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Dirichlet type problem for the system of elliptic equations, which order degenerate at a line

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Abstract. Dirichlet type problem in a bounded domain for the system of linear elliptic equations of second order, which degenerate into first order system at a line crossing the domain, is studied. The existence and uniqueness of a solution of this problem in the Hölder class of functions are proved without any additional condition at line of degeneracy. The only requirement is that the solution is bounded in the considered domain.

Keywords: systems of elliptic equations, degenerating elliptic systems, Dirichlet type problem.

1 Introduction

The boundary value problems for elliptic equations including equations with degeneracy are closely related with non-local problems for mixed type equations, which are elliptic in a part of the considered domain [1]. In such case, there arise usually the question of well-posedness of boundary value problems in the elliptic part. To be more specific, it is very important to formulate properly the conditions for the solution on the part of a boundary, where treated equation change its type. It is well known that sometimes a part of the boundary on which elliptic equation has some degeneracy must be free from any boundary value condition in order to have well possed problem [2, 3]. It will be observed that usually there is stated the requirement for the solution of such problems to be bounded in considered domain.

This article treats of a boundary value problem for elliptic system of PDE, which is degenerate at a line crossing the domain. Specifically, the order of the considered system degenerate at this line. One can approach the degeneracy line as a part of the boundary of the domain in which this system is studied. The aim of the article is to consider the Dirichlet type problem when the degeneracy line must be free from boundary conditions except the boundedness condition. This problem is some generalization of the Dirichlet type problems for elliptic systems with degeneracy at an inner point of considered domain [4–7].

2 Statement of the problem

Let $D \in \mathbb{R}^{n+1}$, $n \geqslant 1$, $\partial D = \Gamma \in C^{2,\alpha}$, $0 < \alpha < 1$, be a bounded domain of points $x = (x_0, x')$, $x' = (x_1, \dots, x_n)$, containing the cylinder $C_R = \{(x_0, x')\colon |x'| < R, 0 < x_0 < h\}$, both bases of which lie on Γ. Thus, line $x' = 0$ is the axis of cylinder C_R and it crossing the domain D and intersecting with Γ by two points $O(0,0) \in \mathbb{R}^{n+1}$ and $P(h,0) \in \mathbb{R}^{n+1}$.

We consider the system of equations

$$\mathcal{L}u := \sum_{i,j=0}^{n} A_{ij}(x) u_{x_i x_j} + \sum_{i=0}^{n} B_i(x) u_{x_i} + C(x)u = F(x), \quad x \in D, \tag{1}$$

assuming that the matrix $A_{ij} = \operatorname{diag}(a_{ij1}^{(1)}, \dots, a_{ij}^{(m)})$, $B_i = \operatorname{diag}(b_{i1}^{(1)}, \dots, b_i^{(m)})$, $C = (c_{kl})$, $k, l = \overline{1,m}$, and right-hand side $F = (f_1, \dots, f_m)$ are bounded in D and $A_{ij} = A_{ji}$, $i, j = \overline{1,n}$.

Let $r = |x'|$. We shall use the following denotations:

$$D_\delta = D \setminus \{x\colon r \leqslant \delta\}, \quad \Gamma_\delta = \Gamma \setminus \{x\colon r \leqslant \delta\}, \quad \delta \in [0,R],$$
$$C_\rho = \{x\colon r < \rho,\ 0 < x_0 < h\}, \qquad C_\rho^0 = C_\rho \setminus \{r = 0\},$$
$$Q_\rho = \{x\colon r = \rho,\ 0 < x_0 < h\}.$$

Further, we denote by Q_ρ the lateral surface of cylinder C_ρ, by Ω and Ω_δ the projections of the respective domains D and D_δ onto the plane $x_0 = 0$, and by S the boundary of domain Ω. Let us note that, in such case, $D_0 = D \setminus \{x' = 0\}$, $\Omega_0 = \Omega \setminus \{x' = 0\}$ and $\Gamma_0 = \Gamma \setminus \{O \cup P\}$.

By $|\cdot|_{l;D}$ and $|\cdot|_{l,\alpha;D}$ we shall denote the norms in the corresponding Banach spaces $C^l(\overline{D})$ and $C^{l;\alpha}(\overline{D})$, where $l \geqslant 0$ is an integer.

We assume that following conditions are fulfilled:

1. There exist continuous in Ω functions a_i, $i = 1,2$, such that $0 < a_1(x') \leqslant a_2(x')$ in $\Omega_0 \cup S$, $\lim_{x' \to 0} a_2(x') = 0$, and the relations

$$a_1(x')|\xi|^2 \leqslant \sum_{i,j=0}^{n} a_{ij}^{(k)}(x)\xi_i\xi_j \leqslant a_2(x')|\xi|^2, \quad k = \overline{1,m}, \tag{2}$$

   hold everywhere in $\overline{D}$ for each $\xi = (\xi_0, \dots, \xi_n) \in \mathbb{R}^{n+1}$.

2. The inequalities

$$c_k(x) := c_{kk}(x) + \sum_{l \neq k} \bigl|c_{kl}(x)\bigr| < 0, \quad k = \overline{1,m}, \tag{3}$$

   hold for each $x \in D_0$.

Therefore, according to (2), system (1) is elliptic in D_0 and its order degenerate at the line $x' = 0$.

Let us introduce the class of vector-functions

$$C^{2,\alpha}_{\mathrm{loc}}(D) := \{u = (u_1, \dots, u_n)\colon\ u \in C^{2,\alpha}(D_\delta)\ \forall\delta > 0,\ |u| < \infty \text{ in } D_0\}.$$

We study the following Dirichlet type problem:

$$\mathcal{L}u = F \quad \text{in } D_0, \tag{4}$$

$$u|_{\varGamma_0} = g, \tag{5}$$

where $u \in C^{2,\alpha}_{\mathrm{loc}}(D_0)$ and $g = (g_1, \dots, g_m)$ is the given vector-function.

In the case where the operator $\mathcal{L}$ degenerates at line $x' = 0$ into ultraparabolic one, problem (4), (5) is discussed in [8, 9].

3 Auxilaries

Here we discuss the properties of operator $\mathcal{L}$ that will imply the uniqueness of the solution of problem (4), (5).

Lemma 1. *Let $D' \subset D$ be any subdomain lying outside of the line $x' = 0$, let $u = (u_1, \dots, u_m) \in C^2(D') \cap C(\overline{D'})$ be a solution of system* (1)*, and let there exists* $\sup_{D_0} |f_i/c_i|$ *for every $i = \overline{1,m}$. If condition* (3) *is fulfilled, then the estimate*

$$|u_j|_{0;D'} \leqslant \max_{1 \leqslant i \leqslant m} \left\{ \max_{\partial D'} |u_i|,\ \sup_{D_0} \left| \frac{f_i}{c_i} \right| \right\}, \quad j = \overline{1,m}, \tag{6}$$

holds.[1]

Proof. It is easily seen that $|u_j|_{0;D'} \geqslant \max_{\partial D'} |u_j|$, $j = \overline{1,m}$. If equality

$$|u_j|_{0;D'} = \max_{\partial D'} |u_j| \tag{7}$$

holds for each $j = \overline{1,m}$, then estimate (6) is evident.

Assume that some components u_j of solution u do not satisfy (7). Let, without a loose of generality, those are first m_0 ($m_0 \leqslant m$) components $u_1, \dots, u_{m_0}$ and let the rest components u_i, $i = \overline{m_0 + 1, m}$, satisfy (7). In such a case, all $|u_j|$, $j = \overline{1, m_0}$, attain its positive maximum at an inner point $x^j \in D'$, correspondingly. Denote by $\bar{u}$ the largest one of the number set $\{|u_j(x^j)|\}_{j=1}^{m_0}$. If

$$\bar{u} \leqslant \max_{\partial D'} |u_i|, \quad i = \overline{m_0 + 1, m}, \tag{8}$$

then according to the choice of $\bar{u}$ and due to the assumption

$$|u_j|_{0;D'} = \max_{\partial D'} |u_j|, \quad j = \overline{m_0 + 1, m}, \tag{9}$$

[1] If $m = 1$, then estimate (6) coinsides with well known maximum principle for the single elliptic equations [10, 11].

we get that

$$|u_j|_{0;D'} \leqslant \max_{m_0+1\leqslant i\leqslant m} |u_i|_{0;D'} = \max_{m_0+1\leqslant i\leqslant m} \Big\{ \max_{\partial D'} |u_i| \Big\} \\ \leqslant \max_{1\leqslant i\leqslant m} \Big\{ \max_{\partial D'} |u_i| \Big\}, \quad j = \overline{1, m_0}.$$

This jointly with (9) yields the inequality

$$|u_j|_{0;D'} \leqslant \max_{1\leqslant i\leqslant m} \Big\{ \max_{\partial D'} |u_i| \Big\}, \quad j = \overline{1, m}. \tag{10}$$

Therefore, estimate (6) under condition (8) holds.

Let us assume that $\bar{u}$ does not satisfy (8), i.e.

$$\bar{u} > \max_{\partial D'} |u_i|, \quad i = \overline{m_0 + 1, m}. \tag{11}$$

We shall show that then assumption (11) either implies the estimate

$$\bar{u} \leqslant \max_{1\leqslant i\leqslant m} \Big\{ \sup_{D_0} \Big| \frac{f_i}{c_i} \Big| \Big\} \tag{12}$$

or produces the contradiction to itself. The latter case will indicate that (11) is incorrect, i.e. condition (8) is valid. Therefore, there holds relation (10), which yields estimate (6).

Let the value $\bar{u}$ be attain by the kth component u_k of solution u, i.e. $\bar{u} = |u_k(x^k)|$, $1 \leqslant k \leqslant m_0$. Note that

$$\big|u_k\big(x^k\big)\big| \geqslant \big|u_l\big(x^l\big)\big| \geqslant \big|u_l(x)\big| \quad \forall x \in D',\ l = \overline{1, m_0},$$

because of the choice of $\bar{u}$ and

$$\big|u_k\big(x^k\big)\big| \geqslant \big|u_l(x)\big| \quad \forall x \in D',\ l = \overline{m_0 + 1, m},$$

in accordance with assumption (11), i.e.

$$\bar{u} = \big|u_k\big(x^k\big)\big| \geqslant \big|u_l(x)\big|, \quad l = \overline{1, m}, \tag{13}$$

everywhere in D' and

$$\bar{u} = \big|u_k\big(x^k\big)\big| \geqslant \big|u_l\big(x^k\big)\big|, \quad l = \overline{1, m}, \tag{14}$$

particularly. Moreover, according to (13), it follows the inequality

$$|u_j|_{0;D'} \leqslant \bar{u} \tag{15}$$

holding for each $j = \overline{1, m}$.

Let $\bar{u} = u_k(x^k) > 0$. Then x^k is the maximum point of function u_k, consequently,

$$\frac{\partial u_k}{\partial x_i}\bigg|_{x=x^k} = 0, \quad i = \overline{1, n}.$$

Moreover, according to condition

$$\sum_{i,j=0}^{n} a_{ij}^{(k)}(x)\xi_i\xi_j > 0, \quad x \in D' \subset D_0,$$

(see (2)) the inequality

$$\sum_{i,j=0}^{n} a_{ij}^{(k)}(x)\frac{\partial^2 u_k}{\partial x_i \partial x_j}\bigg|_{x=x^k} \leqslant 0$$

holds. Therefore, it follows from kth equation of system (1) the inequality

$$u_k\big(x^k\big)c_{kk}\big(x^k\big) + \sum_{l\neq k} u_l\big(x^k\big)c_{kl}\big(x^k\big) \geqslant f_k\big(x^k\big). \tag{16}$$

Thereby, in view of both (3) and (14), we get that

$$\begin{aligned} &u_k\big(x^k\big)c_{kk}\big(x^k\big) + \sum_{l\neq k} u_l\big(x^k\big)c_{kl}\big(x^k\big) \\ &\quad \leqslant u_k\big(x^k\big)c_{kk}\big(x^k\big) + \sum_{l\neq k} \big|u_l\big(x^k\big)\big|\big|c_{kl}\big(x^k\big)\big| \leqslant u_k\big(x^k\big)\Big(c_{kk}\big(x^k\big) + \sum_{l\neq k}\big|c_{kl}\big(x^k\big)\big|\Big) \\ &\quad = \bar{u}c_k\big(x^k\big). \end{aligned}$$

Hence, taking into account (16), we obtain that

$$\bar{u}c_k\big(x^k\big) \geqslant f_k\big(x^k\big). \tag{17}$$

If $f_k(x^k) \geqslant 0$, then this inequality does not hold because of condition (3), i.e. we get above-mentioned contradiction. So, in this case, estimate (6) holds.

If $f_k(x^k) < 0$, then we obtain from (17) that

$$\bar{u} \leqslant \frac{f_k(x^k)}{c_k(x^k)} \leqslant \sup_{x\in D_0}\left|\frac{f_k(x)}{c_k(x)}\right| \leqslant \max_{1\leqslant i\leqslant m}\left\{\sup_{x\in D_0}\left|\frac{f_i(x)}{c_i(x)}\right|\right\}.$$

This jointly with (15) yields the estimate

$$|u_j|_{0;D'} \leqslant \max_{1\leqslant i\leqslant m}\left\{\sup_{x\in D_0}\left|\frac{f_i(x)}{c_i(x)}\right|\right\}, \quad j=\overline{1,m}, \tag{18}$$

hence, estimate (6), too.

Let $\bar{u} = u_k(x^k) < 0$. Then x^k is the minimum point of function u_k. In this case, we get by repeating of the above-made steps the inequality

$$\bar{u} \geqslant -\frac{f_k(x^k)}{c_k(x^k)}$$

instead of (17). If $f_k(x^k) > 0$, then this inequality implies (18). If $f_k(x^k) \leqslant 0$, then there holds (10) doe to contradiction. Both (10) and (18) yield estimate (6), evidently. □

Let $u = (u_1, \dots, u_m) \in C^2(D') \cap C(\overline{D'})$ be a solution of Dirichlet problem

$$\mathcal{L}u = F \quad \text{in } D', \qquad u|_{\partial D'} = g, \tag{19}$$

where function $g = (g_1, \dots, g_m)$ is continuous on $\partial D'$. If condition (3) is fulfilled, then we get from Lemma 1 the estimate

$$|u_j|_{0;D'} \leqslant \max_{1\leqslant i\leqslant m} \left\{ \max_{\partial D'} |g_i|, \sup_{D_0} \left| \frac{f_i}{c_i} \right| \right\}, \quad j = \overline{1, m}.$$

Hence, $u \equiv 0$ in D' if $F \equiv 0$ in D' and $g \equiv 0$ on $\partial D'$. Thus, Lemma 1 implies the following corollary.

Corollary 1. *If condition* (3) *holds, then the solution of Dirichlet problem* (19) *is unique in the class* $C^2(D') \cap C(\overline{D'})$.

Introduce the operator

$$\mathcal{L}_0^{(k)} := \sum_{i,j=1}^{n} a_{ij}^{(k)}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^{(k)}(x) \frac{\partial}{\partial x_i}.$$

Lemma 2. *Let* $u = (u_1, \dots, u_m) \in C^2(D_0) \cap C(D_0 \cup \Gamma_0)$ *be a solution of equation* $\mathcal{L}u = 0$ *and let there exists a positive in* $\Omega_0 \cup S$ *function* ω *satisfying the following conditions*:

$$\omega(x') \to +\infty \quad \textit{uniformly as } x' \to 0, \tag{20}$$

$$\left(\mathcal{L}_0^{(k)} + c_k(x)\right)\omega(x') < 0 \quad \textit{in } D_0,\ k = \overline{1, m}. \tag{21}$$

If solution u *is uniformly bounded in* D_0*, equal to zero on* Γ_0 *and there holds condition* (3)*, then* $u \equiv 0$ *in* D_0.

Proof. Introduce the vector-function $v = (v_1, \dots, v_n)$ by the formula

$$u(x) = \omega(x')v(x). \tag{22}$$

Let $\varepsilon > 0$ be arbitrary. Since u is uniformly bounded in D_0, due to (20), there exits cylinder C_{r_ε} such that

$$|v_i| = \omega^{-1}|u_i| < \varepsilon \quad \text{in } C_{r_\varepsilon}^0 \cup \Gamma_{r_\varepsilon},\ i = \overline{1, m}. \tag{23}$$

We shall show that those inequalities hold also in D_{r_ε}.

Putting (22) into equation $\mathcal{L}u = 0$, we obtain that v satisfies the system of equations

$$\tilde{\mathcal{L}}v := \sum_{i,j=0}^{n} A_{ij}(x) v_{x_i x_j} + \sum_{i=0}^{n} \tilde{B}_i(x) v_{x_i} + \tilde{C}(x) v = 0, \tag{24}$$

where matrices $\tilde{B}_i = \mathrm{diag}(\tilde{b}_i^{(1)}, \dots, \tilde{b}_i^{(m)})$ and $\tilde{C} = (\tilde{c}_{kl})$, $k, l = \overline{1,m}$, are defined by

$$\tilde{b}_i^{(k)}(x) = b_i^{(k)}(x) + 2\sum_{j=1}^{n} a_{ij}^{(k)}(x)\omega_{x_j}(x'), \quad k = \overline{1,m},$$

and by

$$\tilde{c}_{kl}(x) = \begin{cases} \omega^{-1}(x')\mathcal{L}_0^{(k)}\omega(x') + c_{kk}(x) & \text{if } k = l, \\ c_{kl}(x) & \text{if } k \neq l, \end{cases}$$

correspondingly.

So, we have

$$\begin{aligned} \tilde{c}_k(x) &:= \tilde{c}_{kk}(x) + \sum_{l\neq k} \left|\tilde{c}_{kl}(x)\right| \\ &= \omega^{-1}(x')\mathcal{L}_0^{(k)}\omega(x') + c_k(x) < 0 \quad \text{in } D_0,\ k = \overline{1,m}, \end{aligned}$$

because of (21). Therefore, in accordance with Lemma 1, we get for solution $v = (v_1, \dots, v_n)$ of equation (23) the estimate

$$|v_j|_{0;D_{r_\varepsilon}} \leqslant \max_{1\leqslant i\leqslant m}\Big\{\max_{\partial D_{r_\varepsilon}} |v_i|\Big\}, \quad j = \overline{1,m}, \tag{25}$$

where $\partial D_{r_\varepsilon} = \Gamma_{r_\varepsilon} \cup Q_{r_\varepsilon}$. Note that $v|_{\Gamma_{r_\varepsilon}} = 0$ due to assumption of this lemma and due to relation (22), and $|v_i| < \varepsilon$, $i = \overline{1,m}$, on Q_{r_ε} in view of (23), i.e. $\max_{\partial D_{r_\varepsilon}} |v_i| < \varepsilon$, $i = \overline{1,m}$. Therefore, inequalities $|v_i| < \varepsilon$, $i = \overline{1,m}$, hold in D_{r_ε} because of (25). That jointly with (23) implies inequality $|u_i| < \omega^{-1}\varepsilon$, $i = \overline{1,m}$, everywhere in D_0. Consequently, $u \equiv 0$ in D_0, because ε is arbitrarily chosen. □

Let us define the function $\omega(x')$ by

$$\omega(x') = K - \ln r, \quad K = \mathrm{const},$$

assuming that $K > \mathrm{e}^d$, where $d = \max_{x\in\overline{D}} r$. Obviously, then $\omega(x') > 0$ for all $x' \in \Omega_0 \cup S$ and $\omega(x') \to +\infty$, $k = \overline{1,m}$, uniformly as $x' \to 0$, i.e. condition (20) is fulfilled.

We shall indicate in Lemma 3 the sufficient conditions for operator $\mathcal{L}$, under those the defined above function $\omega(x')$ satisfy condition (21).

Lemma 3. *Let there exist*

$$\sup_{D_0} c_k = -\varkappa < 0, \quad k = \overline{1,m},$$

and let one of following conditions be fulfilled:

(a) $a_2(x') = O(r^\mu)$ *in* Ω*, where* μ *is any positive number, and there exist a number* ν*,* $0 \leqslant \nu < \mu$*, and cylinder* $C_\rho^0 \subset D_0$ *such that*

$$\inf_{C_\rho^0} r^{-\nu}\sum_{i=1}^{n} x_i b_i^{(k)}(x) > 0, \quad k = \overline{1,m},$$

for some $\rho > 0$;

(b) $a_2(x') = O(r^\mu)$ *in* Ω *with any* $\mu \geqslant 2$, $b_i^{(k)}(x) = O(r^\nu)$, $k = \overline{1,m}$, *uniformly in* D *with any* $\nu \geqslant 1$.

If K *is large enough, then the function* $\omega(x') = K - \ln r$ *satisfies in* D_0 *condition* (20).

Proof. By direct calculation we obtain that

$$\mathcal{L}_0^{(k)}\omega(x') = r^{-2}\left(2r^{-2}\sum_{i,j=1}^{n} a_{ij}^{(k)}(x)x_i x_j - \sum_{i=1}^{n}\bigl(a_{ii}^{(k)}(x) + x_i b_i^{(k)}(x)\bigr)\right).$$

Note that, in view of (2), the inequalities

$$\sum_{i,j=1}^{n} a_{ij}^{(k)}(x)x_i x_j \geqslant a_2\bigl(x'\bigr)r^2, \quad \sum_{i=1}^{n} a_{ii}^{(k)}(x) > 0, \quad k = \overline{1,m},$$

are valid for each $x \in D_0$. Taking those in account, we get that

$$\mathcal{L}_0^{(k)}\omega\bigl(x'\bigr) < r^{-2}\psi_k(x) \quad \forall x \in D_0,\ k = \overline{1,m}, \tag{26}$$

where

$$\psi_k(x) = 2a_2(x') - \sum_{i=1}^{n} x_i b_i^{(k)}(x).$$

Besides that, we have according to assumption of lemma that

$$c_k(x) \leqslant -\varkappa \quad \text{in } D_0,\ k = \overline{1,m}.$$

Let condition (a) be fulfilled and let

$$\inf_{C_\rho^0} r^{-\nu}\sum_{i=1}^{n} x_i b_i^{(k)}(x) = \beta_k, \quad k = \overline{1,m}.$$

Obviously, then the inequalities

$$\psi_k(x) \leqslant -\beta_k + 2r^{-\nu}a_2(x'), \quad k = \overline{1,m},$$

hold for every $x \in C_\rho^0$. Since $a_2(x') = O(r^\mu)$ in Ω and $\mu > \nu$, we obtain that functions $\psi_k(x)$, $k = \overline{1,m}$, are negative in some cylinder $C_{r_0}^0 \subseteq C_\rho^0$ with small enough r_0, because numbers β_k, $k = \overline{1,m}$, are positive according to assumption of this lemma. Let

$$\overline{\psi}_0 = \max_{1\leqslant k\leqslant m}\Bigl\{\sup_{\overline{D}_{r_0}} |\psi_k|\Bigr\}.$$

Then it follows from (25) that

$$\mathcal{L}_0^{(k)}\omega(x') \leqslant r_0^{-2}\overline{\psi}_0 \quad \text{in } D_0,\ k = \overline{1,m}.$$

Thus, taking in account (27), we get that

$$\big(\mathcal{L}_0^{(k)} + c_k(x)\big)\omega(x') \leqslant r_0^{-2}\overline{\psi}_0 - \varkappa(K - \ln r) < 0$$

in D_0 if $K > \varkappa^{-1} r_0^{-2}\overline{\psi}_0 + \ln d$.

Now assume there holds conditions (b). Let $\gamma = \min\{\mu, \nu\}$. In such a case, the relations

$$\psi_k(x) = O\big(r^{\gamma+2}\big), \quad k = \overline{1,m},$$

hold uniformly in D_0. Hence, the functions $r^{-2}\psi_k(x)$, $k = \overline{1,m}$, are uniformly bounded in D_0. Let us note that

$$\big(\mathcal{L}_0^{(k)} + c_k(x)\big)\omega(x') \leqslant r^{-2}\psi_k(x) + \varkappa(K - \ln r) \leqslant \kappa - \varkappa K + \varkappa \ln r,$$

where

$$\kappa = \max_{1 \leqslant k \leqslant m} \Big\{ \sup_{D_0} r^{-2}|\psi_k| \Big\}.$$

If $K > \varkappa^{-1}\kappa + \ln d$, then it follows from here that

$$\big(\mathcal{L}_0^{(k)} + c_k(x)\big)\omega(x') < 0 \quad \text{in } D_0.$$

Hence, if K is suitably chosen, either the assumption (a) or the assumption (b) imply inequality (21). □

4 The existence and uniqueness of the solution of problem (4), (5)

We shall prove the existence of the solution of problem (4), (5) in the class of functions $C^{2,\alpha}_{\mathrm{loc}}(D_0)$ defined above. Let us assume that

$$A_{ij}, B_i \quad (i,j = \overline{0,n}) \quad \text{and} \quad C \in C^{2,\alpha}_{\mathrm{loc}}(D), \quad F \in C^{0,\alpha}(\overline{D}), \quad g \in C^{2,\alpha}(\overline{D}). \quad (27)$$

(Without the loose of a generality, we suppose here that g is defined not only on $\varGamma$, but also in $\overline{D}$.)

Note that domain D_δ participating in definition of $C^{2,\alpha}_{\mathrm{loc}}(D)$ is not smooth, because it has two edges $\{r = \delta,\ x_0 = 0\}$ and $\{r = \delta,\ x_0 = h\}$, which are, in fact, the $(n-1)$-dimensional spheres.

Let us take the domain D^*_δ with the boundary $\varGamma^*_\delta \in C^{2,\alpha}$ such that $D_\delta \subset D^*_\delta \subset D_0$. Moreover, we chose D^*_δ so that a part of boundary $\varGamma^*_\delta$ coincide with surface $\varGamma_{2\delta}$ and lateral surface Q_δ of cylinder $\{r \leqslant \delta,\ \delta \leqslant x_0 \leqslant h - \delta\}$. The remaining part of $\varGamma^*_\delta$ lie in the cylinder C_δ. It consists from two surfaces $\sigma^{(1)}_\delta$ and $\sigma^{(2)}_\delta$: first of those joins the spheres $\{r = 2\delta,\ x_0 = 0\}$ and $\{r = \delta,\ x_0 = \delta\}$; the second one joins the spheres $\{r = 2\delta,\ x_0 = h\}$ and $\{r = \delta,\ x_0 = h - \delta\}$.

Let $\{\delta_k\}$ be a vanishing sequence of positive numbers δ_k, and let $\{D^*_{\delta_k}\}$ be the sequence of corresponding domains, which are constructed by the rule given above taking $\delta = \delta_k$. Observe that $A_{ij},\ B_i,\ i,j = \overline{0,n}$, and $C \in C^{2,\alpha}(\overline{D^*_{\delta_k}})$, $\partial D^*_{\delta_k} \in C^{2,\alpha}$, and, in accordance with (2), operator $\mathcal{L}$ is uniformly elliptic in each domain $D^*_{\delta_k}$, $k = 1, 2, \ldots$. That jointly with Corollary 1 yield the existence of the unique solution $u^k = (u^k_1, \ldots, u^k_m) \in C^{2,\alpha}(\overline{D^*_{\delta_k}})$ of Dirichlet problem [12–14]

$$\mathcal{L}u = F \quad \text{in } D^*_{\delta_k}, \qquad u|_{\partial D^*_{\delta_k}} = g.$$

Moreover, due to Lemma 1, there holds the estimate

$$|u^k_j|_{0;D^*_{\delta_k}} \leqslant M = \max_{1 \leqslant i \leqslant m} \left\{ |g_i|_{0;D}, \sup_{D_0} \left| \frac{f_i}{c_i} \right| \right\}, \quad j = \overline{1, m}. \tag{28}$$

Let us define the sequence $\{\tilde{u}^k\}$ of vector-functions $\tilde{u}^k = (\tilde{u}^k_1, \ldots, \tilde{u}^k_m)$ by

$$\tilde{u}^k_j(x) = \begin{cases} u^k_j(x) & \text{if } x \in \overline{D^*_{\delta_k}}, \\ g_j(x) & \text{if } x \in D \setminus \overline{D^*_{\delta_k}}. \end{cases}$$

It is easy to see that every term $\tilde{u}^k$ of this sequence is defined in $\overline{D}$ and $\tilde{u}^k \in C^{2,\alpha}(\overline{D})$, $k = 1, 2, \ldots$.

Lemma 4. *Let ε be arbitrary. There exists a subsequence of the sequence $\{\tilde{u}^k\}$ strongly convergent in the space $C^2(\overline{D}_\varepsilon)$.*

Proof. Let $\mathcal{L}\tilde{u}^k = \tilde{F}$, where $\tilde{F} = (\tilde{f}_1, \ldots, \tilde{f}_m)$. Note that $\tilde{F} = F$ if $\delta_k \geqslant \varepsilon$. Applying to operator $\mathcal{L}$ the a priori estimates inclusively to the part $\varGamma_\varepsilon$ of boundary ∂D_ε, we get [12–14] that

$$\sum_{j=1}^m |\tilde{u}^k_j|_{2,\alpha;D_\varepsilon} \leqslant N_\varepsilon \sum_{i=1}^m \bigl(|\tilde{f}_i|_{0,\alpha;D^*_\varepsilon} + \bigl|\tilde{u}^k_i\bigr|_{0,\alpha;D^*_\varepsilon} + |g_i|_{2,\alpha;\varGamma_\varepsilon} \bigr)$$

with a constant N_ε depending on ε.

Let

$$M_1 = \max\bigl\{ |\mathcal{L}g_j|_{0,\alpha;D}, |f_j|_{0,\alpha;D} \bigr\}, \qquad M_2 = |g_j|_{2,\alpha;D}.$$

Due to obvious estimates

$$\bigl|\mathcal{L}\tilde{u}^k_j\bigr|_{0,\alpha;D^*_\varepsilon} \leqslant M_1, \qquad \bigl|\tilde{u}^k_j\bigr|_{0,\alpha;D^*_\varepsilon} \leqslant M, \qquad |g_j|_{2,\alpha;D^*_\varepsilon} \leqslant M_2,$$

we obtain that

$$\bigl|\tilde{u}^k_j\bigr|_{2,\alpha;D_\varepsilon} \leqslant N_\varepsilon(M + M_1 + M_2).$$

Thus, the sequence $\{\tilde{u}^k\}$ is compact in $C^{2,\alpha}(\overline{D}_\varepsilon)$. This yields the existence of a subsequence strongly convergent in the space $C^2(\overline{D}_\varepsilon)$. □

Remark 1. Assume that some subsequence $\{\tilde{u}^{k_i}\} \subset \{\tilde{u}^k\}$ strongly converges in $C^2(\overline{D}_\varepsilon)$ to $u_\varepsilon = (u_{\varepsilon_1}, \ldots, u_{\varepsilon n})$ as $i \to \infty$. Then $\mathcal{L}u_\varepsilon = F$ in D_ε, $u|_{\varGamma_\varepsilon} = g$, evidently, and, in view of Lemma 1, there holds the estimate $|u_{\varepsilon j}|_{0;D_\varepsilon} \leqslant M$, $j = \overline{1, m}$. Moreover, $u_\varepsilon \in C^{2,\alpha}(\overline{D}_\varepsilon)$, since $\tilde{u}^{k_i} \in C^{2,\alpha}(\overline{D}_\varepsilon)$, $i = 1, 2, \ldots$, whereas the space $C^{2,\alpha}(\overline{D}_\varepsilon)$ is complete.

Using the diagonalization method, we shall show that one can choose the subsequence of sequence $\{\tilde{u}^k\}$, which converges to the solution of problem (4), (5).

Let $\{\varepsilon_j\}$ be the a vanishing sequence of positive numbers and let $\{D_{\varepsilon_j}\}$ be sequence of the the corresponding domains. According to Lemma 4, there exist the subsequences $\{\tilde{u}^{jk_i}\} \subset \{\tilde{u}^k\}$, $j = 1, 2, \ldots$, for each $i = 1, 2, \ldots$, which satisfy condition $\{\tilde{u}^{j+1\ k_i}\} \subset \{\tilde{u}^{jk_i}\}$ and strongly convergent in corresponding spaces $C^2(\overline{D}_{\varepsilon_j})$, $j = 1, 2, \ldots$. Let us consider the sequence $\{v^l\}$, where $v^l = \tilde{u}^{lk_l}$.

Theorem 1. *Let the smoothness conditions* (27) *be fulfilled and let the conditions of Lemma* 2 *be satisfied. Then the sequence* $\{v_k\}$ *determined above converges to the solution* u *of problem* (4)*,* (5)*. This solution is unique.*

Proof. Let δ be arbitrarily chosen and let j_0 be such that $\varepsilon_j \leqslant \delta$ for $j \geqslant j_0$. Then $D_\delta \subset D_{\varepsilon_{j_0}}$ and $v^l \in \{\tilde{u}^{j_0k_i}\}$ for all $l \geqslant j_0$, obviously. Thus, sequence $\{v^l\}$ strongly converges in the space $C^2(\overline{D}_{\varepsilon_{j_0}})$ to some limit u because of the choice rule of $\{\tilde{u}^{j_0k_i}\}$. Thereby, $v^k \to u$ strongly in $C^2(\overline{D}_\delta)$ as $k \to \infty$, since $C^2(\overline{D}_\delta) \subset C^2(\overline{D}_{\varepsilon j_0})$. Taking in account Remark 1, we obtain that $\mathcal{L}u = F$ in D_δ, $u|_{\Gamma_\delta} = g$ and $|u|_{0;D_\delta} \leqslant M$. Furthemore, $u \in C^{2,\alpha}(\overline{D}_\delta)$, because $C^{2,\alpha}(\overline{D}_\delta)$ is complete space. Since δ is arbitrary chosen, we get that $\mathcal{L}u = f$ in D_0, $u|_{\Gamma_0} = g$ and $|u| \leqslant M$ in D_0, i.e. u is the solution of problem (4), (5).

The uniqueness of solution u follows from Lemma 2. Indeed, if $u^{(1)}$ and $u^{(2)}$ are two solutions of problem (4), (5), then $(u^{(1)} - u^{(2)})|_{\Gamma_0} = 0$. Since operator $\mathcal{L}$ satisfies either condition (a) or condition (b) of Lemma 2, it follows from this lemma that $u^{(1)} - u^{(2)} \equiv 0$ in D_0. □

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