Positive solutions to a class of random operator equations and applications to stochastic integral equations*

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Abstract. We study the existence of random positive solutions to a random operator equation on ordered Polish spaces. We apply the results obtained in this paper to study the existence of random positive solutions to some classes of stochastic integral equations.

Keywords: random positive solution, Polish space, τ–φ-concave, complete measure space, stochastic integral equation.

1 Introduction and preliminaries

Nonlinear operator equations defined on Banach spaces play an important role in the theory of differential and integral equations and have been extensively studied over the past several decades (see [1–7]).

Random techniques have been crucial in diverse areas from pure mathematics to applied sciences. Špaček [8] and Hanš [9] initiated the study of random operator equations. They proved random fixed point theorems of contraction type. These results were extended and applied to various problems by Hanš [10–12] and Bharucha-Reid [13,14]. In recent years, the theory of random nonlinear operator equations has attracted the attention of many authors (see [15–23]).

In this paper, we consider some classes of random operator equations on ordered Polish spaces. We study the existence of random positive solutions to such problems by using monotone iterative techniques and the properties of cones. As applications, we use the results obtained in this paper to study the existence of random positive solutions to some stochastic integral equations.

Through this paper, E is a real Banach space with norm $\|\cdot\|$, 0_E is the zero element of E, and P is a cone in E. So, a partially ordered relation in E can be defined by

$$x, y \in E, \quad x \preccurlyeq y \quad \Longleftrightarrow \quad y - x \in P.$$

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If $x \preccurlyeq y$ and $x \neq y$, then we denote $x \prec y$ or $y \succ x$. If $x, y \in E$ are such that $x \preccurlyeq y$, we denote by $[x, y]$ the set defined by

$$[x, y] := \{u \in E \mid x \preccurlyeq u \preccurlyeq y\}.$$

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \preccurlyeq y \preccurlyeq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h \succ 0_E$, we denote by P_h the set

$$P_h := \{x \in E \mid x \sim h\}.$$

Definition 1. A cone $P \subset E$ is said to be normal iff there exists a constant $N > 0$ such that

$$x, y \in E, \quad 0_E \preccurlyeq x \preccurlyeq y \quad \Longrightarrow \quad \|x\| \leqslant N\|y\|.$$

For more details about cones in Banach spaces, we refer the reader to [4, 24, 25].

Definition 2. If E is separable, then E is called a Polish space.

Definition 3. A measure space (Ω, Σ, μ) is said to be complete iff

$$S \subseteq N \in \Sigma \quad \text{and} \quad \mu(N) = 0 \quad \Longrightarrow \quad S \in \Sigma.$$

Definition 4. Let (Ω, Σ, μ) be a measure space. A property P of points in Ω holds almost everywhere iff the set of elements for which the property P does not hold is contained within a set of measure zero. In this case, we write $P(\omega)$, $\omega \in \Omega$ a.e.

Remark 1. Clearly, if (Ω, Σ, μ) is a complete measure space, then $P(\omega)$, $\omega \in \Omega$ a.e. iff the set of elements for which the property P does not hold is a measurable set of measure zero.

Through this paper, we suppose that E is a Polish space. We equip E with a σ-algebra β_E of Borel subsets of E so that (E, β_E) becomes a measurable space. We denote by (Ω, Σ, μ) a complete probability measure space, where (Ω, Σ) is a measurable space, Σ is a sigma-algebra of subsets of Ω, and μ is a probability measure.

Definition 5. A function $x : \Omega \to E$ is said to be measurable iff

$$x^{-1}(U) := \{\omega \in \Omega \mid x(\omega) \in U\} \in \Sigma$$

for all open subset U of E. A measurable function is also called a random variable.

For more details about measure theory, we refer to [26].

Definition 6. A mapping $T : \Omega \times P \to P$ is called a random mapping iff, for each fixed $x \in P$, the mapping $T(\cdot, x) : \Omega \to P$ is measurable.

Definition 7. A random mapping $T : \Omega \times P \to P$ is said to be continuous iff, for $\omega \in \Omega$ a.e., the mapping $T(\omega, \cdot) : P \to P$ is continuous.

Definition 8. A random operator $T : \Omega \times P \to P$ is called increasing iff, for any fixed $\omega \in \Omega$, the mapping $T(\omega, \cdot) : P \to P$ is increasing with respect to the partial order $\preccurlyeq$ induced by the cone P, i.e.,

$$\omega \in \Omega,\ x, y \in P, \quad x \preccurlyeq y \quad \Longrightarrow \quad T(\omega, x) \preccurlyeq T(\omega, y).$$

Definition 9. A measurable function $\xi : \Omega \to P$ is said to be a random fixed point of the random mapping $T : \Omega \times P \to P$ iff

$$T\bigl(\omega, \xi(\omega)\bigr) = \xi(\omega), \quad \omega \in \Omega \text{ a.e.}$$

Lemma 1. (*See* [27, 28].) *Let $T : \Omega \times P \to P$ be a continuous random operator. If $x : \Omega \to P$ is measurable, then $\Omega \ni \omega \mapsto T(\omega, x(\omega)) \in P$ is measurable.*

Lemma 2. (*See* [28, 29].) *Let $\{x_n\}$ be a sequence of measurable functions from Ω to P such that $\lim_{n\to\infty} x_n(\omega) = x(\omega)$, $\omega \in \Omega$ a.e., then $x : \Omega \to P$ is measurable.*

For more details about random operators, we refer the reader to [29].

If $\mathcal{A} \subseteq \Omega$, we denote by $\overline{\mathcal{A}}$ the subset of Ω defined by

$$\overline{\mathcal{A}} := \{\omega \in \Omega \mid \omega \notin \mathcal{A}\}.$$

2 Positive solutions to random operator equations

First, we introduce the class of τ–φ-concave random operators.

Definition 10. A random operator $T : \Omega \times P \to P$ is said to be τ–φ-concave iff there exist two positive-valued functions τ and φ defined on bounded interval (a, b) such that

(H1) $\tau : (a, b) \to (0, 1)$ is a surjection;

(H2) $1 \geqslant \varphi(t) > \tau(t)\ \forall t \in (a, b)$;

(H3) For $\omega \in \Omega$ a.e.,

$$T(\omega, \tau(t)x) \succcurlyeq \varphi(t)T(\omega, x) \quad \forall (x, t) \in P \times (a, b).$$

The following result will be useful later.

Lemma 3. *Let $T : \Omega \times P \to P$ be a continuous random operator. Suppose that the following conditions hold*:

(i) *P is a normal cone*;

(ii) *T is random increasing*;

(iii) *T is τ–φ-concave operator*;

(iv) *$\exists (h, \epsilon) \in P \backslash \{0_E\} \times (0, 1) \mid \epsilon h \preccurlyeq T(\omega, h) \preccurlyeq (1/\epsilon)h$, $\omega \in \Omega$ a.e.*

Then there exists $(r, u_0, v_0) \in (0, 1) \times P_h \times P_h$ such that

$$r\, v_0 \preccurlyeq u_0 \prec v_0 \quad \textit{and} \quad u_0 \preccurlyeq T(\omega, u_0) \preccurlyeq T(\omega, v_0) \preccurlyeq v_0, \quad \omega \in \Omega \textit{ a.e.}$$

Proof. Let

$$\Theta := \{\omega \in \Omega \mid T(\omega, \cdot) \text{ is continuous}\},$$
$$\Delta := \left\{\omega \in \Omega \,\middle|\, \epsilon h \preccurlyeq T(\omega, h) \preccurlyeq \frac{1}{\epsilon} h\right\},$$
$$F := \{\omega \in \Omega \mid T(\omega, \tau(t)x) \succcurlyeq \varphi(t) T(\omega, x)\ \forall (x, t) \in P \times (a, b)\}.$$

Using the continuity of the random operator T, conditions (H3) and (iv), we obtain that

$$\mu(\overline{\Theta \cap \Delta \cap F}) = \mu(\overline{\Theta} \cup \overline{\Delta} \cup \overline{F}) = 0. \tag{1}$$

Let

$$\Lambda := \Theta \cap \Delta \cap F. \tag{2}$$

Let $\omega \in \Lambda$. It follows from (H1) and (iv) that there exists $t_0 \in (a, b)$ such that $\tau(t_0) = \epsilon$, and then

$$\tau(t_0) h \preccurlyeq T(\omega, h) \preccurlyeq [\tau(t_0)]^{-1} h. \tag{3}$$

By (H2), we have $\varphi(t_0)/\tau(t_0) > 1$. So, there exists a positive integer k such that

$$\left(\frac{\varphi(t_0)}{\tau(t_0)}\right)^k \geqslant [\tau(t_0)]^{-1}. \tag{4}$$

Let

$$u_0 := [\tau(t_0)]^k h \quad \text{and} \quad v_0 := [\tau(t_0)]^{-k} h. \tag{5}$$

Clearly, we have

$$u_0 = [\tau(t_0)]^{2k} v_0 \prec v_0.$$

Let $r \in (0, [\tau(t_0)]^{2k}]$. Then

$$r \in (0, 1) \quad \text{and} \quad u_0 \succcurlyeq r v_0.$$

Since T is random increasing, we have $T(\omega, u_0) \preccurlyeq T(\omega, v_0)$. Combining (iii), (3) and (4), we get

$$\begin{aligned}
T(\omega, u_0) &= T(\omega, [\tau(t_0)]^k h) = T(\omega, \tau(t_0)[\tau(t_0)]^{k-1} h) \\
&\succcurlyeq \varphi(t_0) T(\omega, [\tau(t_0)]^{k-1} h) = \varphi(t_0) T(\omega, \tau(t_0)[\tau(t_0)]^{k-2} h) \\
&\succcurlyeq [\varphi(t_0)]^2 T(\omega, [\tau(t_0)]^{k-2} h) \succcurlyeq \cdots \succcurlyeq [\varphi(t_0)]^k T(\omega, h) \\
&\succcurlyeq [\varphi(t_0)]^k \tau(t_0) h \succcurlyeq [\tau(t_0)]^k h = u_0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
T(\omega, v_0) &= T(\omega, [\tau(t_0)]^{-k} h) = T(\omega, [\tau(t_0)]^{-1}[\tau(t_0)]^{1-k} h) \\
&\preccurlyeq [\varphi(t_0)]^{-1} T(\omega, [\tau(t_0)]^{1-k} h) = [\varphi(t_0)]^{-1} T(\omega, [\tau(t_0)]^{-1}[\tau(t_0)]^{2-k} h) \\
&\preccurlyeq (\varphi(t_0))^{-2} T(\omega, (\tau(t_0))^{2-k} h) \preccurlyeq \cdots \preccurlyeq [\varphi(t_0)]^{-k} T(\omega, h) \\
&\preccurlyeq \varphi(t_0)^{-k} [\tau(t_0)]^{-1} h \preccurlyeq [\tau(t_0)]^{-k} h = v_0.
\end{aligned}$$

Thus, we proved that

$$rv_0 \preccurlyeq u_0 \prec v_0 \quad \text{and} \quad u_0 \preccurlyeq T(\omega, u_0) \preccurlyeq T(\omega, v_0) \preccurlyeq v_0 \quad \forall \omega \in \Lambda. \tag{6}$$

Finally, Lemma 3 follows from (1) and (6). $\square$

Now, we prove the following result.

Theorem 1. *Let $T : \Omega \times P \to P$ be a continuous random operator satisfying conditions* (i)–(iv). *Suppose that there exists a measurable function $x_0 : \Omega \to P$ such that $x_0(\omega) \in [0_E, lh]$, $\omega \in \Omega$ a.e., where $l \geqslant 0$ is a constant. Then the operator equation*

$$x(\omega) = x_0(\omega) + T(\omega, x(\omega)), \quad \omega \in \Omega \ \text{a.e.} \tag{7}$$

has a measurable solution $\xi : \Omega \to [u_0, v_0]$, where $u_0, v_0 \in P_h$ and $u_0 \prec v_0$. Moreover, if $\eta : \Omega \to P$ is another measurable solution to (7) *satisfying $\eta(\omega) \in P_h$, $\omega \in \Omega$ a.e., then $\xi(\omega) = \eta(\omega)$, $\omega \in \Omega$ a.e.*

Proof. Let

$$\Phi := \{\omega \in \Omega \mid x_0(\omega) \in [0_E, lh]\}, \qquad \Pi := \Lambda \cap \Phi,$$

where Λ is given by (2). By assumptions, we have

$$\mu(\overline{\Pi}) = 0. \tag{8}$$

Define the operator $A : \Omega \times P \to P$ by

$$A(\omega, x) := x_0(\omega) + T(\omega, x) \quad \forall (\omega, x) \in \Omega \times P.$$

Since T is a continuous random operator and x_0 is measurable, then A is a continuous random operator. Since T is increasing, the random operator A is also increasing. On the other hand, for all $(\omega, x, t) \in \Pi \times P \times (a, b)$, we have

$$\begin{aligned} A(\omega, \tau(t)x) = x_0(\omega) + T(\omega, \tau(t)x) &\succcurlyeq x_0(\omega) + \varphi(t)T(\omega, x) \\ &\succcurlyeq \varphi(t)\big(x_0(\omega) + T(\omega, x)\big) = \varphi(t)A(\omega, x). \end{aligned}$$

Thus, we have

$$A(\omega, \tau(t)x) \succcurlyeq \varphi(t)A(\omega, x) \quad \forall (\omega, x, t) \in \Pi \times P \times (a, b). \tag{9}$$

So, from (8), we have, for $\omega \in \Omega$ a.e.,

$$A(\omega, \tau(t)x) \succcurlyeq \varphi(t)A(\omega, x) \quad \forall (x, t) \in P \times (a, b).$$

Thus, A is a τ–φ-concave operator. From (iv), and since $x_0(\omega) \in [0_E, lh]$ for all $\omega \in \Pi$, we get

$$\epsilon h \preccurlyeq A(\omega, h) \preccurlyeq \left(l + \frac{1}{\epsilon}\right)h \quad \forall \omega \in \Pi.$$

Taking $\rho := \min\{(l+1/\epsilon)^{-1}, \epsilon\}$, we get $\rho \in (0,1)$ and

$$\rho h \preccurlyeq A(\omega, h) \preccurlyeq \frac{1}{\rho} h \quad \forall \omega \in \Pi.$$

Therefore, from (8), we have

$$\rho h \preccurlyeq A(\omega, h) \preccurlyeq \frac{1}{\rho} h \quad \omega \in \Omega \ \text{a.e.}$$

Now, Lemma 3 implies that there exists $(r, u_0, v_0) \in (0,1) \times P_h \times P_h$ such that

$$r v_0 \preccurlyeq u_0 \prec v_0 \quad \text{and} \quad u_0 \preccurlyeq A(\omega, u_0) \preccurlyeq A(\omega, v_0) \preccurlyeq v_0, \quad \omega \in \Omega \ \text{a.e.}$$

Without any restriction of the generality, we can suppose that

$$r v_0 \preccurlyeq u_0 \prec v_0 \quad \text{and} \quad u_0 \preccurlyeq A(\omega, u_0) \preccurlyeq A(\omega, v_0) \preccurlyeq v_0 \quad \forall \omega \in \Pi. \tag{10}$$

Consider the iterative sequences $u_n, v_n : \Omega \to P$ defined by

$$u_0(\omega) := u_0, \quad u_n(\omega) := A(\omega, u_{n-1}(\omega)), \quad \omega \in \Omega, \ n = 1, 2, \dots,$$

and

$$v_0(\omega) := u_0, \quad v_n(\omega) := A\bigl(\omega, v_{n-1}(\omega)\bigr), \quad \omega \in \Omega, \ n = 1, 2, \dots.$$

Using the continuity of the random operator A and Lemma 1, we obtain that, for all n, the mappings u_n and v_n are measurable. By the monotonicity of the random operator A, for all $\omega \in \Pi$, we have

$$u_1(\omega) = A(\omega, u_0) \preccurlyeq A(\omega, v_0) = v_1(\omega).$$

Continuing this process, by induction, for all $\omega \in \Pi$, we have

$$u_n(\omega) \preccurlyeq v_n(\omega), \quad n = 0, 1, 2, \dots.$$

It follows from (10) and the monotonicity of the random operator A that, for all $\omega \in \Pi$,

$$u_0(\omega) \preccurlyeq u_1(\omega) \preccurlyeq \cdots \preccurlyeq u_n(\omega) \preccurlyeq \cdots \preccurlyeq v_n(\omega) \preccurlyeq \cdots v_1(\omega) \preccurlyeq v_0(\omega). \tag{11}$$

Since $u_0 \succcurlyeq r v_0$, we can get $u_n(\omega) \succcurlyeq u_0(\omega) \succcurlyeq r v_0(\omega) \succcurlyeq r v_n(\omega)$, $n = 1, 2, \dots, \omega \in \Pi$. Let $\omega \in \Pi$ be fixed. For all positive integer n, let

$$r_n(\omega) := \sup\bigl\{s > 0 \text{ such that } u_n(\omega) \succcurlyeq s v_n(\omega)\bigr\}.$$

Thus, we have $u_n(\omega) \succcurlyeq r_n(\omega) v_n(\omega)$ for all $n = 1, 2, \dots$. Then

$$u_{n+1}(\omega) \succcurlyeq u_n(\omega) \succcurlyeq r_n(\omega) v_n(\omega) \succcurlyeq r_n(\omega) v_{n+1}(\omega), \quad n = 1, 2, \dots.$$

Therefore, we have

$$r_n(\omega) \leqslant r_{n+1}(\omega),$$

which implies that the sequence $\{r_n(\omega)\} \subset [r,1]$ is increasing. So, there exists $r^*(\omega) \in [r,1]$ such that $r_n(\omega) \to r^*(\omega)$ as $n \to \infty$. Suppose that $r \leqslant r^*(\omega) < 1$. By (H1), there exists $t_\omega \in (a,b)$ such that $\tau(t_\omega) = r^*(\omega)$. We distinguish two cases.

Case 1. $r_N(\omega) = r^*(\omega)$ for some positive integer N. In this case, we have $r_n(\omega) = r^*(\omega)$ for all $n \geqslant N$. Then, using (9), for $n \geqslant N$, we have

$$\begin{aligned} u_{n+1}(\omega) &= A\big(\omega, u_n(\omega)\big) \succcurlyeq A\big(\omega, r^*(\omega)v_n(\omega)\big) = A\big(\omega, \tau(t_\omega)v_n(\omega)\big) \\ &\succcurlyeq \varphi(t_\omega)A\big(\omega, v_n(\omega)\big) = \varphi(t_\omega)v_{n+1}(\omega). \end{aligned}$$

By the definition of $r_n(\omega)$, we get

$$r_{n+1}(\omega) = r^*(\omega) \geqslant \varphi(t_\omega) > \tau(t_\omega) = r^*(\omega),$$

which is a contradiction.

Case 2. $r_n(\omega) < r^*(\omega)$ for any integer n. In this case, we have $0 < r_n(\omega)/r^*(\omega) < 1$. By (H1), there exists $\sigma_{n,\omega} \in (a,b)$ such that $\tau(\sigma_{n,\omega}) = r_n(\omega)/r^*(\omega)$. Then, using (9), we have

$$\begin{aligned} u_{n+1}(\omega) &= A\big(\omega, u_n(\omega)\big) \succcurlyeq A\big(\omega, r_n(\omega)v_n(\omega)\big) \\ &= A\left(\omega, \frac{r_n(\omega)}{r^*(\omega)}r^*(\omega)v_n(\omega)\right) = A\big(\omega, \tau(\sigma_{n,\omega})r^*(\omega)v_n(\omega)\big) \\ &\succcurlyeq \varphi(\sigma_{n,\omega})A\big(\omega, r^*(\omega)v_n(\omega)\big) = \varphi(\sigma_{n,\omega})A\big(\omega, \tau(t_\omega)v_n(\omega)\big) \\ &\succcurlyeq \varphi(\sigma_{n,\omega})\varphi(t_\omega)A\big(\omega, v_n(\omega)\big) = \varphi(\sigma_{n,\omega})\varphi(t_\omega)v_{n+1}(\omega). \end{aligned}$$

By the definition of $r_n(\omega)$, we get

$$r_{n+1}(\omega) \geqslant \varphi(\sigma_{n,\omega})\varphi(t_\omega) > \tau(\sigma_{n,\omega})\varphi(t_\omega) = \frac{r_n(\omega)}{r^*(\omega)}\varphi(t_\omega).$$

Letting $n \to \infty$, we obtain that

$$r^*(\omega) \geqslant \varphi(t_\omega) > \tau(t_\omega) = r^*(\omega),$$

which is a contradiction. Thus, we proved that

$$\lim_{n\to\infty} r_n(\omega) = 1. \tag{12}$$

On the other hand, for any positive integer p, we have

$$\begin{aligned} 0_E &\preccurlyeq u_{n+p}(\omega) - u_n(\omega) \preccurlyeq v_{n+p}(\omega) - u_n(\omega) \\ &\preccurlyeq v_n(\omega) - r_n(\omega)v_n(\omega) \preccurlyeq \big(1 - r_n(\omega)\big)v_0(\omega) \end{aligned}$$

and

$$0_E \preccurlyeq v_n(\omega) - v_{n+p}(\omega) \preccurlyeq v_n(\omega) - u_n(\omega) \preccurlyeq \big(1 - r_n(\omega)\big)v_0(\omega).$$

From (12) and the normality of the cone P, there exists a constant $N > 0$ such that

$$\big\|u_{n+p}(\omega) - u_n(\omega)\big\| \leqslant N\big(1 - r_n(\omega)\big)\big\|v_0(\omega)\big\| \to 0 \quad \text{as } n \to \infty$$

and

$$\big\|v_{n+p}(\omega) - v_n(\omega)\big\| \leqslant N\big(1 - r_n(\omega)\big)\big\|v_0(\omega)\big\| \to 0 \quad \text{as } n \to \infty.$$

So, $\{u_n(\omega)\}$ and $\{v_n(\omega)\}$ are Cauchy sequences. Since E is complete, there exist $u^*(\omega) \in P$ and $v^*(\omega) \in P$ such that

$$\lim_{n\to\infty} \big\|u_n(\omega) - u^*(\omega)\big\| = \lim_{n\to\infty} \big\|v_n(\omega) - v^*(\omega)\big\| = 0.$$

By (11), we know that $u_0(\omega) \preccurlyeq u_n(\omega) \preccurlyeq u^*(\omega) \preccurlyeq v^*(\omega) \preccurlyeq v_n(\omega) \preccurlyeq v_0(\omega)$. Then we have

$$0_E \preccurlyeq v^*(\omega) - u^*(\omega) \preccurlyeq v_n(\omega) - u_n(\omega) \preccurlyeq \big(1 - r_n(\omega)\big)v_0(\omega).$$

Thus, we get

$$\big\|v^*(\omega) - u^*(\omega)\big\| \leqslant N\big(1 - r_n(\omega)\big)\big\|v_0(\omega)\big\| \to 0 \quad \text{as } n \to \infty,$$

which implies that $v^*(\omega) = u^*(\omega)$. Let

$$w(\omega) = v^*(\omega) = u^*(\omega) \quad \forall \omega \in \Pi.$$

Hence, we proved that

$$\lim_{n\to\infty} \big\|u_n(\omega) - w(\omega)\big\| = \lim_{n\to\infty} \big\|v_n(\omega) - w(\omega)\big\| = 0 \quad \forall \omega \in \Pi. \tag{13}$$

Define the mapping $\xi : \Omega \to P$ by

$$\xi(\omega) := \begin{cases} w(\omega) & \text{if } \omega \in \Pi, \\ \frac{u_0+v_0}{2} & \text{if } \omega \in \overline{\Pi}. \end{cases}$$

Observe that

$$u_0 \preccurlyeq \xi(\omega) \preccurlyeq v_0 \quad \forall \omega \in \Omega. \tag{14}$$

On the other hand, from (13) and (8), we have $\lim_{n\to\infty} u_n(\omega) = \xi(\omega)$, $\omega \in \Omega$ a.e. By Lemma 2, it follows that $\xi : \Omega \to P$ is a measurable function. Observe also that, for all $\omega \in \Pi$, we have

$$u_{n+1}(\omega) = A\big(\omega, u_n(\omega)\big) \preccurlyeq A\big(\omega, w(\omega)\big) \preccurlyeq A\big(\omega, v_n(\omega)\big) = v_{n+1}(\omega), \quad n = 1, 2, \dots.$$

Letting $n \to \infty$, we get that

$$A\big(\omega, w(\omega)\big) = w(\omega) \quad \forall \omega \in \Pi.$$

By definition of ξ, we obtain that

$$A\big(\omega, \xi(\omega)\big) = \xi(\omega) \quad \text{a.e.},$$

that is,

$$x_0(\omega) + T\big(\omega, \xi(\omega)\big) = \xi(\omega) \quad \text{a.e.} \tag{15}$$

It follows from the measurability of ξ and (15) that ξ is a solution to the operator equation (7) satisfying (14). Now, suppose that $\eta : \Omega \to P$ is another measurable solution to (7) such that $\eta(\omega) \in P_h$, $\omega \in \Omega$ a.e. Without any restriction of the generality, we can suppose that

$$A\big(\omega, \eta(\omega)\big) = \eta(\omega) \in P_h \quad \forall \omega \in \Pi. \tag{16}$$

Let $\omega \in \Pi$. Since $\eta(\omega) \in P_h$, there exist numbers $\gamma_1(\omega), \gamma_2(\omega) > 0$ such that

$$\gamma_1(\omega)h \preccurlyeq \eta(\omega) \preccurlyeq \gamma_2(\omega)h.$$

Similarly, since $\xi(\omega) \in P_h$, there exist numbers $\delta_1(\omega), \delta_2(\omega) > 0$ such that

$$\delta_1(\omega)h \preccurlyeq \xi(\omega) \preccurlyeq \delta_2(\omega)h.$$

Then we obtain

$$\eta(\omega) \succcurlyeq \gamma_1(\omega)h = \frac{\gamma_1(\omega)}{\delta_2(\omega)}\delta_2(\omega)h \succcurlyeq \frac{\gamma_1(\omega)}{\delta_2(\omega)}\xi(\omega).$$

Let

$$p(\omega) := \sup\big\{p > 0 \text{ such that } \eta(\omega) \succcurlyeq p\xi(\omega)\big\}.$$

Clearly, we have

$$0 < p(\omega) < \infty \quad \text{and} \quad \eta(\omega) \succcurlyeq p(\omega)\xi(\omega).$$

Suppose that $0 < p(\omega) < 1$. From condition (H1), there is $z_\omega \in (a, b)$ such that $\tau(z_\omega) = p(\omega)$. Using (9), we get

$$\begin{aligned}\eta(\omega) &= A\big(\omega, \eta(\omega)\big) \succcurlyeq A\big(\omega, p(\omega)\xi(\omega)\big) = A\big(\omega, \tau(z_\omega)\xi(\omega)\big)\\ &\succcurlyeq \varphi(z_\omega)A\big(\omega, \xi(\omega)\big) = \varphi(z_\omega)\xi(\omega).\end{aligned}$$

Since $\varphi(z_\omega) > \tau(z_\omega) = p_\omega$, this contradicts the definition of p_ω. Hence, we have $p_\omega \geqslant 1$. Then we get $\eta(\omega) \succcurlyeq p(\omega)\xi(\omega) \succcurlyeq \xi(\omega)$. Similarly, we can prove that $\xi(\omega) \succcurlyeq \eta(\omega)$. Thus, we proved that

$$\xi(\omega) = \eta(\omega) \quad \forall \omega \in \Pi,$$

which implies from (8) that $\xi(\omega) = \eta(\omega)$, $\omega \in \Omega$ a.e. This makes end to the proof. $\square$

Taking $l = 0$ in Theorem 1, we obtain immediately the following random fixed point result.

Corollary 1. *Let $T : \Omega \times P \to P$ be a continuous random operator satisfying conditions* (i)–(iv). *Then T has a random fixed point $\xi : \Omega \to [u_0, v_0]$, where $u_0, v_0 \in P_h$ and $u_0 \prec v_0$. Moreover, if $\eta : \Omega \to P$ is another random fixed point of T satisfying $\eta(\omega) \in P_h$, $\omega \in \Omega$ a.e., then $\xi(\omega) = \eta(\omega)$, $\omega \in \Omega$ a.e.*

3 Functional random integral equations

We consider a stochastic integral equation of the form

$$x(t,\omega) = q(t,\omega) + \int_0^t f\big(s, x(s,\omega), \omega\big)\,\mathrm{d}s, \tag{17}$$

where $\omega \in \Omega$, Ω is the supporting set of the complete probability measure space (Ω, Σ, μ), $t \in I = [0,1]$, $q : I \times \Omega \to [0,\infty)$ and $f : I \times [0,\infty) \times \Omega \to [0,\infty)$. We denote by $C(I,\mathbb{R})$ the set of all real valued continuous functions on I. We equip the space $C(I,\mathbb{R})$ with the uniform norm

$$\|u\|_\infty := \max_{t\in I} \big|u(t)\big| \quad \forall\, u \in C(I,\mathbb{R}).$$

It is well known that $(C(I,\mathbb{R}), \|\cdot\|_\infty)$ is a separable Banach space. We equip $C(I,\mathbb{R})$ with a σ-algebra $\beta_{C(I,\mathbb{R})}$ of Borel subsets of $C(I,\mathbb{R})$ so that $(C(I,\mathbb{R}), \beta_{C(I,\mathbb{R})})$ becomes a measurable space. We consider the partial order $\preccurlyeq$ on $C(I,\mathbb{R})$ endowed by the cone

$$P := \big\{u \in C(I,\mathbb{R}) \text{ such that } u(t) \geqslant 0 \text{ for all } t \in I\big\}.$$

Let $\mathcal{F}(I,\mathbb{R})$ be the set of all real valued functions on I. If $x : \Omega \to \mathcal{F}(I,\mathbb{R})$ is a given mapping, we denote

$$x(\omega)(t) := x(t,\omega) \quad \forall (\omega, t) \in \Omega \times I.$$

Definition 11. A mapping $x : \Omega \to P$ is said to be a positive solution to (17) iff x is measurable and satisfies (17) for $\omega \in \Omega$ a.e.

We consider the following assumptions:

(A1) For all $\omega \in \Omega$, the mapping $t \mapsto q(t,\omega)$ is continuous on I;

(A2) $\omega \in \Omega \mapsto q(\cdot,\omega) \in P$ is measurable;

(A3) There exist $h \in P$ (a nonzero function) and $l \geqslant 0$ such that, for $\omega \in \Omega$ a.e.,

$$0 \leqslant q(t,\omega) \leqslant lh(t) \quad \forall t \in [0,1];$$

(A4) For all $(\omega, p) \in \Omega \times [0,\infty)$, the mapping $t \mapsto f(t,p,\omega)$ belongs to $L^1(0,1)$;

(A5) The function $\omega \mapsto f(t,p,\omega)$ is measurable for all $(t,p) \in I \times [0,\infty)$;

(A6) There exist constants $c, d > 0$ and $e \in (0,1)$ such that

$$t^{\mathrm{e}}\big|f(t,p,\omega) - f(t,r,\omega)\big| \leqslant c|p-r|^d \quad \forall (t,p,r,\omega) \in I \times [0,\infty) \times [0,\infty) \times \Omega;$$

(A7) There exists $\epsilon \in (0,1)$ such that, for $\omega \in \Omega$ a.e.,

$$\epsilon h(t) \leqslant \int_0^t f\big(s, h(s), \omega\big)\,\mathrm{d}s \leqslant \frac{1}{\epsilon} h(t) \quad \forall t \in I;$$

(A8) $p, q \in [0,\infty)$, $p \leqslant q \Rightarrow f(s,p,\omega) \leqslant f(s,q,\omega)$ for all $(s,\omega) \in I \times \Omega$;
(A9) There exists a constant $\alpha \in (0,1)$ such that, for $\omega \in \Omega$ a.e.,

$$f(s,\mu p,\omega) \geqslant \mu^\alpha f(s,p,\omega) \quad \forall (s,\mu,p) \in I \times (0,1) \times [0,\infty).$$

We have the following result.

Theorem 2. *Suppose that the assumptions* (A1)–(A9) *are satisfied. Then*:
(I) *The integral equation* (17) *has a positive solution* $\xi : \Omega \to [u_0, v_0]$, *where* u_0, $v_0 \in P_h$ *and* $u_0 \prec v_0$;
(II) *If* $\eta : \Omega \to P$ *is another positive solution to* (17) *satisfying* $\eta(\omega) \in P_h$, $\omega \in \Omega$ *a.e., then* $\xi(\omega) = \eta(\omega)$, $\omega \in \Omega$ *a.e.*

Proof. Let

$$x_0(\omega) := q(\cdot,\omega) \quad \forall \omega \in \Omega.$$

From (A1)–(A3), $x_0 : \Omega \to P$ is a measurable function satisfying

$$0_{C(I,\mathbb{R})} \preccurlyeq x_0(\omega) \preccurlyeq lh, \quad \omega \in \Omega \ a.e.$$

For all $(\omega, x) \in \Omega \times P$, let

$$T(\omega,x)(t) := \int_0^t f\bigl(s, x(s), \omega\bigr)\,\mathrm{d}s \quad \forall t \in I.$$

From (A4) and (A8), the mapping $T : \Omega \times P \to P$ is well defined. Now, a measurable function $x : \Omega \to P$ is a positive solution to (17) iff

$$x(\omega) = x_0(\omega) + T\bigl(\omega, x(\omega)\bigr), \quad \omega \in \Omega \text{ a.e.}$$

Let us prove that T is a random continuous operator. Let $x \in P$ be fixed. From (A5), and since $\omega \mapsto \int_0^t f(s,x(s),\omega)\,\mathrm{d}s$ exists for each $\omega \in \Omega$, hence, it is a limit of a finite sum of measurable functions. So, $T(\cdot,x) : \Omega \to P$ is a measurable function. Thus, T is a random operator. Let $\omega \in \Omega$ be fixed. Let $\{x_n\}$ be a sequence in $C(I,\mathbb{R})$ such that $\|x_n - x\|_\infty \to 0$ as $n \to \infty$, where $x \in C(I,\mathbb{R})$. Using (A6), we get that, for all $t \in I$,

$$\bigl|T(\omega,x_n) - T(\omega,x)\bigr|(t) \leqslant \int_0^t \bigl|f\bigl(s,x_n(s),\omega\bigr) - f\bigl(s,x(s),\omega\bigr)\bigr|\,\mathrm{d}s \leqslant M\|x_n - x\|_\infty^d,$$

where M is a constant that depends on e and c. This implies that

$$\bigl\|T(\omega,x_n) - T(\omega,x)\bigr\|_\infty \leqslant c\|x_n - x\|_\infty^d \to 0 \quad \text{as } n \to \infty.$$

Then, for all $\omega \in P$, $T(\omega,\cdot) : P \to P$ is continuous. Hence, we proved that T is a random continuous operator. Now, we shall prove that T is random increasing. Let $\omega \in \Omega$,

$x, y \in P$ such that $x \preccurlyeq y$, that is, $x(s) \leqslant y(s)$ for all $s \in I$. From condition (A8), for all $t \in I$, we have

$$T(\omega, x)(t) = \int_0^t f\bigl(s, x(s), \omega\bigr)\,\mathrm{d}s \leqslant \int_0^t f\bigl(s, y(s), \omega\bigr)\,\mathrm{d}s = T(\omega, y)(t),$$

which implies that $T(\omega, x) \preccurlyeq T(\omega, y)$. Then T is a random increasing operator. Consider now the functions $\tau : (0,1) \to (0,1)$ and $\varphi : (0,1) \to (0,1)$ defined by

$$\tau(\mu) := \mu \quad \text{and} \quad \varphi(\mu) := \mu^\alpha \quad \forall \mu \in (0,1).$$

Since $\alpha \in (0,1)$, the functions τ and μ satisfy the conditions (H1)–(H2). Moreover, from (A9), for $\omega \in \Omega$ a.e., $(t, \mu, x) \in I \times (0,1) \times P$,

$$T\bigl(\omega, \tau(\mu)x\bigr)(t) = \int_0^t f\bigl(s, \mu x(s), \omega\bigr)\,\mathrm{d}s \geqslant \mu^\alpha \int_0^t f\bigl(s, x(s), \omega\bigr)\,\mathrm{d}s = \varphi(\mu)T(\omega, x)(t),$$

which means that for $\omega \in \Omega$ a.e.,

$$T\bigl(\omega, \tau(\mu)x\bigr) \succcurlyeq \varphi(\mu)T(\omega, x) \quad \forall (x, \mu) \in P \times (0,1).$$

This implies that T is a τ–φ-concave operator. Finally, from condition (A7), we have

$$\epsilon h \preccurlyeq T(\omega, h) \preccurlyeq \frac{1}{\epsilon} h, \quad \omega \in \Omega \ a.e.$$

Now, applying Theorem 1, we get (I) and (II). □

We end the paper with the following example.

Example 1. Let (Ω, Σ, μ) be a complete probability measure space. We consider the stochastic integral equation

$$x(t, \omega) = q(t, \omega) + \int_0^t \frac{s^\gamma + \sqrt{x(s, \omega)}}{\sqrt{s}}\,\mathrm{d}s, \tag{18}$$

where $t \in I = [0,1]$, $\gamma > 1/2$ and q satisfies conditions (A1)–(A3) with $h(t) = t$. Consider the function $f : I \times [0, \infty) \times \Omega \to [0, \infty)$ defined by

$$f(s, p, \omega) := \begin{cases} \frac{s^\gamma + \sqrt{p}}{\sqrt{s}} & \text{if } s \neq 0, \\ a & \text{if } s = 0, \end{cases}$$

where a is any positive constant. Then (18) is equivalent to

$$x(t, \omega) = q(t, \omega) + \int_0^t f\bigl(s, x(s, \omega), \omega\bigr)\,\mathrm{d}s.$$

Clearly, since $\gamma > 1/2$, for all $(\omega, p) \in \Omega \times [0,\infty)$, the mapping $t \mapsto f(t,p,\omega)$ belongs to $L^1(0,1)$. So, condition (A4) is satisfied. Condition (A5) follows immediately from the definition of f. Let $(t,p,r,\omega) \in (0,1] \times [0,\infty) \times [0,\infty) \times \Omega$. We have

$$t^{1/2}\bigl|f(t,p,\omega) - f(t,r,\omega)\bigr| = |\sqrt{p} - \sqrt{r}| \leqslant \sqrt{|p-r|}.$$

So, condition (A6) is satisfied. On the other hand, we have

$$\int_0^t f\bigl(s,h(s),\omega\bigr)\,\mathrm{d}s = \int_0^t f(s,s,\omega)\,\mathrm{d}s = \int_0^t \bigl(s^{\gamma-1/2}+1\bigr)\,\mathrm{d}s = \frac{2\gamma+3}{2\gamma+1}t.$$

Taking

$$0 < \epsilon := \frac{2\gamma+1}{2\gamma+3} < 1,$$

we get that

$$\epsilon h(t) \leqslant \int_0^t f\bigl(s,h(s),\omega\bigr)\,\mathrm{d}s \leqslant \frac{1}{\epsilon}h(t).$$

Thus, condition (A7) is satisfied. Condition (A8) can be easily checked. Finally, let $(s,\mu,p) \in I \times (0,1) \times [0,\infty)$. We have

$$f(s,\mu p,\omega) = \frac{s^\gamma + \sqrt{\mu}\sqrt{p}}{\sqrt{s}} \geqslant \sqrt{\mu}\frac{s^\gamma + \sqrt{p}}{\sqrt{s}} = \mu^{1/2} f(s,p,\omega).$$

Then condition (A9) is satisfied with $\alpha = 1/2$. Now, applying Theorem 2, we obtain the following results:

(R1) The integral equation (18) has a positive solution $\xi : \Omega \to [u_0, v_0]$, where $u_0, v_0 \in P_h$ and $u_0 \prec v_0$;

(R2) If $\eta : \Omega \to P$ is another positive solution to (18) satisfying $\eta(\omega) \in P_h$, $\omega \in \Omega$ a.e., then $\xi(\omega) = \eta(\omega)$ and $\omega \in \Omega$ a.e.

Using (4) and (5), we can take $u_0(t) = \epsilon^3 t$ and $v_0(t) = \epsilon^{-3} t$.

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