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Compound orbits break-up in constituents: An algorithm

Jesús San Martín[a], **Antonia González Gómez**[b], **María José Moscoso**[a], **Daniel Rodriguez-Perez**[c]

[a]ETSIDI, Technical University of Madrid (UPM)
Ronda de Valencia 3, 28012 Madrid, Spain
jesus.sanmartin@upm.es; mariajose.moscoso@upm.es

[b]ETSIM, Techical University of Madrid (UPM)
Ciudad Universitaria s/n, 28040 Madrid, Spain
antonia.gonzalez@upm.es

[c]Department of Mathematical Physics and Fluids,
National Distance Education University (UNED)
Senda del Rey 9, 28040 Madrid, Spain
daniel@dfmf.uned.es

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Abstract. In this paper, decomposition of periodic orbits in bifurcation diagrams are derived in unidimensional dynamics system $x_{n+1} = f(x_n; r)$, being f an unimodal function. We prove a theorem, which states the necessary and sufficient conditions for the break-up of compound orbits in their simpler constituents. A corollary to this theorem provides an algorithm for the computation of those orbits. This process closes the theoretical framework initiated in [J. San Martín, M.J. Moscoso, A. González Gómez, Composition law of cardinal ordering permutations, *Physica D*, 239:1135–1146, 2010]. Theorem 1 of present work closes the theoretical frame of composition and decomposition.

Keywords: visiting order permutation, next visiting permutation, decomposition theorem.

1 Introduction

Dynamical systems underlie in any science we can imagine, from mathematical to social sciences. Countless mathematical models have been developed to describe temporal evolution of the world around us: planets orbiting the Sun, flow of water in a river, people waving in a stadium, cells forming tissues in our body, cars moving along a road, etc. As a consequence of the extraordinary variety of phenomena studied, there exists a huge number of possible behaviors. An efficient way to address these issues is using symbolic dynamics [3]. In that case, the dynamical systems are modeled in a discrete space, resulting of a partition of phase space into disjoint regions. Every region is labeled by a symbol. System evolution is given by a sequence of symbols, each of them representing a region of the system. Although one might think that no crucial information about the system may be obtained by this process there are some groundbreaking results in this subject. Special attention should be given to pioneering works by Metropolis et al. [11] about symbolic sequences and by Milnor and Thurston [12] who developed the kneading theory. In this context, Byers seminal work [4] becomes useful for our work as we will show later. Byers states the conditions an application has to fulfill to serve as a "model" for the behavior of a more general set of functions. This result will endow our theorems with further reach and generality than could be thought of at a first sight. In particular, kneading theory is more easily understood when the dynamical system

$$x(n+1) = f\big(x(n)\big) \tag{1}$$

is ruled by an unimodal function, the function we will work with in this paper. The relationship between periodic orbits of unimodal functions (those we will focus on) and kneading theory was given by Jonker [10]. Jonker found the precise relationship between the periodicity of the orbit of a point and the periodicity of the invariant coordinate of that point. Some tools we will need to harness the power of symbolic dynamics are periodic orbits. They have a periodic symbolic sequence and play an important role in dynamical systems, in particular the unstable ones as we will see later.

The composition law of Derrida et al. [7] allows the generation of a symbolic sequence of complex structure from its constituents (periodic orbits). In particular, starting from the symbolic sequence of the supercycle of period one it is possible to build up symbolic sequences of Feigenbaum cascade orbits [8, 9]. So, one of the most important ways of transition to chaos is characterized. But not only that, by using saddle-node bifurcation cascades [15] and symbolic sequences of Feigenbaum cascade orbits, the structure of chaotic bands of the bifurcation diagram is also characterized (see Fig. 1). Working with an unimodal function, the symbolic sequence is obtained as follows: the critical point of unimodal function is denoted by C, points located to the right of C are denoted R (right) and the ones located to its left as L (left). However, if we label the points in the orbit with natural numbers ordering their positions relative to each other, then every periodic orbit can be associated with a permutation. There are permutations that give rise to the visiting order in Feigenbaum cascade orbits [16] and there exists a composition law of permutations [17] replacing the composition law of Derrida et al. Consequently, the characterization of bifurcation diagram structure is given by permutations. We have just outlined how to build up the bifurcation diagram from its constituents. From a mathematical point of view, however, it would be interesting to solve the inverse problem: what are the constituents of a complex structure? More specifically, given a structure we would like to answer two questions:

(i) Can we break down the structure? That is, is the structure made up of smaller constituents?
(ii) If the answer to the first question is in the affirmative, how can we break down the structure and what are its constituents?

In other words, we are looking for the necessary and sufficient conditions that allow a structure to be decomposed into its constituents. That is the goal of this paper.

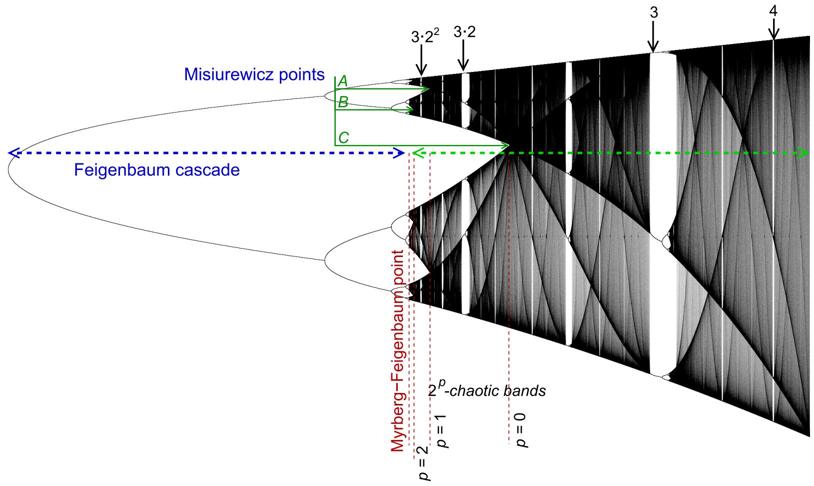

Fig. 1. Canonical bifurcation diagram. 3, $3 \cdot 2$, $3 \cdot 2^2$ and 4-periodic windows are marked. Some Misiurewicz points (A, B, C) where chaotic bands merge are shown above. A fractal structure can be observed in the bifurcation diagram. Compound orbits generating this fractal structure can be splited by using the algorithm in Section 4.

Solving the inverse problem of composition is already interesting because we complete the composition-decomposition problem. But the most important consequence is that the decomposition process is not limited to stable orbits; unstable orbits may also be obtained from such a process. The understanding of unstable orbits (limit cycles) is fundamental because they are the underlying skeleton of chaotic attractors [2, 6]. The shorter the cycles, the better the approximation to the strange attractor [5], that is why it is interesting to split large cycles into smaller constituents. On the other side, the unstable orbits in the skeleton are the corner-stones of many chaos control techniques [13, 14]. To implement theses techniques the unstable orbits need to be determined beforehand.

Orbital decomposition can also be applied to continuous dynamical systems. They can be cast as discrete dynamical systems by using Poincaré section. Points of Poincaré section corresponding to a continuous orbit lay out a periodic orbit in a discrete space. If that orbit can be decomposed then the continuous orbit is a composed orbit. Decomposition of these orbits is crucial to calculate Gutzwiller trace formula [21], which relates spectrum of quantum system with periodic orbits of the equivalent semiclassical system. Roughly speaking, decomposition law of periodic trajectories will be useful every time cycle expansion techniques [1] are being used.

Decomposition law is also important from a practical or experimental point of view. For example, if we have a 12-periodic orbit we may be interested in knowing if the orbit is located in a primary period 12 window or in a period 3 window inside of a period 4 window (see Fig. 2).

The first appearance orbits in the chaotic bands of the logistic map bifurcation diagram follow Sharkovsky's ordering [20]. The decomposition of a $q \cdot 2^p$-periodic orbit within the 2^p-chaotic band (see Fig. 1) will lead to a period q orbit located within the 2^0-chaotic

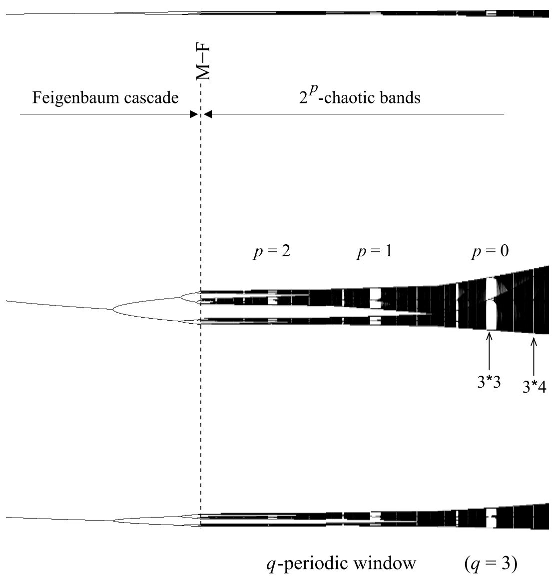

Fig. 2. Highlight of the 3-periodic window of Fig. 1. This window mimics the canonical bifurcation diagram but repeated three times. The $3 \cdot 4$-periodic window is marked.

band and the 2^p-periodic orbit of Feigenbaumn's cascade [15]. It was Jonker [10] who proved Sharkovsky's theorem in the context of kneading theory, showing that kneading theory underlies orbit composition processes.

A very intuitive way of looking at the decomposition process exists. If we have a period hs orbit, that is with hs points, we can imagine that every point is a chair in a room. The chairs are visited according to a permutation β_s. We split the hs chairs into h rooms with s chairs each. We visit the rooms in accordance to one permutation β_h and every time we visit the same room we sit down in a different chair of the room due to another permutation β_s. We must find β_h and β_s from β_{hs}. We are going to solve this task by using of couple of tricks. If we leave only one chair in every room the result would be like an h-periodic orbit such that a point is located at the critical point C of the unimodal function f of (1) and the rest of points are located where f is either increasing or decreasing. The chairs of a room located where f is increasing (decreasing) are mapped into the next room preserving their relative location (flipped from right to left). So, we split the β_{hs} permutation into h rooms of s elements each, in such a way that images of these sets (except one of them) are either preserved or flipped from right to left. The set whose elements are neither preserved nor flipped will be β_s, because they are the chairs of the room associated with the critical point C.

This paper is organized as follows. Definitions and notations are introduced in Section 2. Next, we prove decomposition theorem to solve the mentioned problems. Then we develop an algorithm to implement the theorem. We then finish with our conclusions. We will also show some examples to highlight how the theorems and algorithms work.

2 Definitions and notation

Let $f : I \to I$ be an unimodal map with critical point at C, that is, f is continuous and strictly increasing (decreasing) on $[a, C) = J_{\mathrm{L}}$ and strictly decreasing (increasing) on $(C, b] = J_{\mathrm{R}}$. Without loss of generality it can be assumed the critical point C is a maximum (see Fig. 3). So, f is decreasing in J_{R} and increasing in J_{L}. Let $O_q = \{x_1, \dots, x_q\} = \{C, f(C), \dots, f^q(C)\}$ be a q-periodic supercycle of f and let $\{C^*_{(1,q)}, C^*_{(2,q)}, \dots, C^*_{(q,q)}\}$ be the set that denotes the descending cardinality ordering of the orbit O_q [16]. Let $f(C^*_{(i,q)})$ be the next to $C^*_{(i,q)}$ (see [17]).

Definition 1. The natural number $\beta(i, q)$, $i = 1, \dots, q$, will denote the ordinal position of the cardinal point $f(C_{(i,q)})$, $i = 1, \dots, q$. That is, $f(C_{(i,q)}) = C_{(\beta(i,q),q)}$, $i = 1, \dots, q$ (see Fig. 4).

Remark 1. If c denotes the ordinal position of the critical point C of f as $f(C)$ is in the first position (see [16, Remark 1]), it results that $\beta(c, q) = 1$.

Definition 2. We denote as β_q the permutation $\beta_q = (\beta(1, q)\ \beta(2, q)\ \dots\ \beta(q, q))$. β_q will be called the next visiting permutation of O_q (see Fig. 4).

Remark 2. If the visiting order permutation is such that $f(C_{(i,q)}) = C_{(j,q)}$, that is, $C_{(i,q)} \to C_{(j,q)}$, we write

$$\begin{pmatrix} \cdots & i & \cdots \\ \cdots & j & \cdots \end{pmatrix},$$

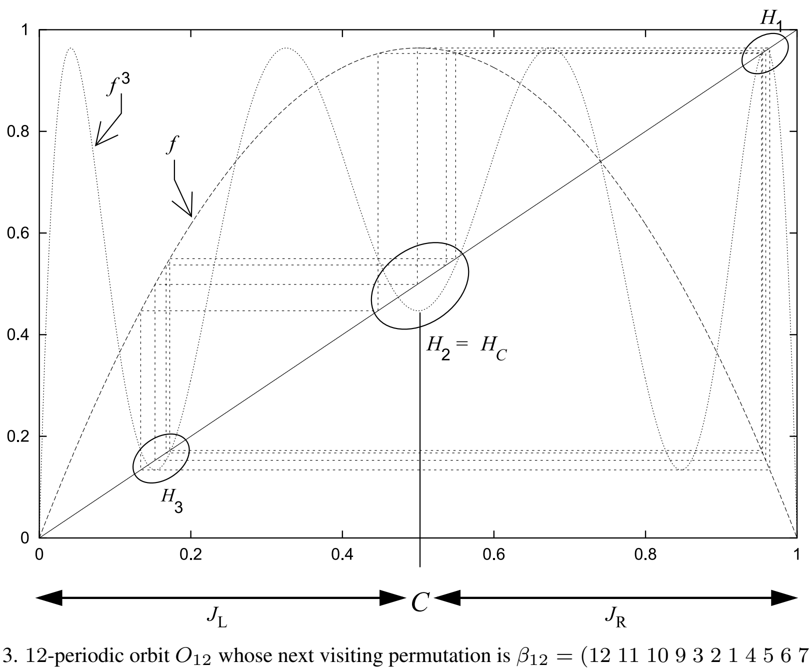

Fig. 3. 12-periodic orbit O_{12} whose next visiting permutation is $\beta_{12} = (12\ 11\ 10\ 9\ 3\ 2\ 1\ 4\ 5\ 6\ 7\ 8)$.

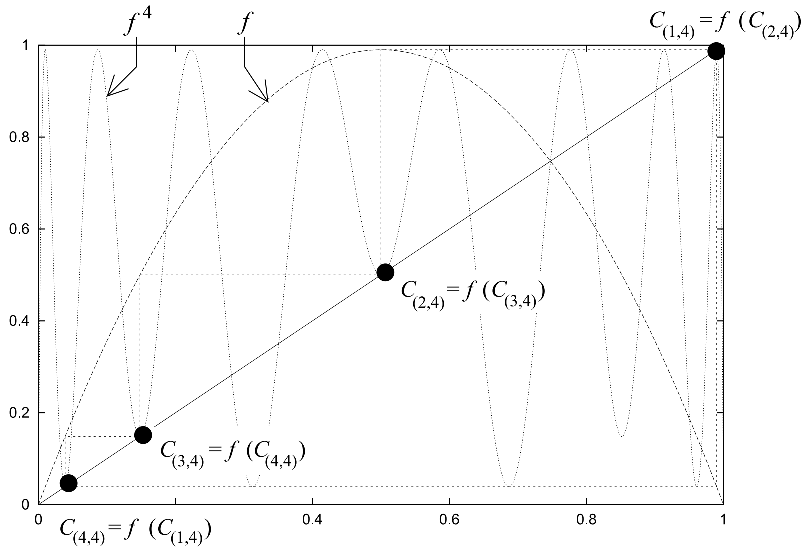

Fig. 4. Given $C_{(i,q)}$, the second label "q" indicates the period of the orbit (in this case, $q = 4$). The first label "i" denotes ordinal position $(1\ 2\ 3\ 4)$ of the points on the orbit. According to the 4-periodic orbit shown in the figure, the visiting order is $C_{(1,4)} \to C_{(4,4)} \to C_{(3,4)} \to C_{(2,4)}$ or, equivalently, $f(C_{(1,4)}) = C_{(4,4)}$, $f(C_{(2,4)}) = C_{(1,4)}$, $f(C_{(3,4)}) = C_{(2,4)}$ and $f(C_{(4,4)}) = C_{(3,4)}$. According to Definition 1, $f(C_{(i,q)}) = C_{(\beta(i,q),q)}$, consequently, $\beta(1,4) = 4$, $\beta(2,4) = 1$, $\beta(3,4) = 2$, $\beta(4,4) = 3$. Hence, according to Definition 2, $\beta_4 = (4\ 1\ 2\ 3)$.

then we reorder the pairs $\binom{i}{j}$ in such a way that the index i has the natural order. For example, let O_4 be a 4-periodic orbit (see Fig. 4) with visiting order permutation

$$1 \to 4 \to 3 \to 2,$$

so, we write

$$\begin{pmatrix} 1 & 4 & 3 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

After reordering we obtain the next visiting permutations

$$\beta_4 = \begin{pmatrix} 4 & 1 & 2 & 3 \end{pmatrix}.$$

Definition 3. Let β_q be the next visiting permutation of O_q and let $q = hs$. We define the j-box of O_q by $H_j = \{(j-1)s + k;\ k = 1, \dots, s\}$ for $j = 1, \dots, h$. We denote by $\beta_q(H_j)$ the set given by $\beta_q(H_j) = \{\beta((j-1)s + k, q);\ k = 1, \dots, s\}$ (see Fig. 3).

In Fig. 3, by taking $h = 3$ and $s = 4$, the cardinals $C_{(1,12)}$, $C_{(2,12)}$, $C_{(3,12)}$ and $C_{(4,12)}$ are located in H_1; cardinals $C_{(5,12)}$, $C_{(6,12)}$, $C_{(7,12)}$ and $C_{(8,12)}$ are located in H_2, and $C_{(9,12)}$, $C_{(10,12)}$, $C_{(11,12)}$ and $C_{(12,12)}$ are located in H_3. From the visiting permutation it results

$$\begin{pmatrix} & H_1 & & & & H_2 & & & & H_3 & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 11 & 10 & 9 & 3 & 2 & 1 & 4 & 5 & 6 & 7 & 8 \\ & \beta_{12}^1 & & & & \beta_{12}^2 & & & & \beta_{12}^3 & & \end{pmatrix}.$$

Definition 4. Let β_q be the next visiting permutation of O_q and let $q = hs$. We denote by (β_q^j) with $j = 1, \dots, h$

$$\beta_q^j = \begin{pmatrix} (j-1)s+1 & \dots & (j-1)s+s \\ \beta((j-1)s+1, q) & \dots & \beta((j-1)s+s, q) \end{pmatrix}$$

and $\beta^j(r, q) = \beta((j-1)s + r, q)$ with $r = 1, \dots, s$ (see Fig. 3).

Definition 5. Let γ_n be a permutation of n elements. We define the inversion permutation of γ_n, denoted by γ_n^*, as the permutation given by $\gamma_n^* = (\gamma^*(1,n) \ \dots \ \gamma^*(n,n))$ with

$$\gamma^*(i,n) = n + 1 - \gamma(i,n), \quad i = 1, \dots, n.$$

Notice that if $I_n = (I(1,n) \ \dots \ I(n,n))$ is the identity permutation, then $I_n^* \circ I_n^* = I_n$.

Definition 6. Let γ_n be a permutation of n elements. We define the conjugated permutation of γ_n, denoted by $\overline{\gamma}_n$, as the permutation given by $\overline{\gamma}_n = (\overline{\gamma}(1,n) \ \dots \ \overline{\gamma}(n,n))$ with

$$\overline{\gamma}(i,n) = n + 1 - \gamma(n+1-i, n), \quad i = 1, \dots, n.$$

3 Theorem of periodic orbit decomposition

In order to obtain the decomposition theorem below, we need to revisit the composition process and state it in terms of next visiting permutations.

Let O_h, O_s be supercycles of a C^2-unimodal map f with next visiting permutations β_h and β_s, respectively. The geometric meaning of composing O_h and O_s involves replacing the h points of O_h by h boxes, with s points each, such that all points of a same box are mapped into the same box. It is important to point out that boxes are visited consecutively according to β_h and that every time the same box is visited then the box points are visited according to β_s if f^h has a maximum and according to $\overline{\beta}_s$ if f^h has a minimum (see [17] for more details). As boxes (see Definition 3) H_i, $i = 1, \dots, h$, are visited according to β_h, we split the visit in two parts:

$$H_c \to H_1, \tag{2}$$

$$H_1 \to \dots \to H_i \to \dots \to H_c. \tag{3}$$

In sequence (3), excluding H_c, boxes are located in J_{R} or J_{L}. Every time the orbit leaves a box located in J_{L} the points in that box are mapped according to the identity permutation, I_s, because f is increasing in J_{L}. On the contrary, every time the orbit leaves a box located in J_{R}, the points in that box are mapped reverted from left to right because f is decreasing in J_{R}, that is, they are linked by I_s^*. As $I_s^* \circ I_s^* = I_s$ it results that H_1 is linked with H_c by I_s or I_s^*. It only remains to know the link between H_c and H_1 of sequence (2) to close the orbit (see Fig. 3)

This is the geometrical mechanism underlying the proof of the following lemma. This lemma is essential to prove the Theorem 1, which is the goal of this paper.

Lemma 1. *Let O_h, O_s be two supercycles of a C^2-unimodal map f with next visiting permutations β_h and β_s, respectively. Let c be such that $\beta(c,h) = 1$. If O_{hs} is the supercycle resulting of composing O_h with O_s, then its next visiting permutation*

$$\beta_{hs} = \bigl(\beta^1(1,hs) \ \ldots \ \beta^1(s,hs) \ \beta^2(1,hs) \ \ldots \ \beta^2(s,hs) \ \ldots \\ \beta^h(1,hs) \ \beta^h(s,hs)\bigr)$$

is given for all $k = 1,\ldots,s$ by:

(a) $\beta^i(k,hs) = \beta(i,h)s - (k-1)$ *if* $i = 1,\ldots,c-1$;

(b) $\beta^c(k,hs) = \begin{cases} \beta(k,s) & \textit{if } i = c \textit{ is odd}, \\ \beta(s+1-k,s) & \textit{if } i = c \textit{ is even}; \end{cases}$

(c) $\beta^i(k,hs) = (\beta(i,h)-1)s + k$ *if* $i = c+1,\ldots,h$.

Proof. As ith box is preceded by $(i-1)$ boxes with s elements each, the elements of ith box are given by $(i-1)s+k$, $k = 1,\ldots,s$.

As ith box is mapped into $\beta(i,h)$th box, it results that the s elements of the ith box are mapped into the s elements of the $\beta(i,h)$th box. In order to know the images of the s elements in the ith box, we have to consider where the ith box is located:

(a) ith box located in J_{R}, that is, $i = 1,\ldots,c-1$.

As f is strictly decreasing in J_{R}, the order of the elements in ith box are reverted from left to right after mapping into $\beta(i,h)$th box, that is,

$$(i-1)s + k \to \beta(i,h)s - (k-1) \ \text{ with } \ k = 1,\ldots,s,$$

so, $\beta^i(k,hs) = \beta(i,h)s - (k-1)$ with $k = 1,\ldots,s$ if $i = 1,\ldots,c-1$.

(b) ith box located in J_{L}, that is, $i = c+1,\ldots,h$.

As f is strictly increasing in J_{L}, the elements of ith box are mapped into the elements of $\beta(i,h)$th box conserving their relative order, that is,

$$(i-1)s + k \to (\beta(i,h)-1)s + k \quad \text{with } k = 1,\ldots,s,$$

so, $\beta^i(k,hs) = (\beta(i,h)-1)s + k$ with $k = 1,\ldots,s$ if $i = c+1,\ldots,h$.

(c) The ith box is H_c, the so-called central box. The proof splits into two steps:

(c1) c is odd. As C is odd, the number of points of O_h located in J_{R} is even (in [17] this is said as the R-parity of $I_1, \ldots, I_{h-1}$ is even [17, Def. 2], so, f^h has a maximum [17, Lemma 3] and then the link of a point of the central box with the next visiting point in this same box is given by β_s as we have just explained above. But the linking of these two points requires visiting all boxes before they connect between themselves. Therefore, as the number of O_h located in J_{R} is even, if we set off H_1 to reach H_c, we will have visited an even number of boxes located in J_{R}. Given that images of points located in J_{R}, where f is decreasing, are reverted from left to right and two reversion are equivalent to an identity, it results that the s elements of H_1 are linked with the s elements of H_c by the identity permutation I_s. So, we have to connect the central box with the first one by an unknown permutation, γ_s, such that $I_s \circ \gamma_s = \beta_s$. Then $\gamma_s = \beta_s$. So, the elements of the central box, given by $(c-1)s + k$, $k = 1, \ldots, s$, are mapped into the elements of first box by β_s, that is,

$$(c-1)s + k \to \beta(k, s) \quad \text{with } k = 1, \ldots, s,$$

so, $\beta^c(k, hs) = \beta(k, s)$ with $k = 1, \ldots, s$ if c is odd.

(c2) c is even. By a similar argument to the one given above, the elements of H_1 and H_c are linked by I_s^* given that there is an odd number of reversions. Furthermore, the link of a point of the central box with the next visiting point in this same box is given by $\overline{\beta}_s$ because f^h has a minimum [17] as we have just explained above. So, we have to connect the central box with the first one by an unknown permutation, γ_s, such that $I_s^* \circ \gamma_s = \overline{\beta}_s$. Then $I_s^* \circ I_s^* \circ \gamma_s = I_s^* \circ \overline{\beta}_s$. Since $\overline{\beta_s} = I_s^* \circ \beta_s \circ I_s^*$, we have $\gamma_s = \beta_s \circ I_s^*$. So, $\beta^c(k, hs) = \beta(s+1-k, s)$ with $k = 1, \ldots, s$ if c is even. □

Remark 3. Notice that, under conditions of Lemma 1, if O_{hs} is the composed supercycle of O_h with O_s, when $\beta(c, h) = 1$ with c even, its next visiting permutation $\beta_{hs} = (\beta(j, hs))$ is given by

$$\left(\begin{array}{c|c|c} (i-1)s + k & (c-1)s + k & (i-1)s + k \\ \underbrace{\beta(i,h)s - (k-1)}_{\substack{k=1,\ldots,s \\ i=1,\ldots,c-1}} & \underbrace{\beta(s+1-k, s)}_{\substack{k=1,\ldots,s \\ i=c}} & \underbrace{(\beta(i,h)-1)s + k}_{\substack{k=1,\ldots,s \\ i=c+1,\ldots,h}} \end{array} \right).$$

Notice also that if $i < c$, then $\beta^i(r+1, hs) = \beta^i(r, hs) - 1$ for all $r = 1, \ldots, s-1$, whereas if $i > c$, then $\beta^i(r+1, hs) = \beta^i(r, hs) + 1$ for all $r = 1, \ldots, s-1$, and that $\{\beta^c(r, hs)\}_{r=1,\ldots,s} \equiv \{1, \ldots, s\}$.

Our next step is to determine necessary and sufficient conditions in order to know whether a periodic orbit is compound or not. Below, an algorithm will be given to break-up periodic orbits into their constituent elements.

Remark 4. $[\,\cdot\,]$ means integer part of a real number.

Theorem 1. *Let O_q be a supercycle of a C^2-unimodal map f with the next visiting permutation $\beta_q = (\beta(1,q)\ \dots\ \beta(q,q))$, and $\beta(z,q) = 1$. Let $h, s \in \mathbb{N}$, be such that $q = hs$. O_q is the composition of two supercycles O_h and O_s if only if β_q is given for all $k = 1, \dots, s$ by*:

(a) $\beta^i(k,q) = \beta^i(1,q) - (k-1)$ *if* $i = 1, \dots, [z/s]$;

(b) $\beta^i(k,q) = \beta^i(1,q) + (k-1)$ *if* $i = [z/s]+2, \dots, h$;

(c) $\beta^i(k,q) = \begin{cases} \beta(k,s) & \textit{if } i = [z/s]+1 \textit{ is odd}, \\ \beta(s+1-k,s) & \textit{if } i = [z/s]+1 \textit{ is even}, \end{cases}$

where $\beta(k,s)$ is the kth element of a next visiting permutation, β_s, of an orbit with period s.

Proof. ($\Rightarrow$) Let O_q be the composition of two supercycles O_h and O_s. Let β_h and β_s be the next visiting permutations of O_h and O_s, respectively. As $\beta(z,q) = 1$ then $\beta([z/s]+1,h) = 1$. If $i \neq [z/s]+1$, by Lemma 1 we have

$$\beta^i(k,q) = \begin{cases} \beta(i,h)s - (k-1) & \text{if } i = 1, \dots, [z/s], \\ (\beta(i,h)-1)s + k & \text{if } i = [z/s]+2, \dots, h. \end{cases} \tag{4}$$

It follows from (4)

$$\beta^i(1,q) = \begin{cases} \beta(i,h)s & \text{if } i = 1, \dots, [z/s], \\ (\beta(i,h)-1)s + 1 & \text{if } i = [z/s]+2, \dots, h. \end{cases} \tag{5}$$

After substituting (5) in Eq. (4), we get for $i \neq [z/s]+1$

$$\beta^i(k,q) = \begin{cases} \beta^i(1,q) - (k-1) & \text{if } i = 1, \dots, [z/s], \\ \beta^i(1,q) + (k-1) & \text{if } i = [z/s]+2, \dots, h. \end{cases} \tag{6}$$

The case $i = [z/s]+1$ follows directly from (b) in Lemma 1.

($\Leftarrow$) We assume that β_q satisfies conditions (a)–(c) of Theorem 1 and want to proof that O_q is the composition of two supercycles O_h and O_s. For this, we will build up two next visiting permutations β_s and β_h whose composition is β_q.

We define $\beta_s = (\beta(1,s)\ \dots\ \beta(s,s))$, where

$$\beta(k,s) = \begin{cases} \beta^{[z/s]+1}(k,q) & \text{if } [z/s]+1 \text{ is odd}, \\ \beta^{[z/s]+1}(s+1-k,q) & \text{if } [z/s]+1 \text{ is even}. \end{cases} \tag{7}$$

As β_q verifies condition (c) in Theorem 1, it results that (7) is a next visiting permutations of a s-periodic orbit O_s.

Now we define $\beta_h = (\beta(1,h)\ \dots\ \beta(h,h))$ with

$$\beta(i,h) = \begin{cases} \beta((i-1)s+1,q)/s, & i = 1,\dots,[z/s], \\ 1, & i = [z/s]+1, \\ \beta((i-1)s+1,q)+(s-1)/s, & i = [z/s]+2,\dots,h. \end{cases} \tag{8}$$

In order to prove that β_h is a next visiting permutation, one of the things we have to prove is that the set $\{\beta(i,h);\ i = 1,\dots,h\}$ coincides with the set $\{1,\dots,h\}$. Let us study the different values of i in (8).

- Let $i = 1,\dots,[z/s]$. According to (8), it results

  $$\beta(i,h) = \frac{\beta((i-1)s+1,q)}{s}. \tag{9}$$

  Given that, for every $i = 1,\dots,h$, there exists only one $j \in \{1,\dots,h\}$ such that $\beta_q(H_i) = H_j$ (see Appendix), it results that

  $$\beta\big((i-1)s+1,q\big) = (j-1)s+r, \quad r = 1,\dots,s. \tag{10}$$

  Taking into account (9) and (10), in order to prove that $\beta(i,h)$ is a natural number, let us see that $\beta((i-1)s+1,q) = (j-1)s+s$. Let us assume it were false, that is,

  $$\beta\big((i-1)s+1,q\big) = (j-1)s+r \quad \text{for some } r = 1,\dots,s-1. \tag{11}$$

  Applying condition (a) for $k = s$ and taking into account Definition 4, it yields

  $$\beta\big((i-1)s+s,q\big) = \beta\big((i-1)s+1,q\big) - (s-1). \tag{12}$$

  Then from Eqs. (11) and (12) it results

  $$\beta\big((i-1)s+s,q\big) = (j-1)s+(r+1-s) \quad \text{for some } r = 1,\dots,s-1. \tag{13}$$

  From (13), given that $r+1-s \leqslant 0$, $\beta((i-1)s+s,q) \notin H_j$, which is in contradiction with $\beta_q(H_i) = H_j$ (see Appendix). So, $\beta((i-1)s+1,q) = (j-1)s+s$ and replacing it in (9), we obtain

  $$\beta(i,h) = \frac{\beta((i-1)s+1,q)}{s} = j \in \{1,\dots,h\}. \tag{14}$$

  According to (c) of this theorem, it holds $\beta_q(H_{[z/s]+1}) = H_1$. Given that $i \leqslant [z/s]$, it results that $j \neq 1$ in (14). Hence, $j \in \{2,\dots,h\}$.

- Let $i = [z/s]+2,\dots,h$. Taking into account Definition 4 and condition (b) of this theorem, it results from (8) that

  $$\beta(i,h) = \frac{\beta((i-1)s+s,q)}{s}. \tag{15}$$

As $\beta((i-1)s+s,q) = (j-1)s+s$ (proof is similar to the case $i \leqslant [z/s]$), it results from (15) that, for every $i \geqslant [z/s]+2$, there exists only one $j \in \{2,\dots,h\}$ such that $\beta(i,h) = j$. Furthermore, these $j \in \{2,\dots,h\}$ are different from those obtained for the case $i \leqslant [z/s]$ (because, for every $i = 1,\dots,h$, there exists only one $j \in \{1,\dots,h\}$ such that $\beta_q(H_i) = H_j$, see Appendix).

- Let $i = [z/s]+1$. According to (c) of this theorem, $\beta_q(H_{[z/s]+1}) = H_1$, that is, $j = 1$.

Consequently, the set $\{\beta(i,h);\ i = 1,\dots,h\}$ coincides with the set $\{1,\dots,h\}$.

Our final goal is to prove that O_q is the composition of O_h and O_s, that is, $O_q \equiv O_{hs}$.

We denote by O_h the h-periodic orbit whose next visiting permutation is given by β_h (see Eq. 8). We denote by O_s the orbit of period s, whose next visiting permutation is given by β_s (see Eq. 7).

According to Lemma 1, for $i \neq [z/s]+1$, it holds

$$\beta^i(k,hs) = \begin{cases} \beta(i,h)s-(k-1), & i = 1,\dots,[z/s], \\ (\beta(i,h)-1)s+k, & i = [z/s]+2,\dots,h. \end{cases} \tag{16}$$

By taking account (8), (16) is rewritten as

$$\beta^i(k,hs) = \begin{cases} \beta((i-1)s+1,q)-(k-1), & i = 1,\dots,[z/s], \\ \beta((i-1)s+1,q)+(k-1), & i = [z/s]+2,\dots,h. \end{cases} \tag{17}$$

According to Lemma 1, for $i = [z/s]+1$, it holds

$$\beta^{[z/s]+1}(k,hs) = \begin{cases} \beta(k,s), & i = [z/s]+1 \text{ is odd}, \\ \beta(s+1-k,s), & i = [z/s]+1 \text{ is even}. \end{cases} \tag{18}$$

By using (7), (18) is rewritten as

$$\beta^{[z/s]+1}(k,hs) = \begin{cases} \beta^{[z/s]+1}(k,q), & i = [z/s]+1 \text{ is odd}, \\ \beta^{[z/s]}(s+1-(s+1-k),q), & i = [z/s]+1 \text{ is even}, \end{cases} \tag{19}$$

so, $\beta^{[z/s]+1}(k,hs) = \beta^{[z/s]+1}(k,q)$.

By hypothesis of the theorem both O_{hs} and O_s are admissible orbits, it remains to be seen that O_h is also an admissible one. By construction the h first elements of the symbolic sequence of O_{hs} coincide with the symbolic sequence of O_h, therefore, by using shift operator and the kneading theory if O_h were not an admissible orbit neither O_{hs} would be $[1,2]$, that is a contradiction, consequently, O_h is an admissible orbit.

Therefore, $\beta_{hs} = \beta_q$. As β_s and β_h are the next visiting permutations of O_s and O_h, respectively, it yields that O_q is the composition of O_h and O_s. □

This decomposition of the logistic map orbits can be obtained from our theorem, taking into account that the visiting order permutation of a periodic orbit of the logistic map remains the same from the appearance of the orbit (in a period doubling bifurcation or in a saddle-node bifurcation) up to its disappearance (and, in particular, it remains the same for the supercycle where our theorem applies).

Notice that Theorem 1 not only provides the decomposition of a compound periodic orbit, but also it allows one to deduce what orbits are not decomposable because they do not originate from the composition of two orbits (see Example 1, below). Given that a compound orbit is associated with a window within a window (in the bifurcation diagram), a non decomposable orbit is associated with a primary window. This is a direct application of the theorem that distinguishes primary windows from windows within windows.

Translation of Theorem 1 to plain language is as follows:

1. Split the next visiting permutation β_q in h sets β_q^i, $i = 1, \dots, h$, with s elements each. The set β_q^i containing $\beta(z, q) = 1$ is denoted as β_q^c.
2. The sets β_q^i located to the left of β_q^c must satisfy condition (a) of Theorem 1. In plain language, condition (a) means that the elements in these sets are successively decreasing natural numbers.
3. The sets β_q^i located to the right of β_q^c must satisfy condition (b) of Theorem 1. In plain language, condition (b) means that the elements in these sets are successively increasing natural numbers.
4. The set β_q^c must satisfy condition (c) of Theorem 1. In plain language, condition (c) says that β_q^c determines the next visiting permutation of an admissible orbit of period s.

Let us see this in detail along the following examples.

Example 1. Let the visiting sequence of the 15-periodic orbit be given by

$$1 \to 15 \to 8 \to 7 \to 9 \to 6 \to 10 \to 5 \to 11 \to 4 \to 12 \to 3 \to 13 \to 2 \to 14,$$

so, its next visiting permutations is

$$\beta_{15} = \begin{pmatrix} 15 & 14 & 13 & 12 & 11 & 10 & 9 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 8 \end{pmatrix}. \tag{20}$$

β_{15} can be decomposed as $\beta_3 \circ \beta_5$ or $\beta_5 \circ \beta_3$.

(a) If $h = 3$ and $s = 5$, we split β_{15} as

$$\begin{array}{c|c|c} 15 \quad 14 \quad 13 \quad 12 \quad 11 & 10 \quad 9 \quad 7 \quad 6 \quad 5 & 4 \quad 3 \quad 2 \quad 1 \quad 8 \\ \beta_{15}^1 & \beta_{15}^2 & \beta_{15}^3 \end{array}.$$

$\beta_{15}^3 \equiv \beta_{15}^c$ because it contains the number 1. From β_{15}^3 it results $\beta^3(5, 15) = 8 > s = 5$. This is not the next visiting permutation of an admissible orbit of period 5, so, this decomposition is not possible.

(b) If $h = 5$ and $s = 3$, we split β_{15} as

$$\begin{array}{c|c|c|c|c} 15 \quad 14 \quad 13 & 12 \quad 11 \quad 10 & 9 \quad 7 \quad 6 & 5 \quad 4 \quad 3 & 2 \quad 1 \quad 8 \\ \beta_{15}^1 & \beta_{15}^2 & \beta_{15}^3 & \beta_{15}^4 & \beta_{15}^5 \end{array}.$$

$\beta^5_{15} \equiv \beta^c_{15}$ because it contains the number 1. The elements of β^3_{15} are not successively decreasing natural numbers, so, this decomposition is not possible either.

Consequently, the O_{15} orbit given is not a composed orbit.

Example 2. Let the visiting sequence of the 12-periodic orbit (see Fig. 3) be given by

$$1 \to 12 \to 8 \to 4 \to 9 \to 5 \to 3 \to 10 \to 6 \to 2 \to 11 \to 7,$$

so, its next visiting permutation of O_{12} is

$$\beta_{12} = \begin{pmatrix} 12 & 11 & 10 & 9 & 3 & 2 & 1 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}. \tag{21}$$

β_{12} could be decomposed as $\beta_2 \circ \beta_6$, $\beta_6 \circ \beta_2$, $\beta_4 \circ \beta_3$ or $\beta_3 \circ \beta_4$.

1. If $h = 2$ and $s = 6$, we split β_{12} as

$$\begin{array}{c|c} 12 \quad 11 \quad 10 \quad 9 \quad 3 \quad 2 & 1 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\ \beta^1_{12} & \beta^2_{12} \end{array}.$$

$\beta^2_{12} \equiv \beta^c_{12}$ because it contains the number 1. The elements of β^1_{12} are not successive, hence, this decomposition is not possible.

2. If $h = 6$ and $s = 2$, we split β_{12} as

$$\begin{array}{c|c|c|c|c|c} 12 \quad 11 & 10 \quad 9 & 3 \quad 2 & 1 \quad 4 & 5 \quad 6 & 7 \quad 8 \\ \beta^1_{12} & \beta^2_{12} & \beta^3_{12} & \beta^4_{12} & \beta^5_{12} & \beta^6_{12} \end{array}.$$

$\beta^4_{12} \equiv \beta^c_{12}$ because it contains the number 1. From β^4_{12} it results $\beta^4(2, 12) = 4 > s = 2$, so, this is not the visiting permutation of an admissible orbit of period 2. This decomposition is not possible.

3. If $h = 4$ and $s = 3$, we split β_{12} as

$$\begin{array}{c|c|c|c} 12 \quad 11 \quad 10 & 9 \quad 3 \quad 2 & 1 \quad 4 \quad 5 & 6 \quad 7 \quad 8 \\ \beta^1_{12} & \beta^2_{12} & \beta^3_{12} & \beta^4_{12} \end{array}.$$

$\beta^1_{12} \equiv \beta^c_{12}$ because it contains the number 1. The elements of β^2_{12} are not successive, so, the decomposition is still not possible.

4. If $h = 3$ and $s = 4$, we split β_{12} as

$$\begin{array}{c|c|c} 12 \quad 11 \quad 10 \quad 9 & 3 \quad 2 \quad 1 \quad 4 & 5 \quad 6 \quad 7 \quad 8 \\ \beta^1_{12} & \beta^2_{12} & \beta^3_{12} \end{array}.$$

$\beta^2_{12} \equiv \beta^c_{12}$ because it contains the number 1. The elements of β^1_{12} are successively decreasing natural numbers. The elements of β^3_{12} are successively increasing natural numbers. $\beta^2_{12} \equiv \beta^c_{12}$ determine an order 4 permutation. It is still left to determine β_3 and β_4 such that $\beta_{12} = \beta_3 \circ \beta_4$. This will be done below after the introduction of the corresponding algorithm (see Section 4).

Notice that the factorization of a natural number is not unique. For instance, the above compound 12-periodic orbit could be associated with: a 3-periodic window inside a 4-periodic one, a 4-periodic inside a 3-periodic, a 2-periodic inside a 6-periodic, or a 6-periodic inside the 2-chaotic band. Although we have not yet given a meaning to β_3 and β_4, the theorem gives the only admissible decomposition, that is, this 12-periodic orbit is located inside a 4-periodic window within a 3-periodic window.

4 Algorithm

The following corollary to Theorem 1 provides a decomposition algorithm for compound orbits.

Corollary 1. *Let O_q be a supercycle of a C^2-unimodal map f with the next visiting permutation β_q. Let z be such that $\beta(z,q) = 1$. If O_q is the result of composing two supercycles O_h and O_s, then the next visiting permutations β_h and β_s are given by*

$$\beta(k,s) = \begin{cases} \beta^{[z/s]+1}(k,q) & \text{if } [z/s]+1 \text{ is odd},\\ \beta^{[z/s]+1}(s+1-k,q) & \text{if } [z/s]+1 \text{ is even} \end{cases}$$

and

$$\beta(i,h) = \begin{cases} \beta((i-1)s+1,q)/s, & i = 1,\dots,[z/s],\\ 1 & i = [z/s]+1,\\ \beta((i-1)s+s,q)/s, & i = [z/s]+2,\dots,h. \end{cases}$$

This corollary is direct consequence from (7), (8) and (15).

Theorem 1 determines how β_q is decomposed as $\beta_q = \beta_h \circ \beta_s$. This corollary gives the explicit expression of β_h and β_s.

Algorithm says, in plain language:

1. Split β_q into h subsets β_q^i, $i = 1,\dots,h$, with s elements each. β_q^i containing $\beta(z,q) = 1$ will be denoted β_q^c.
2. The next visiting permutation β_s is given by the images of β_q^c ($\bar{\beta}_q^c$) if c is odd (even). Being $\bar{\beta}_q^c$ the mirror of β_q^c.
3. The next visiting permutation β_h is obtained as follows:
   (a) If β_q^i is placed to the left of β_q^c, divide the first element in β_q^i by s. Then assign to i (from β_q^i) the number thus obtained.
   (b) If β_q^i is placed to the right of β_q^c, divide the last element in β_q^i by s. Then assign to i (from β_q^i) the number thus obtained.
   (c) The number i such that $\beta_q^i \equiv \beta_q^c$ gets assigned to number 1.

   The permutation thus obtained will be the next visiting permutation β_h.

Example 3. According to Example 2, we know that the orbit O_{12}, whose next visiting permutation is

$$\beta_{12} = \begin{pmatrix} 12 & 11 & 10 & 9 & 3 & 2 & 1 & 4 & 5 & 6 & 7 & 8 \end{pmatrix},$$

can be splitted as $\beta_{12} = \beta_3 \circ \beta_4$. We want to determinate β_3 and β_4 by using Corollary 1 (by using the algorithm in plain language).

According to Example 2, $\beta_{12}^c \equiv \beta_{12}^2 = (3\ 2\ 1\ 4)$, the next visiting permutation is either

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

As $c = 2$ is even, we must take the second permutation, that is,

$$\beta_{s=4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

We also have to calculate β_h. According to the plain language algorithm, as $s = 4$, it results

$$\begin{array}{c|c|c}
12\quad 11\quad 10\quad 9 & 3\quad 2\quad 1\quad 4 & 5\quad 6\quad 7\quad 8 \\
\beta_{12}^1 & \beta_{12}^2 & \beta_{12}^3 \\
\Downarrow & \Downarrow & \Downarrow \\
i = 1 & i = 2 & i = 3 \\
\Downarrow & \Downarrow & \Downarrow \\
\dfrac{\text{First element}}{s} = \dfrac{12}{4} = 3 & 1 & \dfrac{\text{Last element}}{s} = \dfrac{8}{4} = 2 \\
\text{Box to the left of } \beta_{12}^c & \text{Central box} & \text{Box to the right of } \beta_{12}^c
\end{array}$$

So,

$$\beta_{h=3} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \longleftarrow i \quad .$$

Example 4. Let the next visiting permutation of the 12-periodic orbit O_{12} be given by

$$\beta_{12} = \begin{pmatrix} 12 & 11 & 10 & 9 & 8 & 7 & 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix}.$$

In a similar way as we did in Examples 1 and 2, we obtain the decomposition $\beta_{12} = \beta_4 \circ \beta_3$.

We write

$$\begin{array}{c|c|c|c}
12\quad 11\quad 10 & 9\quad 8\quad 7 & 3\quad 1\quad 2 & 4\quad 5\quad 6 \\
\beta_{12}^1 & \beta_{12}^2 & \beta_{12}^3 & \beta_{12}^4
\end{array}.$$

$\beta_{12}^3 \equiv \beta_{12}^c$ because it contains the number 1. As $c \equiv 3$ is odd, it results

$$\beta_{s=3} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

According to the plain language algorithm, as $s = 3$, it results

12 11 10	9 8 7	3 1 2	4 5 6
β_{12}^1	β_{12}^2	$\beta_{12}^c \equiv \beta_{12}^3$	β_{12}^4
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
$i = 1$	$i = 2$	$i = 3$	$i = 4$
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
$\frac{\text{First element}}{s} = \frac{12}{3} = 4$	$\frac{\text{First element}}{s} = \frac{9}{3} = 3$	1	$\frac{\text{Last element}}{s} = \frac{6}{3} = 2$
Box to the left of β_{12}^c	Box to the left of β_{12}^c	Central box	Box to the right of β_{12}^c

So,

$$\beta_{h=4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \begin{matrix} \longleftarrow i \\ \end{matrix} .$$

β_h is an orbit of a period doubling cascade, therefore, $\beta_4 \circ \beta_3$ represents an orbit of a saddle-node bifurcation cascade [15] located in the 2^2-chaotic band (see the $3 \cdot 2^2$ window in Fig. 1). In general, when β_h is an orbit of a period doubling cascade [16, 18], the $\beta_h \circ \beta_s$ represents an orbit of a saddle-node bifurcation cascade. However, with the same β_3 and β_4, the 12-periodic orbit given by $\beta_{12} = \beta_3 \circ \beta_4$ would correspond to a period-doubling cascade orbit within the 3-periodic window. This type of nuances are very important to understand the onset of chaos [19].

5 Conclusion

If we had a compound hs-periodic orbit, we could decompose it in two orbits of periods h and s, respectively, according to Theorem 1. This process is the opposite to that described in [17], where two orbits with periods h and s were composed to generate an hs-periodic orbit. Therefore, Theorem 1 closes the theoretical frame of composition and decomposition.

Theorem 1 states the necessary and sufficient conditions for the decomposition in simpler orbits. Meanwhile, Corollary 1 provides an algorithm for the computation of those orbits. As it was remarked in Section 3, Theorem 1 can be generalized using Byers' results in [4].

The decomposition theorems treated in this paper have an immediate application (through Poincaré section) to those continuous physical systems showing bifurcation diagrams similar that of Fig. 1.

Two periodic orbits (with h and s points in their respective Poincaré sections) can be composed into another periodic orbit having hs points in their Poincaré map in accordance with the composition theorem in [17] (or Lemma 1). Now the opposite result can also be achieved using Theorem 1.

An s-periodic orbit inside the h-periodic window must follow a visiting order in its Poincaré map that can be decomposed using decomposition Theorem 1: from a known periodic orbit another two unique orbits can be described. This link between periodic orbits (not only from simpler to more complex as studied in [17], but also from complex to simpler orbits as studied in this paper) imposes strong restrictions on a physical system dependent on one control parameter, whose underlying origin must be studied.

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Appendix

Theorem A. *Let O_q be an supercycle of a C^2-unimodal map f with the next visiting permutation $\beta_q = (\beta(1,q)\ \beta(2,q)\ \dots\ \beta(q,q))$. If β_q is given by*:

(a) $\beta^i(k,q) = \beta^i(1,q) - (k-1)$ *if* $i = 1, \dots, [z/s]$;

(b) $\beta^i(k,q) = \beta^i(1,q) + (k-1)$ *if* $i = [z/s]+2, \dots, h$;

(c) $\beta^i(k,q) = \begin{cases} \beta(k,s) & \text{if } i = [z/s]+1 \text{ is odd,} \\ \beta(s+1-k,s) & \text{if } i = [z/s]+1 \text{ is even} \end{cases}$

for all $k = 1, \dots, s$.

Then, for each $i = 1, \dots, h$, *there exists only one* $j \in \{1, \dots, h\}$ *such that*

$$\beta_q(H_i) = \bigl\{\beta\bigl((i-1)s+k,q\bigr);\ k = 1, \dots, s\bigr\} = \bigl\{(j-1)s+r;\ r = 1 \dots, s\bigr\} = H_j.$$

Furthermore,

$$\bigcup_{i=1}^{h} \beta_q(H_i) = \bigcup_{j=1}^{h} H_j = \{1, \dots, hs\}.$$

Proof. (i) Let $i = [z/s]+1$. From (c) it results that $\beta_q(H_i) = \beta_q(H_{[z/s]+1}) = H_1$.

(ii) Let $i \neq [z/s]+1$. The proof is by contradiction. We suppose that $\beta_q(H_i) \neq H_j$, $j = 1, \dots, h$.

Let $i < [z/s]+1$ (for $i > [z/s]+1$, the proof is similar). As $\beta_q(H_i) \neq H_j$ and β_q maps s successive elements to s successive elements (see item (a) in Theorem 1), it results

$$\bigl(\beta(i-1)s+1,\ q\bigr) \neq \dot{s} \quad \text{and} \quad \bigl(\beta(i-1)s+s,\ q\bigr) \neq \dot{s}$$

(where $\dot{s}$ denotes a multiple of s), consequently,

$$\bigl(\beta(i-1)s+1,q\bigr) = ns+r, \quad r < s,\ n,r \in \mathbb{N}. \tag{A.1}$$

Taking into account (A.1), item (a) in Theorem 1 and Definition 4, it results

$$\bigl(\beta(i-1)s+s,q\bigr) = ns+r-(s-1), \quad r < s,\ n,r \in \mathbb{N}. \tag{A.2}$$

As β_q takes every value in $\{1, 2, \dots, hs\}$, it results from (A.1) and (A.2) that

$$\{1, \dots, hs\} = A \cup \beta_q(H_i) \cup B,$$

where

$$A = \{1, \dots, ns + r - (s-1) - 1\}, \qquad B = \{ns + r + 1, \dots, hs\},$$
$$\beta_q(H_i) = \{ns + r - (s-1), \dots ns + r\}.$$

Notice that the cardinality of the sets A and B are, respectively, $(n-1)s + r$ and $(h-n)s - (r-1)$. Except for H_i, the images of the other boxes will be mapped into s successive elements either in A or in B (see (a) and (b) in Theorem A). Consequently, the elements of A and B will be exhausted but, for r elements in A and $s - (r-1)$ in B, therefore, the image of some boxes will not be formed by successive elements, which is in contradiction with the definition of β_q.

From items (i) and (ii) above it results

$$\bigcup_{i=1}^{h} \beta_q(H_i) = \bigcup_{j=1}^{h} H_j = \{1, \dots, hs\},$$

where it has been taken into account that, as β_q is a permutation for every $i = 1, \dots, h$, there exists only one $j = 1, \dots, h$ such that $\beta_q(H_i) = H_j$. □

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