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On the stability of a weighted finite difference scheme for wave equation with nonlocal boundary conditions*

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Abstract. We consider the stability of a weighted finite difference scheme for a linear hyperbolic equation with nonlocal integral boundary condition. By studying the spectrum of the transition matrix of the three-layered difference scheme we obtain a sufficient stability condition in a special matrix norm.

Keywords: integral conditions, hyperbolic equation, weighted difference scheme, spectrum analysis.

1 Introduction

Nonlocal problems is a major research area in many branches of modern physics, biotechnology, chemistry and engineering, which arises when it is impossible to determine the boundary values of unknown function and its derivatives. Increasingly often, there arise problems with nonlocal integral boundary conditions, especially in particle diffusion [1] and heat conduction [2, 3]. Partial differential equations of the hyperbolic type with integral conditions often occur in problems related to fluid mechanics [4] (dynamics and elasticity), linear thermoelasticity [5], vibrations [6] etc.

Hyperbolic problems with nonlocal conditions have not been studied so broadly as, say, parabolic or elliptic problems. The paper [7] dealt with the new technique (Adomian Decomposition Method) for solving wave equation with integral boundary conditions.

Author consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < 1,\ 0 < t \leqslant T,$$

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subject to initial condition

$$u(x,0) = r(x), \quad u_t(x,0) = s(x), \quad 0 \leqslant x \leqslant 1,$$

and the nonlocal boundary conditions

$$u(0,t) = p(t), \quad \int\limits_0^1 u(x,t)\,\mathrm{d}x = q(t), \quad 0 < t \leqslant T,$$

where r, s, p and q are known functions, f is sufficiently smooth to produce a smooth classical solution.

The author applied the Adomian decomposition method for the solution of the wave equation. This algorithm is simple and easy to implement. The obtained results confirmed a good accuracy of the method and the calculations are simpler and faster than traditional techniques.

The stability of difference schemes for the nonlocal hyperbolic problems was studied in [8]. Multidimensional hyperbolic equation with Dirichlet condition is considered

$$\begin{aligned}
&\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{m} \bigl(a_r(x) u_{x_r}\bigr)_{x_r} = f(t,x), \quad x = (x_1,\dots,x_m) \in \varOmega,\ 0 < t < 1,\\
&u(0,x) = \sum_{j=1}^{n} \alpha_j u(\lambda_j, x) + \phi(x), \quad u_t(0,x) = \sum_{k=1}^{n} \beta_k u_t(\lambda_k, x) + \psi(x), \quad x \in \overline{\varOmega},\\
&u(t,x) = 0, \quad x \in S,
\end{aligned}$$

here $\varOmega := \{x = (x_1,\dots,x_m)\colon\ 0 < x_j < 1,\ 1 \leqslant j \leqslant m\}$ is the open unit cube in the m-dimensional Euclidean space $\mathbb{R}^m$ with boundary S, $\overline{\varOmega} = \varOmega \cup S$. Stability conditions in a special norm $\|\cdot\|_{L_{2h}}$ were obtained and numerical analysis was made.

The spectrum and characteristic functions for eigenvalues of Sturm–Liouville problem are widely investigated in [9–12]. For example, the following problems

$$-u'' = \lambda u, \quad t \in (0,1),$$

with one classical boundary condition $u(0) = 0$ and other nonlocal boundary condition

$$u(1) = \gamma u(\xi) \quad \text{or} \quad u(1) = \gamma \int\limits_0^\xi u(t)\,\mathrm{d}t \quad \text{or} \quad u(1) = \gamma \int\limits_\xi^1 u(t)\,\mathrm{d}t,$$

where $\gamma \in \mathbb{R}$, $0 < \xi < 1$, were investigated. There were found eigenvalues of two types: the first type eigenvalues do not depending on γ, and the second type eigenvalues which do not depend on γ and exist only for some rational ξ. The authors introduced a method of generalized characteristic functions [13]. The complex eigenvalues exist for these problems. Complex eigenvalues of these Sturm–Liouvile problems (in the case integral boundary condition) were investigated in [14, 15].

In the present paper, we give a sufficient condition for the stability of a weighted difference scheme for a hyperbolic equation with nonlocal integral boundary conditions. By using a method applied earlier to explicit difference scheme for hyperbolic equations with nonlocal boundary conditions [16], we rewrite a three-layer difference scheme in the form of an equivalent two-layer scheme. By analyzing the spectrum of the transition matrix of the two-layer scheme, we obtain sufficient conditions for the stability of the three-layer scheme depending on the parameters occurring in the integral boundary conditions and not depending on the weight parameter used in scheme. We have generalized and clarified the results presented in [16]. We note that for the investigated problem all eigenvalues are real.

To obtain the stability estimates of a difference nonlocal hyperbolic problem, we use a weighted three-layer difference scheme and approximate the nonlocal integral conditions by the trapezoid quadrature formula. By representing this scheme in the form of a second-order operator-difference equation and by using some transformations, one can obtain a two-layer scheme equivalent to this three-layer scheme [17, p. 364]. To study the spectrum of the transition matrix of the two-layer scheme, we define special norms of matrices and vectors. The analysis of the structure of the spectrum of the transition matrix [18] and the use of a generalized nonlinear eigenvalue problem permit one to state the main result of the present paper, a sufficient condition for the stability of a weighted difference scheme for hyperbolic equations with integral boundary conditions.

2 A weighted finite difference scheme for nonlocal hyperbolic problem

2.1 Differential problem with integral conditions

Consider the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1}$$

where $\Omega = (0, L)$, with the classical initial conditions

$$u|_{t=0} = \phi(x), \quad x \in \overline{\Omega} := [0, L], \tag{2}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x), \quad x \in \overline{\Omega}, \tag{3}$$

and the additional nonlocal integral boundary conditions

$$u(0,t) = \gamma_0 \int_0^L u(x,t)\,\mathrm{d}x + v_l(t), \quad t \in [0,T], \tag{4}$$

$$u(1,t) = \gamma_1 \int_0^L u(x,t)\,\mathrm{d}x + v_r(t), \quad t \in [0,T], \tag{5}$$

where $f(x,t)$, $\phi(x)$, $\psi(x)$, $v_l(t)$, and $v_r(t)$ are given functions, and γ_0 and γ_1 are given real parameters. We are interested in sufficiently smooth solutions of the nonlocal problem (1)–(5). Later we use notation $\gamma := \gamma_0 + \gamma_1$.

2.2 Notations

We introduce grids

$$\begin{aligned}
\overline{\omega}^h &:= \{x_i\colon x_i = ih,\ i = \overline{0,N}\}; \quad h = \frac{L}{N},\\
\overline{\omega}^h_{1/2} &:= \left\{x_{i-1/2} = \frac{x_{i-1}+x_i}{2},\ i = \overline{1,M},\ x_{-1/2} = x_0,\ x_{M+1/2} = x_M\right\};\\
&\quad h_{i+1/2} = x_{i+1/2} - x_{i-1/2}, \quad i = \overline{0,M},\\
\overline{\omega}^\tau &:= \{t^j\colon t^j = j\tau,\ j = \overline{0,M}\}; \quad \tau = \frac{T}{M},\\
\omega^h &:= \{x_1,\dots,x_{N-1}\}, \quad \widetilde{\omega}^\tau := \{t^1,\dots,t^M\}, \quad \omega^\tau := \{t^1,\dots,t^{M-1}\},
\end{aligned}$$

where $N+1$ and $M+1$ are the numbers of grid points for x and t directions, accordingly, and $N, M \geqslant 2$.

We use the notation $U_i^j := U(x_i, t^j)$ for the function defined on the grid (or parts of the grid) $\overline{\omega}^h \times \overline{\omega}^\tau$. Instead of writing indices, we denote $\check{U}^j := U^{j-1}$ and $\widehat{U}^j := U^{j+1}$ on grids $\widetilde{\omega}^\tau$ and $\omega^\tau \cup \{t_0\}$, respectively. Later in this paper, we use the following notations: $U^{(\sigma)} = \sigma\check{U} + (1-2\sigma)U + \sigma\widehat{U}$, $\sigma \in \mathbb{R}$. We define a space grid operator

$$\delta_x^2 : \overline{\omega}^h \to \omega^h, \quad \left(\delta_x^2 U\right)_i := \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2},$$

and the time grid operators

$$\begin{aligned}
\overline{\partial}_t &: \overline{\omega}^\tau \to \widetilde{\omega}^\tau, \quad \overline{\partial}_t U := \frac{U - \check{U}}{\tau},\\
\overline{\partial}_t^2 &: \overline{\omega}^\tau \to \omega^\tau, \quad \overline{\partial}_t^2 U := \frac{\check{U} - 2U + \widehat{U}}{\tau^2}.
\end{aligned}$$

Let $\overline{H}$ and H be a spaces of grid functions on $\overline{\omega}^h$ and ω^h, respectively.
We define the inner products

$$\begin{aligned}
[U,V] &:= \sum_{i=0}^{N} U_i V_i h_{i+1/2}, \quad U, V \in \overline{H},\\
(U,V) &:= \sum_{i=1}^{N-1} U_i V_i h, \quad U, V \in H.
\end{aligned}$$

We can investigate problem (1)–(5) in the interval $[0,1]$ instead of $[0,L]$ using transformation $x = Lx'$. Then new $c' = c/L$. Further we consider $c' = 1$ for simplicity.

We use inner products for functions $\mathrm{e}^{\pm \imath z x_i}$, $z \in \mathbb{C}$, $x \in \overline{\omega}^h$:

$$\big[1, \mathrm{e}^{\pm \imath z x}\big] = \big[1, \mathrm{e}^{\pm \imath z(1-x)}\big] = h \sin\frac{z}{2} \tan^{-1}\frac{zh}{2}\,\mathrm{e}^{\pm \imath z/2}.$$

As a result, we obtain formulas

$$\big[1, \sin\big(z(1-x)\big)\big] = \big[1, \sin(zx)\big] = h\frac{\sin^2(z/2)\cos(zh/2)}{\sin(zh/2)}, \tag{6}$$

and, using the fact that trapezoid formula is exact for linear polynomials, we also have

$$[1,1] = 1, \qquad [1,x] = \frac{1}{2}. \tag{7}$$

Later we also use some inner products with discrete functions $U_i = (-1)^i$ and $U_i = (-1)^i x_i$:

$$\big[1, (-1)^i\big] = 0, \qquad \big[1, (-1)^i x_i\big] = \frac{1}{4}h^2\big((-1)^N - 1\big). \tag{8}$$

We use the following vector notation: $\mathbf{U} = (U_1, U_2, \ldots, U_{N-1})^\intercal$. Let $\mathbf{P}$ be a nonsingular matrix ($\det \mathbf{P} \neq 0$); we define the norm of any $m \times m$ matrix $\mathbf{M}$ as follows: $\|\mathbf{M}\|_* = \|\mathbf{P}^{-1}\mathbf{M}\mathbf{P}\|_2$, where $\|\mathbf{M}\|_2 = (\max_{1\leqslant i\leqslant m} \lambda_i(\mathbf{M}^*\mathbf{M}))^{1/2}$ is the classical matrix norm and $\mathbf{M}^*$ is the adjoint matrix. We define the associated vector norm by the formula

$$\|\mathbf{V}\|_* = \big\|\mathbf{P}^{-1}\mathbf{V}\big\|_2 = \left(\sum_{i=1}^m |\widetilde{\mathbf{V}}_i|^2\right)^{1/2}, \tag{9}$$

where the $\widetilde{\mathbf{V}}_i$ are the coordinates of the vector $\mathbf{P}^{-1}\mathbf{V}$.

If all eigenvectors $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_m$ of any nonsymmetric $m\times m$ matrix $\mathbf{S}$ are linearly independent, then we form the nonsingular matrix $\mathbf{T} = (\mathbf{V}_1, \ldots, \mathbf{V}_m)$. We have the relation

$$\|\mathbf{S}\|_* = \big\|\mathbf{T}^{-1}\mathbf{S}\mathbf{T}\big\|_2 = \|\mathbf{J}\|_2 = \max_{1\leqslant i\leqslant m}\big\|\mu_i(\mathbf{S})\big\| = \varrho(\mathbf{S}), \tag{10}$$

where $\mathbf{J} = \operatorname{diag}(\mu_1, \ldots, \mu_m)$, μ_i, $i = \overline{1,m}$, are the eigenvalues of $\mathbf{S}$ and $\varrho(\mathbf{S})$ is its spectral radius.

The vector norm associated with the matrix norm (10) is defined by identity (9) with $i = \overline{1,m}$. We use the theorems proved in [19, p. 168] and define the norms of matrices and vectors to be used in the stability analysis of the difference scheme.

2.3 Three-layer finite difference scheme

Now we state a difference analogue of the differential problem (1)–(5). We define a weighted finite difference scheme (FDS) approximating the original differential equation (1):

$$\overline{\partial}_t^2 U - \delta_x^2 U^{(\sigma)} = F, \quad (x_i, t_j) \in \omega^h \times \omega^\tau, \tag{11}$$

where σ is a weight parameter. The initial conditions are approximated as follows:

$$U^0 = \Phi, \quad x_i \in \overline{\omega}^h, \tag{12}$$

$$\overline{\partial}_t U^1 = \Psi, \quad x_i \in \overline{\omega}^h. \tag{13}$$

We rewrite the boundary conditions using the defined inner product:

$$U_0 = \gamma_0[1, U] + V_l, \quad t^j \in \widetilde{\omega}^\tau \backslash \{t^1\}, \tag{14}$$

$$U_N = \gamma_1[1, U] + V_r, \quad t^j \in \widetilde{\omega}^\tau \backslash \{t^1\}. \tag{15}$$

In problem (11)–(15), we approximate functions f, ϕ, ψ, v_l, and v_r by grid functions F, Φ, Ψ, V_l, and V_r.

Remark 1. Properly choosing right hand-side functions in (11)–(15), one can obtain required approximation accuracy. For example, if $\Psi = \psi + 0.5\tau(\delta_x^2 U^0 + f^0)$, the differential problem (1)–(5) is approximated by (11)–(15) with accuracy $\mathcal{O}(\tau^2 + h^2)$.

2.4 Equivalence of the three-layer scheme to a two-layer scheme

Equations (14)–(15) is a system of two linear equations for unknowns U_0 and U_N. We express these unknowns via inner points U_i, $i = \overline{1, N-1}$, and obtain

$$U_0 = \widetilde{\gamma}_0(1, U) + \widetilde{V}_l, \tag{16}$$

$$U_N = \widetilde{\gamma}_1(1, U) + \widetilde{V}_r, \tag{17}$$

where $\widetilde{\gamma}_0 = \gamma_0 d^{-1}$, $\widetilde{\gamma}_1 = \gamma_1 d^{-1}$, $d = 1 - h\gamma/2 > 0$; $\widetilde{V}_l = (V_l + hc)d^{-1}$, $\widetilde{V}_r = (V_r - hc)d^{-1}$, $c = (\gamma_0 V_r - \gamma_1 V_l)/2$. By substituting expressions (16) and (17) into Eq. (11) for $i = 1$ and $i = N - 1$, we rewrite it in the form

$$\mathbf{A}\widehat{\mathbf{U}} + \mathbf{B}\mathbf{U} + \mathbf{C}\check{\mathbf{U}} = \tau^2 \mathbf{F}, \tag{18}$$

$$\mathbf{A} = \mathbf{C} = \mathbf{I} + \tau^2\sigma\mathbf{\Lambda}, \qquad \mathbf{B} = -2\mathbf{I} + \tau^2(1 - 2\sigma)\mathbf{\Lambda}, \tag{19}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and

$$\mathbf{\Lambda} = \frac{1}{h^2}\begin{pmatrix} 2 - \widetilde{\gamma}_0 h & -1 - \widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h & \dots & -\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -\widetilde{\gamma}_1 h & -\widetilde{\gamma}_1 h & -\widetilde{\gamma}_1 h & \dots & -\widetilde{\gamma}_1 h & -1 - \widetilde{\gamma}_1 h & 2 - \widetilde{\gamma}_1 h \end{pmatrix} \tag{20}$$

are $(N-1) \times (N-1)$ matrices, $\mathbf{I}$ is the identity matrix, $\mathbf{0}$ is a zero matrix. Finally, $\mathbf{F} = (\widetilde{F}_1, \dots, \widetilde{F}_{N-1})^\intercal$, where $\widetilde{F}_i = F_i$, $i = \overline{2, N-2}$, and $\widetilde{F}_i = \widetilde{F}_i(F_i, V_l, V_r)$, $i = 1, N-1$.

Remark 2. We suppose that all eigenvalues of matrix $\mathbf{\Lambda}$ are real. In this case, $\det \mathbf{A} > 0$ if the following condition is satisfied:

$$-\frac{1}{\tau^2 \max(0, \lambda_{\max})} < \sigma < -\frac{1}{\tau^2 \min(0, \lambda_{\min})}.$$

Matrix $\mathbf{A}^{-1}$ exists for such σ.

We represent the three-layer scheme (18) as an equivalent two-layer scheme (e.g., see [17])

$$\widehat{\mathbf{W}} = \mathbf{SW} + \mathbf{G}, \tag{21}$$

using notations

$$\mathbf{W} = \begin{pmatrix} \mathbf{U} \\ \check{\mathbf{U}} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \tau^2 \mathbf{A}^{-1}\mathbf{F} \\ \mathbf{0} \end{pmatrix}. \tag{22}$$

According to [20, 21], one can study the stability conditions for the two-layer difference scheme (21) by analyzing the spectrum of the matrix $\mathbf{S}$. Note that the matrices $\mathbf{S}$ and $\mathbf{\Lambda}$ are nonsymmetric (matrix $\mathbf{\Lambda}$ is nonsymmetric except the classical case $\gamma_1 = 0$ and $\gamma_2 = 0$).

3 The structure of the spectrum of the matrix $\mathbf{\Lambda}$

Eigenvalue problem

$$\mathbf{\Lambda U} = \lambda \mathbf{U}$$

for $(N-1) \times (N-1)$ matrix $\mathbf{\Lambda}$ is in general equivalent to the eigenvalue problem for the difference operator with nonlocal boundary conditions

$$-\delta_x^2 U = \lambda U, \quad U \in \omega^h, \tag{23}$$

$$U_0 = \gamma_0[1, U], \quad U_N = \gamma_1[1, U]. \tag{24}$$

Lemma 1. (*See* [22].) *For arbitrary values of the parameters $\gamma_0, \gamma_1 \in \mathbb{R}$, all eigenvalues λ of the matrix $\mathbf{\Lambda}$ are real and simple, moreover, the following assertions hold*:

1) *if $\gamma = \gamma_0 + \gamma_1 < 2$, then all eigenvalues are positive*;
2) *if $\gamma = 2$, then there exists one zero eigenvalue, other eigenvalues are positive*;
3) *if $2 < \gamma < 2/h$, then there exists one negative eigenvalue, all other are positive.*

Remark 3. Let us specify the additional properties of eigenvalues λ, which are not stated in Lemma 1. First, we enumerate all the eigenvalues $\lambda_1 < \cdots < \lambda_{N-1}$ of problem (23)–(24) in the ascending order using the classical case $\gamma_0 = 0, \gamma_1 = 0$ (in this case, $\gamma = 0$). Now we formulate some additional properties of the eigenvalues:

1) all eigenvalues are simple and real;
2) if $\gamma < 2$, then $\lambda \in (0, 4/h^2]$;

3) if $\gamma \nearrow 2/h$, then $\lambda_1 \to -\infty$;
4) if $\gamma = 2/h$, then boundary conditions (14)–(15) are not equivalent to conditions (16)–(17);
5) if $\gamma \searrow 2/h$, then $\lambda_1 \to +\infty$;
6) if $\gamma > 2/h$, then all the eigenvalues λ are positive.

We further clarify above mentioned properties in brief. Instead of investigating eigenvalues $\lambda \in \mathbb{C}_\lambda := \mathbb{C}$, we use a bijection $\lambda = \lambda(q)$ from complex plane $\mathbb{C}_q$ to $\mathbb{C}_\lambda$:

$$\lambda = \frac{4}{h^2}\sin^2\frac{qh}{2}, \quad q := \alpha + \imath\beta, \tag{25}$$

where $\mathbb{C}_q = \{q = \alpha\colon 0 < \alpha < \pi/h\} \cup \{q = \imath\beta\colon \beta \geqslant 0\} \cup \{q = \pi/h + \imath\beta\colon \beta \geqslant 0\}$. The points $q = 0$ and $q = \pi/h$ are the branch points of the map (25). Therefore, every eigenvalue $\lambda_i = \lambda(q_i)$ conforms to q_i, $i = \overline{1, N-1}$, and vice versa. A numeration of $\{q_1, \dots, q_{N-1}\}$ coincides with the numeration of $\{\lambda_1, \dots, \lambda_{N-1}\}$ ($\{\lambda_2, \dots, \lambda_{N-1}\}$ and $\{q_2, \dots, q_{N-1}\}$ for $\gamma = 2/h$).

Now we investigate the spectrum of matrix $\mathbf{\Lambda}$ in detail.

Case (i): $q \neq 0$, $q \neq \pi/h$. The general solution of (23) is

$$U = C_0\cos(qx) + C_1\sin(qx), \quad x \in \overline{\omega}^h.$$

By substituting it into (24), we have

$$\begin{aligned}
&\bigl(\gamma_0\bigl[1,\cos(qx)\bigr] - 1\bigr)C_0 + \gamma_0\bigl[1,\sin(qx)\bigr]C_1 = 0,\\
&\bigl(\gamma_1\bigl[1,\cos(qx)\bigr] - \cos q\bigr)C_0 + \bigl(\gamma_1\bigl[1,\sin(qx)\bigr] - \sin q\bigr)C_1 = 0.
\end{aligned} \tag{26}$$

A nontrivial solutions of system (26) exist if its determinant is equal to zero:

$$\gamma_0\bigl[1, \sin(qx)\cos q - \cos(qx)\sin q\bigr] - \gamma_1\bigl[1,\sin(qx)\bigr] + \sin q = 0,$$

or simplifying this formula, we have

$$-\gamma_0\bigl[1, \sin\bigl(q(1-x)\bigr)\bigr] - \gamma_1\bigl[1,\sin(qx)\bigr] + \sin q = 0.$$

Using expression (6), we get an equation for q:

$$\gamma h\frac{\sin(q/2)\cos(qh/2)}{\sin(qh/2)} = \sin q. \tag{27}$$

In this formula, functions $\sin(qh/2)$ and $\cos(qh/2)$ are never equal to zero in $\mathbb{C}_q \backslash \{0, \pi/h\}$ (since a sine function has only real zero points in a complex plane and function $\sin(qh/2)$ has no zero points in the interval $(0, \pi/h)$). We rewrite Eq. (27) in the form

$$\sin\frac{q}{2}\left(\gamma h\sin\frac{q}{2} - 2\cos\frac{q}{2}\tan\frac{qh}{2}\right) = 0. \tag{28}$$

The roots of Eq. (28) can be found from two equations:

$$\sin\frac{q}{2} = 0, \tag{29a}$$

$$\gamma h \sin\frac{q}{2} - 2\cos\frac{q}{2}\tan\frac{qh}{2} = 0. \tag{29b}$$

The roots of the first type satisfy Eq. (29a). They are called *constant points* (see [13]) because they don't depend on γ:

$$q_{2k} = 2k\pi, \quad k = \overline{0, N_0}, \quad N_0 := \left\lfloor \frac{N-1}{2} \right\rfloor. \tag{30}$$

If $\sin(q/2) = 0$, then $\cos(q/2) = \pm 1$. So, the roots of Eq. (29b) depend on γ. Such type of roots is called second type roots. Now we divide this equation by $\sin(q/2)$ and get expression for γ:

$$\gamma = 2h^{-1}\tan^{-1}\frac{q}{2}\tan\frac{qh}{2}. \tag{31}$$

A function $\gamma = \gamma(q)$ is called *complex-real characteristic function* [13]. The roots of the second type q_{2k+1}, $k = \overline{0, N_1}$, $N_1 := \lfloor N/2 \rfloor$ can be found as γ-values of the characteristic function (31).

Case (ii): $\lambda = q = 0$. In this case, the general solution of (23) is

$$U_i = C_0 + C_1 ih.$$

By substituting it into (24), we have

$$\begin{aligned} &(\gamma_0 - 1)C_0 + \frac{1}{2}\gamma_0 C_1 = 0, \\ &(\gamma_1 - 1)C_0 + \left(\frac{1}{2}\gamma_1 - 1\right)C_1 = 0. \end{aligned} \tag{32}$$

So, we have zero eigenvalue when $\gamma = 2$.

Case (iii): $q = \pi/h$ ($\lambda = 4/h^2$). Now the general solution of (23) is

$$U_i = (-1)^i(C_0 + C_1 ih).$$

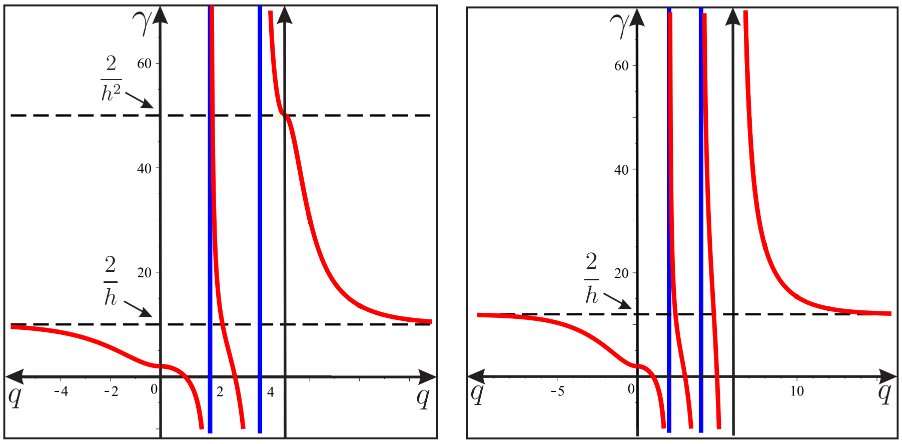
By substituting it into Eq. (24), for the case when N is odd (when N is even, there are no nonzero solutions), we obtain:

$$\begin{aligned} &C_0 + \gamma_0\frac{h^2}{2}C_1 = 0, \\ &C_0 + \left(1 - \frac{h^2}{2}\gamma_1\right)C_1 = 0. \end{aligned}$$

Solving this system using formulas (8), we obtain:

$$2(\gamma_1 + \gamma_0) = \frac{4}{h^2}. \tag{33}$$

The solution of Eq. (33) is defined only if N is odd: $\gamma = 2/h^2$.



(a) The case of odd number of grid points, $N = 5$ ($h = 1/5$).

(b) The case of even number of grid points, $N = 6$ ($h = 1/6$).

Fig. 1. Generalized characteristic function $\gamma(\pi q)$.

Remark 4. The *generalized characteristic function* [13], is plotted on Fig. 1. All the $N-1$ roots of Eq. (28) belong to a union of three intervals $\{q = \alpha \in [0, \pi/h]\} \cup \{q = \imath\beta\colon\ \beta \geqslant 0\} \cup \{q = \pi/h + \imath\beta\colon\ \beta \geqslant 0\}$ (if $\gamma = 2/h$, then we have $N-2$ roots). So, all the eigenvalues λ are real. We plot a graph of function (31) in each interval: $\gamma = \gamma(\alpha)$, $\alpha \in [0, \pi/h]$; $\gamma := \gamma_-(\beta) = \gamma(\imath\beta)$, $\beta \geqslant 0$; $\gamma := \gamma_+(\beta) = \gamma(\pi/h + \imath\beta)$, $\beta \geqslant 0$. We combine them on one graph of *real characteristic function* [13]. Finally, we add vertical lines $q = \alpha = 2\pi k$, $k = \overline{0, N_1}$, which are the roots of the first type, and get generalized characteristic function. As one can see real characteristic function asymptotes coincide with the roots of the first type. We plot characteristic function $\gamma(\pi\alpha)$ graph in Fig. 1 (instead of $\gamma(\alpha)$). So, in the classical case ($\gamma_0 = 0$, $\gamma_1 = 0$), graph intersects α axis in the integer values (the index of eigenvalues). We note that characteristic functions are continuous at the points $q = 0$ and $q = \pi/h$.

In general (except the case of $\gamma = 2/h$), the eigenvectors are real and form the complete eigenvector system $\{\mathbf{V}_1, \dots, \mathbf{V}_{N-1}\}$ (we have $N-2$ eigenvectors $\{\mathbf{V}_2, \dots, \mathbf{V}_{N-1}\}$ when $\gamma = 2/h$). We call two eigenvectors equal if they are linearly dependent. These eigenvectors can be expressed by general formula:

$$V_{ki} = \sin(q_k x_i) - \gamma_0\big[1, \sin\big(q_k(x_i - x)\big)\big], \quad k \in \overline{1, N-1}. \tag{34}$$

Note that $q_k = q_k(\gamma)$. So, V_{ki} also depends on γ. Eq. (34) can be rewritten as

$$V_{ki} = \sin\big(q_k(1 - x_i)\big) - \gamma_1\big[1, \sin\big(q_k(x_i - x)\big)\big], \quad k \in \overline{1, N-1}.$$

Remark 5. For the different values of q, we can rewrite Eq. (34) in such forms:

(a) if $q_k = \alpha_k \in (0, \pi/h)$, then

$$V_i = \sin(\alpha_k x_i) - \gamma_0\big[1, \sin\big(\alpha_k(x_i - x)\big)\big]; \tag{35}$$

(b) if $q_1 = \imath\beta$ ($\gamma \in (2, 2/h)$), then

$$V_{1i} = \sinh(\beta x_i) - \gamma_0\big[1, \sinh\big(\beta(x_i - x)\big)\big]; \tag{36}$$

(c) if $q_1 = \pi/h + \imath\beta$ ($\gamma \in (2/h, 2/h^2)$ if N is odd; $\gamma \in (2/h, \infty)$ if N is even), then

$$V_{1i} = (-1)^i\big(\sinh(\beta x_i) - \gamma_0\big[1, \sinh\big(\beta(x_i - x)\big)\big]\big); \tag{37}$$

(d) if $q_1 = 0$ ($\gamma = 2$), then

$$V_{1i} = x_i - \gamma_0[1, x_i - x] = x_i - \gamma_0\left(x_i - \frac{1}{2}\right); \tag{38}$$

(e) if $q_1 = \pi/h$ ($\gamma = 2/h^2$), then

$$V_{1i} = (-1)^i\big(-\gamma_0 h^2 + 2x_i\big). \tag{39}$$

Expressions (38)–(39) are the limit versions of formula (34) at the points $q = 0$ and $q = \pi/h$ (as well as Eqs. (35)–(37)).

4 The stability of finite difference scheme

4.1 The structure of the spectrum of the matrix S

First, we note one important property of the three-layer scheme (18) with $(N-1)\times(N-1)$ matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ defined by Eqs. (19)–(20). We use notation $\lambda_k(\mathbf{A})$, $\lambda_k(\mathbf{B})$, $\lambda_k(\mathbf{C})$ for the kth eigenvalue of matrix $\mathbf{A}$, $\mathbf{B}$ or $\mathbf{C}$ accordingly. We investigate the case of the complete $N-1$ eigenvector system $\{\mathbf{V}_1, \ldots, \mathbf{V}_{N-1}\}$ (in the case $\gamma \neq 2/h$).

Lemma 2. *The matrices* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *have a common system of eigenvectors. More precisely, the eigenvectors of the matrix* $\mathbf{\Lambda}$ *are the eigenvectors of the matrices* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$.

Proof. The eigenvectors of the matrix $\mathbf{\Lambda}$ are also the eigenvectors of the unit matrix $\mathbf{I}$. So, since $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are the linear combination of matrices $\mathbf{I}$ and $\mathbf{\Lambda}$, the formulated lemma is valid. □

Let μ be the eigenvalue of the $2(N-1) \times 2(N-1)$ matrix $\mathbf{S}$ (see Eq. (22)). We consider the eigenvalue problem

$$\det(\mathbf{S} - \mu\mathbf{I}) = \det\begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} - \mu\mathbf{I} & -\mathbf{I} \\ \mathbf{I} & -\mu\mathbf{I} \end{pmatrix} = \det\big(\mathbf{A}\mu^2 + \mathbf{B}\mu + \mathbf{C}\big) = 0. \tag{40}$$

We simplify determinant in Eq. (40) and get a characteristic equation for the eigenvalues of the generalized nonlinear eigenvalue problem

$$\big(\mu^2\mathbf{A} + \mu\mathbf{B} + \mathbf{C}\big)\mathbf{U} = 0, \quad \mathbf{U} \neq \mathbf{0}. \tag{41}$$

Problem (41) is rather well studied for the case of symmetric matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ (e.g., see [23, p. 23]). We note that the eigenvalues μ of the matrix $\mathbf{S}$ coincide with the eigenvalues of the generalized nonlinear eigenvalue problem (41). The number of eigenvalues of problem (41) is $2(N-1)$. Let us clarify the relationship between the eigenvalues μ of the matrix $\mathbf{S}$ and the eigenvalues λ of the matrix $\mathbf{\Lambda}$.

By substituting an eigenvector $\mathbf{V}_k$ of matrix $\mathbf{\Lambda}$ (see Eq. (34)) into Eq. (41), we obtain

$$\big(\mu^2\mathbf{A} + \mu\mathbf{B} + \mathbf{C}\big)\mathbf{V}_k = \big(\mu^2\lambda_k(\mathbf{A}) + \mu\lambda_k(\mathbf{B}) + \lambda_k(\mathbf{C})\big)\mathbf{V}_k = 0. \tag{42}$$

So, eigenvalues of the matrix $\mathbf{S}$ satisfy the quadratic equation

$$\mu^2\lambda_k(\mathbf{A}) + \mu\lambda_k(\mathbf{B}) + \lambda_k(\mathbf{C}) = 0, \quad k = \overline{1, N-1}. \tag{43}$$

Lemma 3. *Each eigenvalue $\lambda_k\big(\mathbf{\Lambda}\big)$, $k = \overline{1, N-1}$ corresponds to two eigenvalues μ_k^1 and μ_k^2 of the matrix $\mathbf{S}$:*

$$\mu_k^m = -b_k \pm \sqrt{b_k^2 - 1}, \quad m = 1, 2, \quad b_k = \frac{-1 + \tau^2(1/2-\sigma)\lambda_k}{1 + \tau^2\sigma\lambda_k}, \quad k = \overline{1, N-1}. \tag{44}$$

Proof. Using relations (19), we calculate $\lambda_k(\mathbf{A}) = \lambda_k(\mathbf{C}) = 1 + \tau^2\sigma\lambda_k$, $\lambda_k(\mathbf{B}) = -2 + \tau^2(1 - 2\sigma)\lambda_k$. By substituting these values into (43) and solving the resulting equation, we obtain relations (44) for eigenvalues of matrix $\mathbf{S}$. □

Remark 6. Equation (44) determines the relation between eigenvalues μ_k^m and λ_k. The value of μ_k^m can be complex as well as real, depending on the parameters σ, τ and eigenvalues λ_k.

Lemma 4. *Let λ_k and $\mathbf{V}_k$ be an eigenvalue and an eigenvector of the matrix $\mathbf{\Lambda}$, respectively. Let μ_k^1 and μ_k^2 be the eigenvalues of matrix $\mathbf{S}$ corresponding to λ_k, $\mu_k^1 \neq \mu_k^2$. Then*

$$\mathbf{W}_k^m = \begin{pmatrix} \mathbf{V}_k \\ (\mu_k^m)^{-1}\mathbf{V}_k \end{pmatrix}, \quad m = 1, 2,\ k = \overline{1, N-1}, \tag{45}$$

are linearly independent eigenvectors of the matrix $\mathbf{S}$.

Proof. Consider the eigenvalue problem $\mathbf{SW} = \mu_k^m\mathbf{W}$, $m = 1$ or $m = 2$. Using definition of matrix $\mathbf{S}$ (see Eq. (22)), we have

$$\begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} = \mu_k^m \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix}, \quad m = 1, 2,\ k = \overline{1, N-1}, \tag{46}$$

where $\mathbf{W} = \big(\mathbf{W}_1, \mathbf{W}_2\big)^\mathsf{T}$ is an eigenvector. So, two equalities are valid:

$$-\mathbf{A}^{-1}\mathbf{B}\mathbf{W}_1 - \mathbf{W}_2 = \mu_k^m\mathbf{W}_1, \tag{47}$$

$$\mathbf{W}_1 = \mu_k^m\mathbf{W}_2. \tag{48}$$

Substituting Eq. (48) into Eq. (47) and multiplying it by $\mu_k^m\mathbf{A}$, we get an analogue of formula (42): $((\mu_k^m)^2\mathbf{A} + \mu_k^m\mathbf{B} + \mathbf{A})\mathbf{W}_1 = 0$. Every $\mathbf{V}_k$, $k = \overline{1, N-1}$, satisfies Eq. (42) with $\mu = \mu_k^m$. So, we can take $\mathbf{W}_1 = \mathbf{V}_k$, $k = \overline{1, N-1}$. Then from Eq. (48) it follows that $\mathbf{W}_2 = (\mu_k^m)^{-1}\mathbf{V}_k$. □

Remark 7. If $\mu_k^1 \neq \mu_k^2$, $k = \overline{1, N-1}$, then we have $2(N-1)$ linear independent eigenvectors $\mathbf{W}_k^m$, $m = 1, 2$, $k = \overline{1, N-1}$, which form a complete eigenvector system. If eigenvalues μ_k^m, $m = 1, 2$ are complex, then eigenvectors $\mathbf{W}_k^m$ are also complex.

4.2 The main result

A polynomial satisfies the *root condition* if all the roots of this polynomial are in the closed unit disc of complex plane and roots of magnitude 1 are simple [24, 25]. For polynomial of the second order

$$A\mu^2 + B\mu + C, \quad A \neq 0,\ B, C \in \mathbb{C}, \tag{49}$$

the following theorem is valid.

Theorem 1. (*See* [26].) *The roots of the second order polynomial are in the closed unit disc of complex plane and those roots of magnitude* 1 *are simple if*

$$|C|^2 + |\bar{A}B - \bar{B}C| \leqslant |A|^2, \tag{50a}$$

$$|B| < 2|A|. \tag{50b}$$

We rewrite the quadratic equation (43) in a form

$$p(\mu) := a\mu^2 - 2(a-\eta)\mu + a = 0, \tag{51}$$

where $a = 1 + \tau^2\sigma\lambda \in \mathbb{R}$, $\eta = \tau^2\lambda/2 \in \mathbb{R}$. For this real polynomial $p(\mu)$, inequality (50a) is trivial, and $p(\mu)$ has two complex roots of magnitude 1. The strong inequality (50b) ensures that these roots are simple [26]. So, polynomial $p(\mu)$ satisfies the root condition if and only if

$$|a - \eta| < |a| \tag{52}$$

(see (50b)).

Remark 8. If polynomial (43) satisfies the root condition, then $\rho(\mathbf{S}) = 1$.

Now we formulate the main result of this article.

Theorem 2. *If* $\gamma < 2$ *and*

$$\sigma > \frac{1}{4} - \frac{1}{\tau^2\lambda_{\max}}, \tag{53}$$

then the weighted FDS (11)–(15) *is stable.*

Proof. Let us analyze condition (52). If $a \leqslant 0$, then $a < a - \eta < -a$. In this case, we have $\eta < 0$ or $\lambda < 0$, which contradicts the assumption $\gamma < 2$. If $a > 0$, then $-a < a - \eta < a$. If $\gamma < 2$, then $\lambda_k > 0$, $k = \overline{1, N-1}$, and inequality $\eta/2 > 0$ is valid. So, we have $a > \eta/2 > 0$. We rewrite the inequality $a > \eta/2$ as

$$\sigma > \frac{1}{4} - \frac{1}{\tau^2\lambda}. \tag{54}$$

If $\sigma > 1/4 - 1/(\tau^2\lambda_{\max})$, then (54) is valid for all λ_k, $k = \overline{1, N-1}$. □

Remark 9. The obtained inequality (53) is an analogue of the stability inequality for three-layered difference schemes with classical Dirichlet boundary conditions (see [17]).

Remark 10. While $\gamma < 2$, the eigenvalues λ_k, $k = \overline{1, N-1}$, are in the interval $(0, 4/h^2)$. So, we can use inequality

$$\sigma \geqslant \frac{1}{4} - \frac{h^2}{4\tau^2}$$

instead of the condition (53). If $\sigma \geqslant 1/4$, then the weighted FDS in unconditionally stable. If $\sigma = 0$, then the FDS is stable under the condition $\tau < h$.

Conclusions and final remarks

- The sufficient stability condition ($\gamma < 2$ and $\sigma > 1/4 - 1/(\tau^2 \lambda_{\max})$) for the three-layered weighted finite difference scheme is obtained.
- The weighted FDS in unconditionally stable under the condition $\sigma \geqslant 1/4$ ($\gamma < 2$).
- The (stability) condition for the weight σ is the same as in the classical case $\gamma_0 = \gamma_1 = 0$.
- The spectrum of the matrix $\mathbf{\Lambda}$ is investigated. Eigenvalues are real, and eigenvectors form a complete system (except the case of $\gamma = 2/h$).
- The spectrum of $\mathbf{\Lambda}$ is qualitatively different (in some sence) for the cases of odd and even number of grid points N.
- If $\gamma > 2/h^2$ and the number of grid points N is odd, then the spectrum of matrix $\mathbf{\Lambda}$ is in the interval $(0, 4/h^2)$ (as well as in the case of $\gamma < 2$).
- If $\gamma > 2/h$, then all the eigenvalues λ_k, $k = \overline{1, N-1}$, are positive, but eigenvalue $\lambda_{\max}$ could be greater than $4/h^2$. This affects the condition on σ.

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