|^2) dx,$$

we obtain the existence of an optimal control as follows.

Theorem 5. *There exists a pair of optimal controls in \mathcal{E} that maximize the function $J(E_1, E_2)$.*

Next, we calculate the sensitivities, and have

Theorem 6. *The mapping $(E_1, E_2) \in \mathcal{E} \rightarrow (u, v)$ is differentiable, i.e.,*

$$\begin{aligned} \frac{u(E_1 + \varepsilon h, E_2) - u(E_1, E_2)}{\varepsilon} &\rightarrow \psi_1, & \frac{v(E_1 + \varepsilon h, E_2) - v(E_1, E_2)}{\varepsilon} &\rightarrow \psi_2, \\ \frac{u(E_1, E_2 + \varepsilon h) - u(E_1, E_2)}{\varepsilon} &\rightarrow \phi_1, & \frac{v(E_1, E_2 + \varepsilon h) - v(E_1, E_2)}{\varepsilon} &\rightarrow \phi_2, \end{aligned}$$

as $\varepsilon \rightarrow 0$ for any $(E_1, E_2) \in \mathcal{E}$ and $h = (h_1, h_2) \in L^\infty(Q)$ such that $(E_1 + \varepsilon h_1, E_2 + \varepsilon h_2) \in \mathcal{E}$ for ε small. Also, ψ_1, ψ_2 satisfy

$$\begin{aligned} (\psi_1)_t - D_1 \Delta \psi_1 &= r\psi_1 - \frac{2r}{K}u\psi_1 - \frac{u}{A+u}\psi_2 - \frac{Av}{(A+u)^2}\psi_1 - E_1\psi_1 - h_1u, \\ (\psi_2)_t - D_2 \Delta \psi_2 &= -B\psi_2 + \frac{u}{A+u}\psi_2 + \frac{Av}{(A+u)^2}\psi_1 - E_2\psi_2, \\ \frac{\partial \psi_1}{\partial \nu} = 0 = \frac{\partial \psi_2}{\partial \nu}, & \quad \psi_1(x, 0) = 0 = \psi_2(x, 0), \end{aligned} \tag{26}$$

and ϕ_1, ϕ_2 satisfy

$$\begin{aligned} (\phi_1)_t - D_1 \Delta \phi_1 &= r\phi_1 - \frac{2r}{K}u\phi_1 - \frac{u}{A+u}\phi_2 - \frac{Av}{(A+u)^2}\phi_1 - E_1\phi_1, \\ (\phi_2)_t - D_2 \Delta \phi_2 &= -B\phi_2 + \frac{u}{A+u}\phi_2 + \frac{Av}{(A+u)^2}\phi_1 - E_2\phi_2 - h_2v, \\ \frac{\partial \phi_1}{\partial \nu} = 0 = \frac{\partial \phi_2}{\partial \nu}, & \quad \phi_1(x, 0) = 0 = \phi_2(x, 0). \end{aligned} \tag{27}$$

To derive the optimal system and to characterize the optimal control, we need adjoint variables and adjoint operator associated with ψ_1, ψ_2 . We write the ψ_1, ψ_2 PDE system as

$$\mathcal{L} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -h_1u \\ 0 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} L_1\psi_1 \\ L_2\psi_2 \end{pmatrix} + M \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

and

$$M = \begin{pmatrix} -r + \frac{2r}{K}u + \frac{Av}{(A+u)^2} + E_1 & \frac{u}{A+u} \\ -\frac{Av}{(A+u)^2} & B + E_2 - \frac{u}{A+u} \end{pmatrix}.$$

Similarly, we rewrite the ϕ_1, ϕ_2 PDE systems as

$$\mathcal{L} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -h_2 v \end{pmatrix}.$$

We define the adjoint PDE system as

$$\mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} K_1 E_1 \\ K_2 E_2 \end{pmatrix}, \quad \text{where } \mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} L_1^* p \\ L_2^* q \end{pmatrix} + M^T \begin{pmatrix} p \\ q \end{pmatrix},$$

and

$$L_1^* p = -p_t - \Delta p, \quad L_2^* q = -q_t - \Delta q.$$

For the adjoint system, we have the appropriate bounded conditions, namely, zero Neumann conditions and the transversality conditions $p(x, T) = 0$, and $q(x, T) = 0$ for $x \in \Omega$.

Furthermore, assume (E_1^*, E_2^*) is an optimal pair and (u_*, v_*) is its corresponding solution pair. Consider $(E_1^* + \varepsilon h_1, E_2^*) \in \mathcal{E}$ with associated solution $u_\varepsilon, v_\varepsilon$. Since the adjoint equations are linear, there exists p, q satisfying (26) and (27).

$$\begin{aligned} 0 &\geq \lim_{\varepsilon \rightarrow 0} \frac{J(E_1^* + \varepsilon h_1, E_2^*) - J(E_1^*, E_2^*)}{\varepsilon} \\ &\geq \lim_{\varepsilon \rightarrow 0} \int_Q \left\{ K_1 E_1^* \left(\frac{u_\varepsilon - u_*}{\varepsilon} \right) + K_1 h_1 u_\varepsilon + K_2 E_2^* \left(\frac{v_\varepsilon - v_*}{\varepsilon} \right) \right. \\ &\quad \left. - 2M_1 h_1 E_1^* - M_1 \varepsilon h_1^2 \right\} dx dt \\ &= \int_Q \{ K_1 E_1^* \psi_1 + K_1 h_1 u_* + K_2 E_2^* \psi_2 - 2M h_1 E_1^* \} dx dt \\ &= \int_Q \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} K_1 E_1^* \\ K_2 E_2^* \end{pmatrix} + h_1 (K_1 u_* - 2M_1 E_1^*) \right\} dx dt \\ &= \int_Q h_1 \{ -p_1 u_* + K_1 u_* - 2M_1 E_1^* \}. \end{aligned}$$

Hence,

$$E_1^* = \min \left\{ \left(\frac{K_1 - p}{2M_1} u_* \right)^+, \Gamma_1 \right\}, \quad E_2^* = \min \left\{ \left(\frac{K_1 - q}{2M_2} v_* \right)^+, \Gamma_2 \right\}.$$

Acknowledgements

The authors are grateful to the anonymous referees for their helpful comments and valuable suggestions.

Appendix

Now, we consider the system (3) with $u^* > K/2$, $H = H_0$ and $D_1 = D_2 = 0$. Thus, $u_+ = u^*$. Let $\hat{u} = u - u_+$, $\hat{v} = v - v_+$, also let u and v denote \hat{u} and \hat{v} , respectively. System (3) is transformed into the following equivalent system:

$$\begin{aligned}\frac{du(t)}{dt} &= ru \left(1 - \frac{2u_+}{K}\right) - \frac{r}{K}u^2 - \frac{u + u_+}{A + u + u_+}v, \\ \frac{dv(t)}{dt} &= v \left(\frac{u + u_+}{A + u_+ + u} - B\right).\end{aligned}\quad (\text{A.1})$$

Linearizing system (A.1) about $(0, 0)$, we obtain the associated matrix

$$\begin{pmatrix} R & -B \\ 0 & 0 \end{pmatrix},$$

where $R := r(1 - 2u_+/K)$. And the eigenvalues are particularly easy to compute and are given by $\lambda_1 = 0$, $\lambda_2 = R$, with eigenvectors

$$\begin{pmatrix} 1 \\ \frac{R}{B} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}.\quad (\text{A.2})$$

Using the eigenbasis (A.2), we obtain the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{R}{B} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with inverse

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & \frac{B}{R} \\ 1 & -\frac{B}{R} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

which transforms (A.1) into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x \frac{(1-B)(x+y)}{A+u_++x+y} \\ Ry - \frac{r}{K}(x+y)^2 + \frac{(R+B)(B-1)x(x+y)}{B(A+u_++x+y)} \end{pmatrix}.\quad (\text{A.3})$$

From center manifold theory, there exists a center manifold for system (A.1) which can locally be represented as follows,

$$W^c(0) = \{(x, y) \in \mathbb{R}^2: y = h(x), |x| < \delta, h(0) = Dh(0) = 0\}$$

for δ sufficiently small. we now assume that $h(x)$ has the form

$$h(x) = ax^2 + bx^3 + \mathcal{O}(x^4),\quad (\text{A.4})$$

and the equation for the center manifold is given by

$$\mathcal{N}(h(x)) = Dh(x)[A_{\mathcal{N}}x + f_{\mathcal{N}}(x, h(x))] - B_{\mathcal{N}}h(x) - g_{\mathcal{N}}(x, h(x)) = 0.\quad (\text{A.5})$$

Let $B^* := 1/(A + u^*) = (1 - B)/A$ and by Taylor' formula:

$$\begin{aligned} \frac{1}{A + u_+ + x + y} &= \frac{1}{A + u_+} - \frac{x + y}{(A + u_+)^2} + \frac{x^2 + y^2}{(A + u_+)^3} + \dots \\ &= B^* - (B^*)^2(x + y) + (B^*)^3(x^2 + y^2) + \dots, \end{aligned}$$

from Eq. (A.3), we have for any $(x, y) \in \mathbb{R}^2$,

$$A_{\mathcal{N}} = 0, \quad B_{\mathcal{N}} = R,$$

$$\begin{aligned} f_{\mathcal{N}}(x, y) &= x \frac{(1 - B)(x + y)}{A + u_+ + x + y} = B^*(1 - B)x^2 + x^3(1 - B)B^*(a - B^*) + \dots, \\ g_{\mathcal{N}}(x, y) &= -\frac{r}{K}(x + y)^2 + \frac{(R + B)(B - 1)x(x + y)}{B(A + u_+ + x + y)} \\ &= x^2 \left(-\frac{r}{K} + \frac{B^*}{B}(R + B)(B - 1) \right) \\ &\quad + x^3 \left(-\frac{2ar}{K} + (R + B)(B - 1)(a - B^*) \frac{B^*}{B} \right) + \dots. \end{aligned}$$

Substituting (A.4) into (A.5), we obtain

$$\begin{aligned} \mathcal{N}(h(x)) &= (2ax + 3bx^3)(B^*(1 - B)x^2 + x^3(1 - B)B^*(a - B^*)) \\ &\quad - R(ax^2 + bx^3) - x^2 \left(-\frac{r}{K} + \frac{B^*}{B}(R + B)(B - 1) \right) \\ &\quad - x^3 \left(-\frac{2ar}{K} + (R + B)(B - 1)(a - B^*) \frac{B^*}{B} \right) + \dots = 0. \end{aligned}$$

Thus, the coefficients of each power of x must be zero, and equating coefficients on each power of x to zeros gives

$$\begin{aligned} x^2: \quad Ra - \left(\frac{r}{K} + \frac{(R + B)(1 - B)B^*}{B} \right) &= 0 \\ \implies a &= \frac{1}{R} \left(\frac{r}{K} + \frac{(R + B)(1 - B)^2}{AB} \right), \\ x^3: \quad bR - \left(\frac{2ar}{K} + \frac{(R + B)(1 - B)B^*}{B}(a - B^*) \right) &= 0 \\ \implies b &= \frac{1}{R} \left(2a(1 - B)B^* + \frac{2ar}{K} + \frac{(R + B)(1 - B)^2}{AB} \left(a - \frac{(1 - B)}{A} \right) \right). \end{aligned}$$

From $u^* = AB/(1 - B) > K/2$, we have $AB > (1 - B)K/2$ and

$$\begin{aligned} a &= \frac{1}{RKAB} (rAB(2B - 1) + K(r + B)(1 - B)^2), \\ rAB(2B - 1) + K(r + B)(1 - B)^2 &> \frac{rK(1 - B)}{2} + KB(1 - B)^2 > 0, \quad R < 0. \end{aligned}$$

Hence $a < 0$, and $h(x) = (1/R)(r/K + (R + B)(1 - B)^2/(AB))x^2 + \mathcal{O}(x^3)$. For x sufficiently small, $x = 0$ is thus stable. Hence, $(x, y) = (0, 0)$ is stable in Eq. (A.3), and $E_1 = (u_+, 0)$ is stable in system (A.1).

References

1. G. Hu, W. Li, Hopf bifurcation analysis for a delayed predator-prey system with diffusion effects, *Nonlinear Anal., Real World Appl.*, **11**, pp. 819–826, 2010.
2. Y. Qu, J. Wei, Bifurcation analysis in a time-delay model for prey-predator growth with stage-structure, *Nonlinear Dyn.*, **49**, pp. 285–294, 2007.
3. Y. Song, J. Wei, Local Hopf bifurcation and global periodic solutions in a delayed predator-prey system, *J. Math. Anal. Appl.*, **301**, pp. 1–21, 2005.
4. Y. Song, Y. Peng, J. Wei, Bifurcations for a predator-prey system with two delays, *J. Math. Anal. Appl.*, **377**, pp. 466–479, 2008.
5. D. Xiao, S. Ruan, Multiple bifurcations in a delayed predator-prey system with nonmonotonic functional response, *J. Differ. Equations*, **176**, pp. 494–510, 2001.
6. J. Zhang, W. Li, X. Yan, Multiple bifurcations in a delayed predator-prey diffusion system with a functional response, *Nonlinear Anal., Real World Appl.*, **11**, pp. 2708–2725, 2010.
7. S. Chen, J. Shi, Global stability in a diffusive Holling–Tanner predator-prey model, *Appl. Math. Lett.*, **25**, pp. 614–618, 2012.
8. R.M. May, Limit cycles in predator-prey communities, *Science*, **177**, pp. 900–902, 1972.
9. A.B. Medvinsky, S.V. Petrovskii, I.A. Tikhonova, H. Malchow, B.L. Li, Spatiotemporal complexity of plankton and fish dynamics, *SIAM Rev.*, **44**, pp. 900–902, 2002.
10. R. Peng, J. Shi, Non-existence of non-constant positive steady states of two Holling type-II predator-prey systems: Strong interaction case, *J. Differ. Equations*, **247**, pp. 866–886, 2009.
11. M.L. Rosenzweig, R. MacArthur, Graphical representation and stability conditions of predator-prey interaction, *Am. Nat.*, **97**, pp. 209–223, 1963.
12. F. Yi, J. Wei, J. Shi, Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator-prey system, *J. Differ. Equations*, **246**, pp. 1944–1977, 2009.
13. D. Xiao, L. Jennings, Bifurcations of a ratio-dependent predator-prey system with constant rate harvesting, *SIAM J. Appl. Math.*, **65**, pp. 737–753, 2005.
14. D. Xiao, W. Li, M. Han, Dynamics in a ratio-dependent predator-prey model with predator harvesting, *J. Math. Anal. Appl.*, **324**, pp. 14–29, 2006.
15. P. Yodzis, Predator-prey theory and management of multispecies fisheries, *Ecol. Appl.*, **4**, pp. 51–58, 1994.
16. A. Martin, S. Ruan, Predator-prey models with delay and prey harvesting, *J. Math. Biol.*, **43**, pp. 247–267, 2001.

17. Y. Qu, J. Wei, Bifurcation analysis in a predator-prey system with stage-structure and harvesting, *J. Franklin Inst.*, **347**, pp. 1097–1113, 2010.
18. D. Xiao, S. Ruan, Bogdanov–Takens bifurcations in predator-prey systems with constant rate harvesting, *Fields Inst. Commun.*, **21**, pp. 493–506, 1999.
19. L. Zhang, W. Wang, Y. Xue, Spatiotemporal complexity of a predator-prey system with constant harvest rate, *Chaos Solitons Fractals*, **41**, pp. 38–46, 2009.
20. J.R. Beddington, J.G. Cooke, Harvesting from a prey-predator complex, *Ecol. Model.*, **14**, pp. 155–177, 1982.
21. F. Brauer, A.C. Soudack, Stability regions and transition phenomena for harvested predator-prey systems, *J. Math. Biol.*, **7**, pp. 319–337, 1979.
22. G. Dai, M. Tang, Coexistence region and global dynamics of a harvested predator-prey system, *SIAM J. Appl. Math.*, **58**, pp. 193–210, 1998.
23. W.L. Hogarth, J. Norbury, I. Cuning, K. Sommers, Stability of a predator-prey model with harvesting, *Ecol. Model.*, **62**, pp. 83–106, 1992.
24. M.R. Myerscough, B.E. Gray, W.L. Hogarth, J. Norbury, An analysis of an ordinary differential equation model for a two-species predator-prey system with harvesting and stocking, *J. Math. Biol.*, **30**, pp. 389–411, 1992.
25. F. Brauer, A.C. Soudack, Stability regions in predator-prey systems with constant-rate prey harvesting, *J. Math. Biol.*, **8**, pp. 55–71, 1979.
26. F. Brauer, A.C. Soudack, Coexistence properties of some predator-prey systems under constant rate harvesting and stocking, *J. Math. Biol.*, **12**, pp. 101–114, 1981.
27. J. Xia, Z. Liu, R. Yuan, S. Ruan, The effects of harvesting and time delay on predator-prey systems with Holling type II functional response, *SIAM J. Appl. Math.*, **70**, pp. 1178–1200, 2009.
28. T. Faria, L.T. Magalhães, Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation, *J. Differ. Equations*, **122**, pp. 181–200, 1995.
29. T. Faria, Stability and bifurcation for a delayed predator-prey model and the effect of diffusion, *J. Math. Anal. Appl.*, **254**, pp. 433–463, 2001.
30. H. Thieme, X. Zhao, A non-local delayed and diffusive predator-prey model, *Nonlinear Anal., Real World Appl.*, **2**, pp. 145–160, 2001.
31. Y. Wang, Asymptotic behavior of solutions for a class of predator-prey reaction diffusion systems with time delays, *J. Math. Anal. Appl.*, **328**, pp. 137–150, 2007.
32. X. Yan, Stability and Hopf bifurcation for a delayed prey-predator system with diffusion effects, *Appl. Math. Comput.*, **192**, pp. 137–150, 2007.
33. X. Yan, C. Zhang, Asymptotic stability of positive equilibrium solution for a delayed prey-predator diffusion system, *Appl. Math. Modelling*, **34**, pp. 184–199, 2010.
34. T. Faria, Normal forms and Hopf bifurcation for partial differential equations with delays, *Trans. Am. Math. Soc.*, **352**, pp. 2217–2238, 2000.

35. J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996.
36. B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, 1981.
37. S. Ruan, J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, *Dyn. Contin. Discrete Impulsive Syst.*, **10**, pp. 863–874, 2003.
38. K.R. Fister, Optimal control of harvesting in a predator-prey parabolic system, *Houston J. Math.*, **23**, pp. 341–355, 1997.
39. J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, Berlin, 1977.