

## Adaptive motion/force control of nonholonomic mechanical systems with affine constraints\*

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**Abstract.** The motion/force tracking control of nonholonomic mechanical systems with affine constraints is investigated in this paper. In the procedure of control design, by flexibly using the algebra processing technique, constraint forces are successfully canceled in the dynamic equations, and then an integral feedback compensation strategy and an adaptive scheme are applied to identify the dynamic uncertainty. It is shown that the proposed controller ensures the position state of the closed-loop system asymptotically tracks the desired trajectory and the force tracking error has a controllable bound. Finally, a boat on a running river as a simulation example is given to show the effectiveness of the control scheme.

**Keywords:** tracking control, nonholonomic mechanical systems, affine constraints, adaptive control.

### 1 Introduction

Nonholonomic constraints arise in many mechanical systems when there is a rolling or sliding contact, such as wheeled mobile robots,  $n$ -trailer systems, space robots, underwater vehicles, multi-fingered robotic hands, and so on. Although great progress [1–9] has been made for nonholonomic systems during the last decades, controller design for these systems still has a challenge to control engineers.

It is worth pointing out that most existing results [10–15] aimed at the classic nonholonomic linear constraints (i.e.  $J(q)\dot{q} = 0$ ). In fact, there is another large class of constraints, which are affine in velocities, called affine constraints [16–19] (i.e.  $J(q)\dot{q} = A(q)$ ), such as a boat on a running river with the varying stream, ball on rotating table with

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invariable angular velocity, underactuated mechanical arm, etc. Kai defined rheonomous affine constraints in [16] and explained a geometric representation method for them, and derived a necessary and sufficient condition for complete nonholonomicity of the rheonomous affine constraints. In [17], Kai derived very good results about nonholonomic dynamic systems with affine constraints. To be specific, the local accessibility and local controllability were developed based on both Sussmann's theorem and linear approximation approaches, and a necessary and sufficient condition for complete nonholonomicity of the a rheonomous affine constraints were presented at last.

The tracking problem for mechanical systems, as a much more interesting issue in practice, is to make the entire state of the closed-loop system track to a given desired trajectory. For example, [20] presented an adaptive robust control strategies for a class of mechanical systems with both holonomic and nonholonomic constraints. Based on physical properties, adaptive robust motion/force control for wheeled inverted pendulums is investigated in [21]. For mobile manipulators under both holonomic and nonholonomic constraints, [22] and [23] proposed state-feedback control strategies by introducing an appropriate state transformation, and an adaptive robust output-feedback force/motion control strategies, respectively, and so on. However, it should be noted that the aforementioned works were reported on nonholonomic systems with classical linear constraints, and to date, no solutions have been done on tracking control of nonholonomic systems subjected to affine constraints. Hence, researching the tracking problem for such nonholonomic mechanical systems is an innovatory and significant work. Xian presented a new continuous control mechanism that compensated for uncertainty in a class of high-order, multiple-input multiple-output nonlinear systems in [24]. Based on this control strategy, Makkar considered modeling and compensation for parameterizable friction effects for a class of mechanical systems [25]. However, the upper bound of the uncertainties must be known as a prerequisite. Base on Xian's compensatory scheme, this paper investigates the tracking control problem for a class of uncertainty nonholonomic mechanical systems. At the same time, the constraint that the upper bound of the uncertainties is known is relaxed by constructing an adaptive update law. To achieve the tracking objective, by flexibly using the Algebra processing technique, we firstly triumphantly reduce the number of state variables, which provide a motion complying with affine constraints, and an integral feedback compensation term is used to identify the dynamic uncertainties. The main contributions of the paper are briefly characterized by the following specific features:

- (i) We establish the dynamical model of the nonholonomic control systems with affine constraints. Since affine constraints are introduced to mechanical systems, it is difficult to find linearly independent vector fields to cancel the constraint forces  $J(q)\lambda$  in dynamic equation. Hence, how to deal with them is the main innovation of this paper.
- (ii) Based on the asymptotic tracking idea for uncertain multi-input nonlinear systems, the related adaptive theory and the compensatory strategy for the uncertainties, an adaptive tracking controller is designed such that the trajectory tracking error asymptotically tends to zero and the force tracking error is bounded with a controllable bound.

- (iii) As a practical application, a boat on a running river with varying stream is presented to illustrate the reasonability of the assumptions and the effectiveness of the control strategy.

**Notations.**  $\|x\|$  denotes the Euclidean norm of  $x$ ;  $C^i$  denotes the set of all functions with continuous  $i$ th partial derivative on  $\mathbb{R}^n$ ;  $\text{sgn}(\cdot)$  denotes the standard signum function; a continuous function  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is said to be a  $\mathcal{K}$  function if it is strictly increasing and vanish at zero. For simplicity, sometimes the arguments of functions are dropped.

## 2 System description and control design

### 2.1 Dynamics model

According to Euler–Lagrange formulation, equations of nonholonomic mechanical systems are described by

$$M(q)\ddot{q} + V(q, \dot{q})\dot{q} + G(q) = f + B(q)\tau, \quad (1)$$

where  $q = [q_1, \dots, q_n]^T$  is the generalized coordinates, and  $\dot{q}, \ddot{q} \in \mathbb{R}^n$  represent the generalized velocity vector, acceleration vector, respectively;  $M(q) \in \mathbb{R}^{n \times n}$  is the inertia matrix;  $V(q, \dot{q})\dot{q} \in \mathbb{R}^n$  presents the vector of centripetal, Coriolis forces;  $G(q) \in \mathbb{R}^n$  represents the vector of gravitational forces;  $\tau$  denotes the  $r$ -vector of generalized control inputs;  $B(q) \in \mathbb{R}^{n \times r}$  is a known input transformation matrix ( $r < n$ ) with full rank;  $f \in \mathbb{R}^n$  denotes the vector of constraint forces.

Consider the situation where kinematic constraints are imposed, which represented by analytical relations between the generalized coordinates  $q$  and velocity vector  $\dot{q}$ , it can be described by

$$J^T(q)\dot{q} = A(q), \quad (2)$$

where  $J(q) = [j_1(q), \dots, j_m(q)] \in \mathbb{R}^{n \times m}$  is full of constraint matrix,  $A(q) = [a_1(q), \dots, a_m(q)]^T \in \mathbb{R}^m$  is known.

The constraint equation (2) is regarded as affine constraints. When it is imposed on the mechanical systems (1), the constraint forces are given by

$$f = J(q)\lambda,$$

where  $\lambda \in \mathbb{R}^m$  is a Lagrangian multiplier corresponding to  $m$  nonholonomic affine constraints.

**Remark 1.** (i) It's worth pointing out that the system studied in this paper is much more general than that in some existing literatures such as [10, 14, 20–23], in which dynamic equations satisfy the classical linear constraints. In fact, by taking  $A(q) = 0$ ,  $J^T(q)\dot{q} = A(q)$  can be transformed into linear constraints, whose tracking problems have been extensively studied during the last two decades.

(ii) With affine constraints introduced to mechanical systems, traditional methods [20–23] are hardly applied to such systems. Hence, how to deal with constraint forces is also a difference between this paper and the existing literatures.

## 2.2 Reduced dynamics and state transformation

This part mainly focuses on reducing the number of state variables, which provides motion complying with the affine constraints.

It is easy to find a full-rank matrix  $S \in \mathbb{R}^{n \times (n-m)}$  satisfying

$$J^T(q)S(q) = 0. \quad (3)$$

Noticing that  $S(q)$  is full of rank, there must exist a full-rank matrix  $S_1(q) \in \mathbb{R}^{(n-m) \times n}$  satisfying  $S_1(q)S(q) = I$ , where  $I$  is an identity matrix. If defining  $\xi(t) = [q, -t]^T$ , then (2) can be expressed concisely as

$$[J^T(q), A(q)]\dot{\xi} = 0. \quad (4)$$

For the sake of convenience, we define

$$E(q) = \begin{bmatrix} S(q) & \eta(q) \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n-m+1)},$$

where  $\eta(q) \in \mathbb{R}^n$  satisfies  $J^T(q)\eta(q) = A(q)$ . One can deduce that  $E$  is a full of rank and satisfies

$$[J^T(q), A(q)]E(q) = 0. \quad (5)$$

From (4) and (5), we know that there exists an  $(n-m+1)$ -dimensional vector  $\bar{z}$  such that

$$\dot{\xi} = E\dot{z} = \begin{bmatrix} S(q) & \eta(q) \\ 0 & -1 \end{bmatrix} \dot{z}, \quad (6)$$

where  $\dot{z} = [\dot{z}^T, \dot{z}_{n-m+1}]^T$ ,  $z = [z_1, \dots, z_{n-m}]^T$ .

In view of the relationship (6), one can obtain  $\dot{z}_{n-m+1} = 1$ , and the generalized velocity vectors can be written as

$$\dot{q} = S(q)\dot{z} + \eta(q). \quad (7)$$

$z$  corresponds to the internal state variable, and  $(q, z)$  is sufficient to describe the constrained motion. Equation (7) represents the kinematics of a nonholonomic mechanical system.

Substituting (7) into (1), the dynamics of the mechanical system (1) with affine constraints (2) can be described clearly as

$$\bar{M}(q)\ddot{z} + \bar{V}(q, \dot{q})\dot{z} + \bar{G}(q, \dot{q}) = J(q)\lambda + B(q)\tau, \quad (8)$$

where  $\bar{M}(q) = M(q)S(q)$  and  $\bar{V}(q, \dot{q}) = M(q)\dot{S}(q) + V(q, \dot{q})S(q)$ ,  $\bar{G}(q, \dot{q}) = M(q)\dot{\eta}(q) + V(q, \dot{q})\eta(q) + G(q)$  can be seen as the generalized gravitational force vector.

**Remark 2.** The aforementioned transform methods differ from the traditional ones mentioned in existing papers [20–22]. More specifically, when the affine constraints are imposed on the mechanical systems, it is difficult to find linearly independent vector fields to proceed a simple transformation for canceling the constraint forces in dynamic equations. Hence, we present a new transformation to achieve this goal.

**Remark 3.** The transformation consists of (3) and (7) can ensure system (8) still satisfy constraint equation (2), and possesses the practical physical meaning, this can also be confirmed by the practical example in Section 5.

### 2.3 Error system development

In practice, the complexity and unpredictability of the structure of uncertainties usually appear in the dynamics of the mechanical systems, we assume that  $\bar{M}(q)$ ,  $\bar{V}(q, \dot{q})$  and  $\bar{G}(q, \dot{q})$  are expressed in the form

$$\begin{aligned}\bar{M}(q) &= M_0(q) + \nabla M(q), \\ \bar{V}(q, \dot{q}) &= V_0(q, \dot{q}) + \nabla V(q, \dot{q}), \\ \bar{G}(q, \dot{q}) &= G_0(q, \dot{q}) + \nabla G(q, \dot{q}),\end{aligned}$$

where  $M_0$ ,  $V_0$ ,  $G_0$ , as the nominal matrices, are assumed to be known exactly, and  $\nabla M$ ,  $\nabla V$ ,  $\nabla G$  represent the uncertainties in system matrices. Then, the dynamic model (8) can be rewritten as

$$M_0(q)\ddot{z} + V_0(q, \dot{q})\dot{z} + G_0(q, \dot{q}) + \Phi_u(q, \dot{q}, \dot{z}, \ddot{z}) = J(q)\lambda + B(q)\tau, \quad (9)$$

where  $\Phi_u(q, \dot{q}, \dot{z}, \ddot{z}) = \nabla M(q)\ddot{z} + \nabla V(q, \dot{q})\dot{z} + \nabla G(q, \dot{q}) \in \mathbb{R}^{n-m}$ .

Pre-multiplying  $S^T(q)$  on both sides of (9), and noting  $J^T(q)S(q) = 0$ , the following transformed system can be received:

$$M_1(q)\ddot{z} + V_1(q, \dot{q})\dot{z} + G_1(q, \dot{q}) + \bar{\Phi}_1(q, \dot{q}, \dot{z}, \ddot{z}) = B_1(q)\tau,$$

where  $M_1(q) = S^T(q)M_0(q)$ ,  $V_1(q, \dot{q}) = S^T(q)V_0(q, \dot{q})$ ,  $B_1(q) = S^T(q)B(q)$ ,  $G_1(q, \dot{q}) = S^T(q)G_0(q, \dot{q})$ ,  $\bar{\Phi}_1(q, \dot{q}, \dot{z}, \ddot{z}) = S^T(q)\Phi_u(q, \dot{q}, \dot{z}, \ddot{z})$ . According to Masahiro Oya' statement [14], there exists a coordinate transformation  $q = \Psi(z)$  such that  $\Phi_1(z, \dot{z}, \ddot{z}) = \bar{\Phi}_1(q, \dot{q}, \dot{z}, \ddot{z})|_{q=\Psi(z)}$ . Let  $\Phi_1$  replace  $\bar{\Phi}_1$  in above equation, the following is obtained:

$$M_1(q)\ddot{z} + V_1(q, \dot{q})\dot{z} + G_1(q, \dot{q}) + \Phi_1(z, \dot{z}, \ddot{z}) = B_1(q)\tau. \quad (10)$$

The control objective of this paper is specified as: A given desired trajectory  $z_d(t)$  satisfying that  $z_d^{(i)}(t)$ ,  $i = 0, \dots, 4$ , exist and are bounded, a desired constraint force  $f_d(t)$  or a desired multiplier  $\lambda_d(t)$ , determine a adaptive control law for system (1) such that:

- (i) All the states of the closed-loop system are globally bounded.
- (ii) The position and velocity tracking error  $z(t) - z_d(t)$ ,  $\dot{z}(t) - \dot{z}_d(t)$  converge to zero as  $t \rightarrow \infty$ , respectively.
- (iii) The tracking error of constraint force  $f - f_d$  is bounded for all  $t \geq 0$ .

The subsequent development is based on the assumption that  $\Phi_1$  is an  $C^2$  nonlinear vector function. In order to solve the previous problem, we make the following assumptions.

**Assumption 1.** (See [26].) The matrix  $M_1$  is symmetric, positive definite and satisfies

$$a\|x\|^2 \leq x^T M_1(x)x \leq \bar{a}(\|x\|)\|x\|^2,$$

where  $a$  is a known positive constant,  $\bar{a}(x)$  is a known positive function.

**Assumption 2.** If  $q(t) \in \mathcal{L}_\infty$ , then  $\partial M_1(q)/\partial q$  exists and is bounded. Moreover, if  $q(t), \dot{q}(t), \ddot{q}(t) \in \mathcal{L}_\infty$ , then  $V_1(q, \dot{q})$  and  $\partial V_1(q, \dot{q})/\partial q$  exist and are all bounded.

For practical mechanical systems, the Assumption 1 and 2 are reasonable. Next, we develop the following tracking error system, which will be used in the subsequent controller design and stability analysis:

$$e_1 = z_d - z, \quad (11)$$

$$e_\lambda = \lambda - \lambda_d, \quad (12)$$

where  $e_1 \in \mathbb{R}^{n-m}$ ,  $e_\lambda \in \mathbb{R}^m$ . To achieve the desired control objective, the following filtered tracking errors [25, 27], denoted by  $e_2, \rho \in \mathbb{R}^{n-m}$ , are defined as

$$\begin{aligned} e_2 &= \dot{e}_1 + \alpha_1 e_1, \\ \rho &= \dot{e}_2 + \alpha_2 e_2, \end{aligned} \quad (13)$$

where  $\alpha_1 > 0$ ,  $\alpha_2 > 0$  are designed constants.

In view of (9), (11) and (13), pre-multiplying  $M_0$  on both sides of the second formula of (13), the following expression can be arrived at:

$$\begin{aligned} M_0 \rho &= M_0 \ddot{z}_d + V_0 \dot{z}_d + G_0 + \Phi_u - B\tau - J(q)\lambda \\ &\quad + \alpha_1 M_0 \dot{e}_1 + \alpha_2 M_0 e_2 - V_0 \dot{e}_1. \end{aligned} \quad (14)$$

Based on the expression (14), a control torque input is designed as follows:

$$B\tau = M_0 \ddot{z}_d + V_0 \dot{z}_d + G_0 - J(q)\lambda_c + S_1^T \mu, \quad (15)$$

where the force term  $\lambda_c$  is defined as  $\lambda_c = \lambda_d - k_\lambda e_\lambda$ ,  $k_\lambda$  is a constant of force control feedback gain, and  $\mu(t) \in \mathbb{R}^{n-m}$  denotes a subsequently designed control term. Substituting (15) into (14), we can further get

$$M_0 \rho = \Phi_u + J(q)(\lambda_c - \lambda) - S_1^T \mu + \alpha_1 M_0 \dot{e}_1 + \alpha_2 M_0 e_2 - V_0 \dot{e}_1. \quad (16)$$

After pre-multiplying  $S^T$ , noting  $S^T(q)J(q) = 0$  and  $S_1(q)S(q) = I$ , the above equation becomes

$$M_1 \rho = \Phi_1 - \mu + \alpha_1 M_1 \dot{e}_1 + \alpha_2 M_1 e_2 - V_1 \dot{e}_1. \quad (17)$$

To facilitate the design of  $\mu(t)$ , differentiating (17) yields

$$M_1 \dot{\rho} = \dot{\Phi}_1 - \dot{\mu} - \dot{M}_1 \rho + \Upsilon, \quad (18)$$

where  $\Upsilon = \alpha_1 \dot{M}_1 e_2 + \alpha_1 M_1 \rho - \alpha_1 \alpha_2 M_1 e_2 - \alpha_1^2 \dot{M}_1 e_1 - \alpha_1^2 M_1 e_2 + \alpha_1^3 M_1 e_1 + \alpha_2 \dot{M}_1 e_2 + \alpha_2 M_1 \rho - \alpha_2^2 M_1 e_2 - \dot{V}_1 e_2 - V_1 \rho + \alpha_2 V_1 e_2 + \alpha_1 \dot{V}_1 e_1 + \alpha_1 V_1 e_2 - \alpha_1^2 V_1 e_1$ .

Based on the method of compensation for uncertain dynamic [24],  $\mu(t)$  is designed as follows:

$$\begin{aligned} \mu(t) &= (k_s + 1)e_2(t) - (k_s + 1)e_2(0) \\ &\quad + \int_0^t ((k_s + 1)\alpha_2 e_2(s) + \hat{\Theta}(s) \operatorname{sgn}(e_2(s))) \, ds \end{aligned} \quad (19)$$

with the adaptive update law

$$\dot{\hat{\Theta}}(t) = \frac{1}{\gamma} \operatorname{sgn}(e_2^T(t)) \rho(t), \quad (20)$$

where design parameters  $k_s, \gamma \in \mathbb{R}$  are positive control gains, and  $\hat{\Theta}(t) \in \mathbb{R}$  is the parameter estimation of  $\Theta$ , which will be specified later. The second term in (19) is used to ensure that  $\mu(0) = 0$ .  $\mu(t)$  does not depend on the unmeasurable filtered tracking error term  $\rho$ , but its time derivative can be expressed as a function of  $\rho$ . Taking the time derivative of  $\mu(t)$ , one has

$$\begin{aligned} \dot{\mu}(t) &= (k_s + 1)\dot{e}_2(t) + (k_s + 1)\alpha_2 e_2(t) + \dot{\hat{\Theta}}(t) \operatorname{sgn}(e_2(t)) \\ &= (k_s + 1)\rho(t) + \dot{\hat{\Theta}}(t) \operatorname{sgn}(e_2(t)). \end{aligned} \quad (21)$$

Substituting (21) into (18) results in

$$M_1(q)\dot{\rho} = -(k_s + 1)\rho - \dot{\hat{\Theta}}(t) \operatorname{sgn}(e_2(t)) - \frac{1}{2}\dot{M}_1(q)\rho - e_2 + \Gamma(z, \dot{z}, t), \quad (22)$$

where  $\Gamma(z, \dot{z}, t) = \dot{\Phi}_1 + \Upsilon - (1/2)\dot{M}_1\rho + e_2 \in \mathbb{R}^{n-m}$ . Now, defining

$$\Gamma_d = \frac{\partial \Phi_1(z_d, \dot{z}_d, \ddot{z}_d)}{\partial z_d} \dot{z}_d + \frac{\partial \Phi_1(z_d, \dot{z}_d, \ddot{z}_d)}{\partial \dot{z}_d} \ddot{z}_d + \frac{\partial \Phi_1(z_d, \dot{z}_d, \ddot{z}_d)}{\partial \ddot{z}_d} z_d^{(3)}.$$

Noting that  $\Phi_1$  is an  $\mathcal{C}^2$  vector function and  $z_d^{(i)}, i = 0, \dots, 4$ , are all bounded, there must exist two unknown positive constants  $B_1$  and  $B_2$  such that

$$\|\Gamma_d\| \leq B_1, \quad \|\dot{\Gamma}_d\| \leq B_2.$$

Defining  $\tilde{\Gamma}(t) = \Gamma(t) - \Gamma_d(t)$ , the closed-loop error system (22) can be rewritten as

$$M_1(q)\dot{\rho} = -(k_s + 1)\rho - \dot{\hat{\Theta}}(t) \operatorname{sgn}(e_2(t)) - \frac{1}{2}\dot{M}_1(q)\rho - e_2 + \tilde{\Gamma}(t) + \Gamma_d(t). \quad (23)$$

**Remark 4.** It is worth highlighting that  $\hat{\Theta}(t)$  does not depend on the unmeasurable signal  $\dot{z}(t)$ . In fact, integrating both sides of formula (20), one can get  $\hat{\Theta}(t) = \hat{\Theta}(t_0) + (1/\gamma) \int_{t_0}^t \operatorname{sgn}(e_2^T(s)) \dot{e}_2(s) \, ds + (\alpha_2/\gamma) \int_{t_0}^t \operatorname{sgn}(e_2^T(s)) e_2(s) \, ds$ . Thereby  $\hat{\Theta}(t)$  only depends on the measurable signal  $e_2(t)$ .

### 3 Main results

Now, we are ready to present the following theorem, which summarizes the main results of the paper.

**Theorem 1.** *Consider the nonholonomic mechanical system described by (1) and (2), subjects to Assumptions 1 and 2. Given a desired trajectory  $z_d(t)$ , which satisfies the constraint equation (2), using the control laws (15), (19) and (20), the following hold:*

- (i) *All the states of the closed-loop system are globally bounded.*
- (ii) *The tracking error  $e_1$  and  $\dot{e}_1$  converge to zero as  $t \rightarrow \infty$ .*
- (iii)  *$e_\lambda$  is bounded for all  $t \geq 0$ .*

*Proof.* Let  $D \in \mathbb{R}^{3(n-m)+2}$  be a domain containing  $y(t) = 0$ , where  $y(t) \in \mathbb{R}^{3(n-m)+2}$  is defined as  $y(t) = [x^T(t), \tilde{\Theta}(t), \sqrt{P(t)}]^T$ ,  $x(t) \in \mathbb{R}^{3(n-m)}$  is defined as  $x(t) = [e_1^T, e_2^T, \rho^T]^T$ , and  $\tilde{\Theta}(t) = \Theta - \hat{\Theta}(t)$  represents the parameter estimation error. The function  $P(t) \in \mathbb{R}$  is defined as

$$P(t) = \Theta \|e_2(0)\| - e_2(0)^T \Gamma_d(0) - \int_0^t L(s) ds,$$

where the auxiliary function  $L(t)$  is defined as

$$L(t) = \rho^T (\Gamma_d(t) - \Theta \operatorname{sgn}(e_2)).$$

Selecting  $\Theta = B_1 + (1/\alpha_2)B_2 + 1$ , by taking the same manipulations as Appendix A in [24], there is

$$\int_0^t L(s) ds \leq \Theta \|e_2(0)\| - e_2(0)^T \Gamma_d(0).$$

Hence,  $P(t) \geq 0$ .

Now, choose a continuously differentiable, positive definite and radially unbounded function

$$V(y, t) = e_1^T e_1 + \frac{1}{2} e_2^T e_2 + \frac{1}{2} \rho^T M_1 \rho + P + \frac{\gamma}{2} \tilde{\Theta}^2. \quad (24)$$

Taking the time derivative of  $V$  along solutions of (10), noting the definition of  $\dot{\tilde{\Theta}}$  and substituting (11), (13) and (23) into it, we immediately get

$$\begin{aligned} \dot{V} &= 2e_1^T \dot{e}_1 + e_2^T \dot{e}_2 + \rho^T M_1 \dot{\rho} + \frac{1}{2} \rho^T \dot{M}_1 \rho + \dot{P} - \gamma \tilde{\Theta} \dot{\tilde{\Theta}} \\ &\leq -2\alpha_1 \|e_1\|^2 - \alpha_2 \|e_2\|^2 - (k_s + 1) \|\rho\|^2 + 2e_1^T e_2 + \rho^T \tilde{F}. \end{aligned} \quad (25)$$

Since  $\Gamma(t)$  is continuously differentiable, with the the help of mean value theorem, one can acquire the upper bound of  $\tilde{F}$  as follows [24]:

$$\|\tilde{F}\| \leq \varphi(\|x\|) \|x\|,$$



where  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an appropriate  $\mathcal{K}$  function. By using the fact that  $2e_1^T e_2 \leq \|e_1\|^2 + \|e_2\|^2$ ,  $\dot{V}$  can be simplified as

$$\dot{V} \leq -\lambda\|x\|^2 - k_s\|\rho\|^2 + \varphi(\|x\|)\|\rho\|\|x\|, \quad (26)$$

where  $\lambda = \min\{2\alpha_1 - 1, \alpha_2 - 1, 1\}$ , and  $\alpha_1, \alpha_2$  must be chosen to satisfy  $\alpha_1 > 1/2, \alpha_2 > 1$ .

Completing the squares for the third term in (26), it follows that

$$\varphi(\|x\|)\|\rho\|\|x\| \leq k_s\|\rho\|^2 + \frac{\varphi^2(\|x\|)\|x\|^2}{4k_s},$$

with this inequality in mind, the following expression can be obtained:

$$\dot{V} \leq -\lambda\|x\|^2 + \frac{\varphi^2(\|x\|)\|x\|^2}{4k_s}. \quad (27)$$

Now, we define a compact set

$$N_1 = \{y \in \mathbb{R}^{3(n-m)+2} \mid \|y\| \leq \varphi^{-1}(2\sqrt{\lambda k_s})\}.$$

Inequality (27) shows  $V(t) \leq V(0)$  in  $N_1$ , hence, all the the signals  $e_1, e_2, \rho, \tilde{\Theta}$  on the right-hand side of function (24) are bounded in  $N_1$ . From the definition of  $e_1, e_2, \rho, \tilde{\Theta}$ , we know  $\dot{e}_1 = e_2 - \alpha_1 e_1, \dot{e}_2 = \rho - \alpha_2 e_2, \dot{\tilde{\Theta}} = \Theta - \hat{\Theta}$ , therefore, we can further get  $\dot{e}_1, \dot{e}_2, \dot{\tilde{\Theta}} \in \mathcal{L}_\infty$  in  $N_1$ . The assumption that  $z_d, \dot{z}_d, \ddot{z}_d$  are bounded can be used to conclude that  $z, \dot{z}, \ddot{z} \in \mathcal{L}_\infty$  in  $N_1$ . With  $M_1, V_1, G_1$  are all known and bounded in  $N_1$ . Thereby  $\tau_1, \mu \in \mathcal{L}_\infty$  in  $N_1$  can be further obtained from (15) and (19).

Then, let  $N_2 \subset N_1$  denotes a set defined as follows:

$$N_2 = \{y(t) \subset N_1 \mid \delta_2(y)\|y\|^2 < \delta_1(\varphi^{-1}(2\sqrt{\lambda k_s}))^2\},$$

where  $\delta_1 = (1/2) \min\{1, a\}$ ,  $\delta_2(y) = \max\{1, (1/2)\bar{a}(y)\}$ , and the definitions of  $a$  and  $\bar{a}(y)$  have been given in Assumption 1. From expression (27), one can obtain that there must exist an appropriate positive semidefinite function  $U(y) = c\|x(t)\|^2$  such that

$$\dot{V} \leq -U(y).$$

With invariance-like theorem (Theorem 8.4 of [28]) in mind, one can further get

$$U(y) = c\|x(t)\|^2 \rightarrow 0, \quad t \rightarrow \infty, \quad \forall y(0) \in N_2.$$

Based on the definitions of  $x(t)$ , one can finally gain  $e_1(t), e_2(t), \rho(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all  $y(0) \in N_2$ . From (13), we then know  $\dot{e}_1(t), \dot{e}_2(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all  $y(0) \in N_2$ .

On the other hand, from (17), it is evident that if  $\rho(t), e_2(t)$  and  $\dot{e}_1(t)$  are all bounded, then  $\mu(t) - \Phi_1$  is bounded. According to the boundedness of  $S_1(q), S_1^T(\mu(t) - \Phi_1)$  is

bounded. Substituting the control laws (15) and (19) into reduced order dynamic model (9) yields

$$\begin{aligned} J(q)(\lambda - \lambda_c) &= (1 + k_\lambda)J(q)e_\lambda = \Phi_u - S_1^T \mu - M_0 \ddot{e}_1 - V_0 \dot{e}_1 \\ &= S_1^T (\Phi_1 - \mu(t)) - M_0 \ddot{e}_1 - V_0 \dot{e}_1 = \omega(q, \dot{q}, \dot{z}, \ddot{z}, \dot{z}_d, \ddot{z}_d). \end{aligned}$$

$\omega(q, \dot{q}, \dot{z}, \ddot{z}, \dot{z}_d, \ddot{z}_d)$  be a bounded function vector. Therefore, the force tracking error ( $f - f_d$ ) is bounded and can be adjusted by changing the feedback gain  $k_\lambda$ . Thus, the theorem is proved completely.  $\square$

#### 4 Simulation

Consider a boat with payload on a running river [17, 19] (see Fig. 1). The  $x$ -axis and  $y$ -axis denote the transverse direction and the downstream direction of the river, respectively. Here, we suppose the stream of the river only depends on transverse position  $x$  in the simulation. According to the motion of boat on the river, one can get the following kinematic equations:

$$\begin{aligned} \dot{x} &= u \cos \theta - C(x) \cos \theta \sin \theta, \\ \dot{y} &= u \sin \theta + C(x) \cos^2 \theta, \end{aligned}$$

where  $C(x)$  denotes the stream of the river. After some simple calculations, the affine constraints can be obtained as follows:

$$\cos \theta \dot{y} - \sin \theta \dot{x} = C(x) \cos \theta.$$

We assume that the traveling direction velocity and the angular velocity of the boat can be controlled. Hence, the control input  $\tau = [\tau_1, \tau_2]^T$  is defined by the transformation matrix

$$B(q) = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{bmatrix}.$$

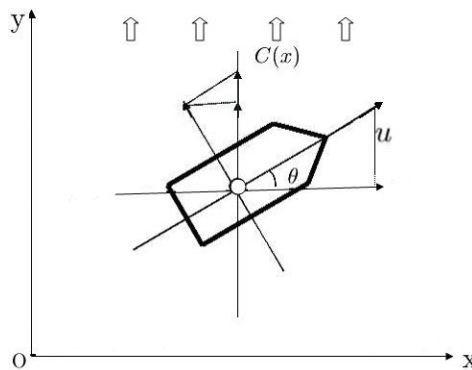


Fig. 1. Boat on a running river.

The standard forms are given as follows:

$$q = [y, x, \theta]^T, \quad M(q) = \begin{bmatrix} m + m_0 & 0 & 0 \\ 0 & m + m_0 & 0 \\ 0 & 0 & I + I_0 \end{bmatrix},$$

$$V(q, \dot{q}) = 0, \quad G(q) = 0,$$

$$J^T(q) = [\cos q_3, -\sin q_3, 0], \quad A(q) = C(q_2) \cos q_3,$$

where  $m$  is the mass of the boat and  $I$  is the inertia of the boat,  $m_0$  denotes the unknown mass of the payload and  $I_0$  denotes the unknown inertia of the payload. For the sake of simplicity, select  $m = 1$ ,  $I = 1$ ,  $C(q_2) = q_2$ .

One can choose

$$S(q) = \begin{bmatrix} \sin q_3 & 0 \\ \cos q_3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \eta(q) = [q_2, 0, 0]^T.$$

It follows from the procedure of the aforementioned diffeomorphism transformation that

$$\begin{aligned} \dot{q}_1 &= \dot{z}_1 \sin q_3 + q_2, \\ \dot{q}_2 &= \dot{z}_1 \cos q_3, \\ \dot{q}_3 &= \dot{z}_2. \end{aligned} \quad (28)$$

After imposing the constraint forces, the original system can be converted into the following form:

$$\begin{aligned} & \begin{bmatrix} \sin q_3 & 0 \\ \cos q_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} \dot{q}_3 \cos q_3 & 0 \\ -\dot{q}_3 \sin q_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} \dot{q}_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} m_0 II_1 \\ m_0 II_2 \\ I_0 \ddot{z}_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos q_3 \\ -\sin q_3 \\ 0 \end{bmatrix} \lambda + \begin{bmatrix} \cos q_3 & 0 \\ \sin q_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}, \end{aligned}$$

where  $II_1 = \ddot{z}_1 \sin z_2 + \dot{z}_1 \dot{z}_2 \cos z_2 + \dot{z}_1 \cos z_2$ ,  $II_2 = \ddot{z}_1 \cos z_2 - \dot{z}_1 \dot{z}_2 \sin z_2$ .

For the given  $J(q)$ ,  $S(q)$  and  $\eta(q)$ , the desired trajectory  $q_d = [\sin t - \cos t, \sin t, \pi/4]^T$  satisfies kinematic constraint  $J^T(q_d)\dot{q}_d = A(q_d)$  and diffeomorphism transform  $\dot{q}_d = S(q_d)\dot{z}_d + \eta(q_d)$  with  $z_d = [\sqrt{2} \sin t + 2, \pi/4]^T$ . The control objective is to determine an adaptive feedback control so that the trajectory  $z$  follows  $z_d$ , and  $\lambda$  is bounded.

Based on the previous design procedure, we get the actual controller

$$\begin{aligned} B\tau &= \begin{bmatrix} \sin q_3 & 0 \\ \cos q_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{z}_{1d} \\ \ddot{z}_{2d} \end{bmatrix} + \begin{bmatrix} \dot{q}_3 \cos q_3 & 0 \\ -\dot{q}_3 \sin q_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_{1d} \\ \dot{z}_{2d} \end{bmatrix} + \begin{bmatrix} \dot{q}_2 \\ 0 \\ 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} \cos q_3 \\ -\sin q_3 \\ 0 \end{bmatrix} \lambda_c + \begin{bmatrix} \sin q_3 & 0 \\ \cos q_3 & 0 \\ 0 & 1 \end{bmatrix} \mu(t), \end{aligned}$$

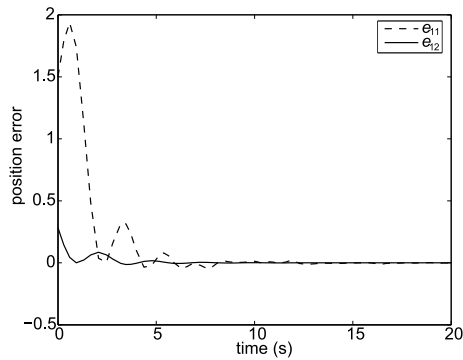


Fig. 2. The trajectories of  $e_1(t)$ .

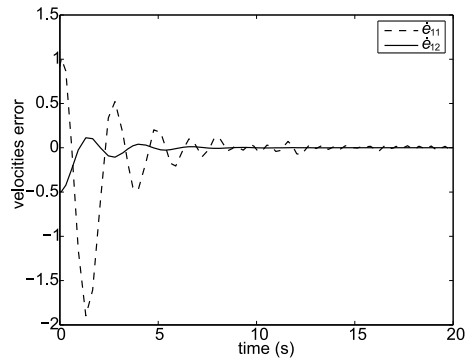


Fig. 3. The trajectories of  $\dot{e}_1(t)$ .

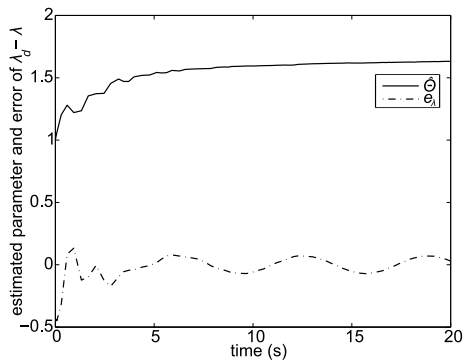


Fig. 4. The trajectories of  $\hat{\Theta}(t)$  and  $e_\lambda(t)$ .

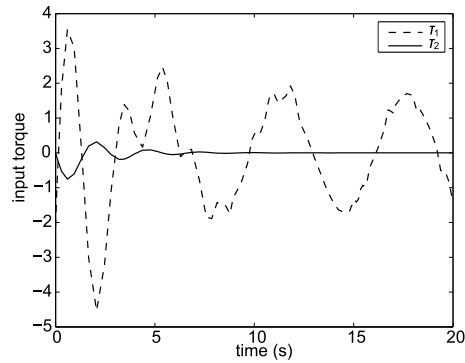


Fig. 5. The trajectories of  $\tau(t)$ .

where

$$\lambda_c = 5 - k_\lambda(\lambda - 5),$$

$$\mu(t) = (k_s + 1)e_2(t) - (k_s + 1)e_2(0) + \int_0^t ((k_s + 1)\alpha_2 e_2(s) + \hat{\Theta}(s) \operatorname{sgn}(e_2(s))) ds$$

with the adaptive update law

$$\dot{\hat{\Theta}}(t) = \frac{1}{\gamma} \operatorname{sgn}(e_2^T(t)) \rho(t),$$

where  $e_1 = [e_{11}, e_{12}]^T = [\sqrt{2} \sin t - z_1 + 2, \pi/4 - z_2]^T$ ,  $e_2 = [e_{21}, e_{22}]^T = [-\dot{z}_1 - z_1 + \sqrt{2} \sin t + \sqrt{2} \cos t + 2, -\dot{z}_2 - z_2 + \pi/4]^T$ ,  $\rho = [-\dot{z}_1 - 3z_1 - 2z_1 + \sqrt{2} \sin t + 3\sqrt{2} \cos t + 4, -\dot{z}_2 - 3z_2 - 2z_2 + \pi/2]^T$ .

In the simulation, suppose  $m_0 = 0.1$ ,  $I_0 = 0.1$ , chose  $\alpha_1 = 1$ ,  $\alpha_2 = 2$ ,  $k_s = 1$ ,  $k_\lambda = 2$ ,  $\gamma = 10$ , and select  $z_1(0) = z_2(0) = 0.5$ ,  $\dot{z}_1(0) = \dot{z}_2(0) = 0.5$ ,  $\hat{\Theta}(0) = 1$ . The results of the simulation are shown in Figs. 2–5. Figure 2 shows the position tracking errors of  $z(t) - z_d(t)$  converge to zero, Fig. 3 shows the velocity tracking errors of  $\dot{z}(t) - \dot{z}_d(t)$  converge to zero, Fig. 4 shows both state  $\hat{\Theta}(t)$  and the tracking error of  $e_\lambda$  are

bounded. It can be seen that the control inputs shown in Fig. 5 are bounded. At the last of the simulation, we should explain why the given signal  $z_{2d}$  is a constant. In fact, according to transformation (28), we know  $z_2 = \theta$ , so the control torques ensure asymptotical tracking all the time in the unchanged yaw angle with the different velocity of flow. Hence, the practical simulation example confirms the validity of the proposed algorithm.

## 5 Conclusions

In this paper, the trajectory and force tracking problem is addressed for a class of uncertain nonholonomic mechanical systems. The controller guarantees that the configuration state of the system semi-global asymptotically tracks to the desired trajectory and the force tracking error is bounded with a controllable bound. A practical mechanical model is constructed to confirm the reasonability of assumptions and the effectiveness of the control scheme.

## References

1. Y.Q. Wu, B. Wang, G.D. Zong, Finite-time tracking controller design for nonholonomic systems with extended chained form, *IEEE Trans. Circuits Syst., II, Express Briefs*, **50**(11): 798–802, 2005.
2. Z.P. Jiang, H. Nijmeijer, A recursive technique for tracking control of nonholonomic systems in chained form, *IEEE Trans. Autom. Control*, **44**(2):265–279, 1999.
3. M. Markus, Robust tracking control of nonholonomic dynamic systems with application to the bi-steerable mobile robot, *Automatica*, **44**(10):2588–2592, 2008.
4. A.M. Bloch, R. Reyhanoglu, N.H. McClamroch, Control and stabilization of nonholonomic dynamic systems, *IEEE Trans. Autom. Control*, **37**(11):1746–1757, 1992.
5. X.Y. Zheng, Y.Q. Wu, Adaptive output feedback stabilization for nonholonomic systems with strong nonlinear drifts, *Nonlinear Anal., Theory Methods Appl.*, **70**(2):904–920, 2009.
6. W.J. Dong, W.L. Xu, W. Huo, Trajectory tracking control of dynamic nonholonomic systems with unknown dynamics, *Int. J. Robust Nonlinear Control*, **9**:905–922, 1999.
7. Y. Zhao, J.B. Yu, Y.Q. Wu, State-feedback stabilization for a class of more general high order stochastic nonholonomic systems, *Int. J. Adapt. Control Signal Process*, **25**:687–706, 2011.
8. T. Fukao, H. Nakagawa, N. Adachi, Adaptive tracking control of a nonholonomic mobile robot, *IEEE Trans. Robot. Autom.*, **16**(5):609–615, 2000.
9. G.L. Ju, Y.Q. Wu, W.H. Sun, Adaptive output feedback asymptotic stabilization of nonholonomic systems with uncertainties, *Nonlinear Anal., Theory Methods Appl.*, **71**(11):5106–5117, 2009.
10. W.J. Dong, W.L. Xu, Adaptive tracking control of uncertain nonholonomic dynamic system, *IEEE Trans. Autom. Control*, **46**(3):450–454, 2001.
11. Z.J. Li, J. Luo, Adaptive robust dynamic balance and motion controls of mobile wheeled inverted pendulums, *IEEE Trans. Control Syst. Technol.*, **17**(1):233–241, 2009.

12. W.J. Dong, K.D. Kuhnert, Robust adaptive control of nonholonomic mobile robot with parameter and nonparameter uncertainties with parameter and nonparameter uncertainties, *IEEE Trans. Robot.*, **21**(2):261–266, 2005.
13. S.S. Ge, T.H. Lee, J. Wang, Adaptive impedance control of mechanical systems with classical nonholonomic constraints, in: *Proceedings of the 2001 IEEE International Conference on Control Applications (September 5–7, 2001, Mexico City, Mexico)*, 2001, pp. 960–965.
14. M. Oya, C.Y. Su, R. Katoh, Robust adaptive motion/force tracking control of uncertain nonholonomic mechanical systems, *IEEE Trans. Autom. Control*, **19**(1):175–181, 2003.
15. C.Y. Su, Y. Stepanenko, Robust motion/force control of mechanical systems with classical nonholonomic constraints, *IEEE Trans. Autom. Control*, **39**(3):609–614, 1994.
16. T. Kai, Modeling and passivity analysis of nonholonomic Hamiltonian systems with rheonomous affine constraints, *Commun. Nonlinear Sci. Numer. Simul.*, **17**(8):3446–3460, 2012.
17. T. Kai, H. Kimura, S. Hara, Nonlinear control analysis on nonholonomic dynamic systems with affine constraints, in: *Proceedings of the 44th IEEE Conference on Decision and Control (December 12–15, 2005, Seville, Spain)*, 2005, pp. 1459–1464.
18. T. Kai, Generalized canonical transformations and passivity-based control for nonholonomic Hamiltonian systems with affine constraints: control of a coin on a rotating table, in: *Proceedings of the 46th IEEE Conference on Decision and Control (December 12–14, 2007, New Orleans, LA, USA)*, 2007, pp. 3369–3374.
19. T. Kai, Mathematical modelling and theoretical analysis of nonholonomic kinematic systems with a class of rheonomous affine constraints, *Appl. Math. Modelling*, **36**(7):3189–3200, 2012.
20. Z.P. Wang, S.S. Ge, T.H. Lee, Robust motion/force control of uncertain holonomic/non-holonomic mechanical systems, *IEEE/ASME Trans. Mechatronics*, **9**(1):118–123, 2004.
21. Z.J. Li, Y.N. Zhang, Robust adaptive motion/force control for wheeled inverted pendulums, *Automatica*, **46**(8):1346–1353, 2010.
22. W.J. Dong, On trajectory and force tracking control of constrained mobile manipulators with parameter dynamic, *Automatica*, **38**(9):1475–1484, 2002.
23. Z.J. Li, S.S. Ge, M Adams, Adaptive robust output-feedback motion/force control of electrically driven nonholonomic mobile manipulators. *IEEE Trans. Control Syst. Technol.*, **16**(6):1308–1315, 2008.
24. B. Xian, D.M. Dawson, M.S. Queiroz, A continuous asymptotic tracking control strategy for uncertain multi-input nonlinear systems, *IEEE Trans. Autom. Control*, **49**(7):1206–1211, 2004.
25. C. Makkar, G. Hu, W.G. Sawyer, Lyapunov-based tracking control in the presence of uncertain nonlinear parameterizable friction, *IEEE Trans. Autom. Control*, **52**(10):1988–1994, 2007.
26. S.S. Ge, T.H. Lee, C.J. Harris, *Adaptive Neural Network Control of Robust Manipulators*, World Scientific, Singapore, 1998.
27. P.M. Patre, W. MacKunis, K. Dupree, W.E. Dixon, Modular Adaptive control of uncertain euler-lagrange systems with additive disturbances, *IEEE Trans. Autom. Control*, **56**(1):155–160, 2011.
28. H.K. Khalil, *Nonlinear Systems*, 3rd edition, Prentice Hall, New Jersey, 2002.