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Necessary optimality conditions for optimal distributed and (Neumann) boundary control of Burgers equation in both fixed and free final horizon cases*

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Abstract. In this paper, we investigate the optimal control of the Burgers equation. For both optimal distributed and (Neumann) boundary control problems, the Dubovitskii and Milyutin functional analytical approach is adopted in investigation of the Pontryagin maximum principles of the systems. The necessary optimality conditions are, respectively, presented for two kinds of optimal control problems in both fixed and free final horizon cases, four extremum problems in all. Moreover, in free final horizon case, the assumptions of admissible control set on convexity and non-empty interior are removed so that it can be any set including an interesting case contains only finite many points. Finally, a remark on how to utilize the obtained results is also made for the illustration.

Keywords: distributed parameter system, optimal control, distributed/boundary control, Pontryagin's maximum principle, necessary optimality condition, Burgers equation.

1 Introduction

The Burgers equation, as one dimensional simple mathematical model for the convection-diffusion phenomena which are often governed by Navier–Stokes equations, is given the considerable investigations due to its importance in the fluids or combustion. It can also be used in modelling of gas dynamics, traffic flow as well as describing wave processes in acoustics and hydrodynamics.

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Since the past two decades, to impose the control on these phenomena above by the Burgers equation has become one of active topics in applied mathematics and engineering. Here, due to our emphasis on optimization problems for the said equation, more attention is paid to its optimal control observations. So far, in this aspect, lots of contributions are available. An optimal control framework for the viscous Burgers equation is constructed in [5] and initial results for both distributed as well as boundary control (Dirichlet and Neumann) are presented using a continuous-adjoint formulation. Burns and Marrekchi in [2] investigate the optimal fixed-finite-dimensional compensator problem for the Burgers equation with unbounded input/output operators. Agarwal et al. in [1] discuss the optimal and robust control of the Burgers equation with disturbance, in which the quadratic performance index is employed to find the optimal controller for the Burgers equation. Vedantham in [18] develops a technique to utilize the Cole–Hopf transformation to solve an optimal control problem for the Burgers equation. Adjoint techniques are studied in [14] for the optimal control of the Burgers equation with Neumann boundary control. By the optimal control techniques, Leredde et al. in [13] carry out the investigation for the Burgers equation and find the best parameters of the model which ensure the closest simulation to the observed values. de los Reyes and Kunisch [7] do the comparison of three different numerical methods for the constrained optimal control of the Burgers equation and develop the principal ideas of the different strategies considered. The suboptimal feedback control procedure is applied to the stochastic Burgers equation in [6] and several cases of controls are numerically simulated. Kucuk and Sadek in [12] give an efficient computational method for the optimal control problem for the Burgers equation.

In this paper, we are concerned with necessary optimality conditions for optimal control of the Burgers equations and four optimal control problems on the said equation are, respectively, investigated. They are optimal distributed control problem in fixed final horizon case, optimal distributed control problem in free final horizon case, optimal Neumann boundary control problem in fixed final horizon case, and optimal Neumann boundary control problem in free final horizon case. By the Dubovitskii and Milyutin functional analytical approach [10], we respectively obtain the Pontryagin maximum principles of the systems in these four cases. And the necessary optimality conditions are presented for the optimal control problems of the distributed parameter systems.

True enough, the feedback control of dynamical systems has many merits comparing to the open-loop control. However, an undeniable fact is that the latter, the open-loop control has its own advantages in investigation of infinite dimensional systems, such as the efficiency and accuracy of the open-loop control algorithms as well as the robustness aspect of investigational systems [15]. Just as Ho and Pepyne [11] said in “The No Free Lunch Theorem of Optimization (NFLT)”, a general-purpose universal optimization strategy is impossible. Therefore, the open-loop control investigation to the Burgers equation is both necessary and interesting.

Comparing with those existing references, this paper has some noticeable features deserving to be addressed here. Firstly, in two cases of distributed and boundary control, the cost functionals of optimal control problems are quite general and they contain most practically concerned ones like quadratic cost functional that is often adopted in previous observations of the Burgers equation. Secondly, we have the investigations for two kinds of control in free final horizon case, which is, to the best of our knowledge, new and never touched by people. Moreover, in this case, the assumptions for the cost functional are few and the cost functional does not need to be differentiable with respect to the control variable. The admissible control set does neither need to be convex nor contains interior points. In fact, it can be any set.

This paper is organized as follows. Next section, Section 2, contributes to the optimal distributed control problem formulation in fixed final horizon case. The weak solution issue of the system is recalled. The Dubovitskii–Milyutin theorem of optimal control problem in this case is presented. In the first three subsections of Section 3, which consists of four subsections, the cone of directions of decrease, the cone of feasible directions and the cone of tangent directions as well as their dual cones are derived, respectively. Section 3.4, the last subsection of Section 3, is devoted to the proof of the Pontryagin maximum principle of optimal distributed control problem in fixed final horizon case. In Section 4, the optimal distributed control system in free final horizon case is investigated and the corresponding Pontryagin maximum principle is obtained. The optimal (Neumann) boundary control problems are considered in Section 5, in which both fixed and free final horizon cases are investigated. The Pontryagin maximum principles in these two cases are, respectively, given by the same approach. Section 6 contributes to make an illustrative remark to show how to use the obtained maximum principle of the extremum problem. Finally, in Section 7, the section of conclusions, the main results obtained in this paper are highlighted and the general ideas which state these theorems are reviewed.

2 Optimal distributed control problem

Let $T > 0$, $Q_T = (0,T) \times (0,1)$, $V = H_0^1(0,1)$ and $H = L^2(0,1)$. Take the Hilbert space

$$W(0,T;V) = \{\gamma \mid \gamma \in L^2(V),\ \gamma_t \in L^2(V^*)\}$$

equipped with the norm

$$\|\gamma\|_{W(0,T;V)} = \sqrt{\|\gamma\|^2_{L^2(V)} + \|\gamma_t\|^2_{L^2(V^*)}},$$

where $V^* = H^{-1}(0,1)$ is the dual space of V.

Consider the following Burgers equation:

$$\begin{aligned} &y_t(t,x) - \nu y_{xx}(t,x) + y(t,x)y_x(t,x) = \tilde{f}(t,x) \in L^2(V^*), \\ &y(t,0) = y(t,1) = 0, \quad t \in [0,T] \text{ a.e.}, \\ &y(0,x) = y_0(x) \in H, \end{aligned} \tag{1}$$

in which $\nu > 0$ is the viscosity parameter.

For the nonlinear partial differential equations (1), Volkwein in [21] presents the definition of weak solution below and proves the existence of the unique weak solution $y(t,\cdot) \in W(0,T;V)$ as in the proof for the unsteady Navier–Stokes equations in [19].

A function $y(t,\cdot) \in W(0,T;V)$ is a weak solution to (1) if

$$\big\langle y_t(t,\cdot), \zeta(\cdot)\big\rangle_{V^*,V} + \nu\big\langle y(t,\cdot), \zeta(\cdot)\big\rangle_V + b\big(y(t,\cdot), y(t,\cdot), \zeta(\cdot)\big) = \big\langle \tilde{f}(t,\cdot), \zeta(\cdot)\big\rangle_{V^*,V} \quad (2)$$

for all $\zeta(\cdot) \in V$, $t \in [0,T]$ a.e. and $y(0,\cdot) = y_0(\cdot) \in H$, in which $b(\varphi,\psi,\varpi)$, the continuous trilinear form, is

$$b(\varphi,\psi,\varpi) = \frac{1}{3}\int_0^1 (\varphi\psi)'\varpi + \varphi\psi'\varpi \,\mathrm{d}x$$

for $\varphi, \psi, \varpi \in H^1(0,1)$. Here and thereafter $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ is the inner product of Hilbert space $\mathcal{X}$. Letting $\mathcal{Y}$ be a real normed linear space and $\mathcal{Y}^*$ its dual space, we have $\langle \bar{f}, \bar{v}\rangle_{\mathcal{Y}^*,\mathcal{Y}}$ for the duality pairing of $\bar{f} \in \mathcal{Y}^*$ and $\bar{v} \in \mathcal{Y}$.

Now consider the optimal control issues of the investigated system. Unless otherwise stated, in what follows when we speak of a solution of (1), we shall always mean the weak solution in the sense of (2).

Firstly, the attention is paid to optimal distributed control of system (1) in fixed final horizon case. For $T > 0$, take $\tilde{f}(t,x) = f(t,x) + u(t)$, in which $f(t,x) \in L^2(0,T;V^*)$ and $u(t) \in L^2(0,T)$ is the control. Consider an optimal control problem for system (1) with the general cost functional

$$\min_{u(\cdot)\in U_{\mathrm{ad}}} J(y,u) = \min_{u(\cdot)\in U_{\mathrm{ad}}} \int_0^T \int_0^1 L\big(y(t,x), u(t), t, x\big)\,\mathrm{d}x\,\mathrm{d}t \quad (3)$$

and the control constraint U_{ad} that is a non-empty closed convex set of $L^2(0,T)$. Here the cost function L is quite general in the sense that it contains most practically concerned ones like quadratic cost functional of the following form:

$$J(y,u) = \int_0^T \int_0^1 \ell_1\big|y(t,x) - y^\dagger(t,x)\big|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \ell_2\big|u(t) - u^\dagger(t)\big|^2\,\mathrm{d}t, \quad (4)$$

where $\ell_i > 0$, $i = 1,2$, are constants, and $y^\dagger$, $u^\dagger$ are, respectively, the predesigned optimal state and control, which is exactly the main object of [21]'s interest.

Take $y(t,\cdot) \in W(0,T;V)$. The control space is $L^2(0,T)$ and the control function satisfies a convex constraint $u(\cdot) \in U_{\mathrm{ad}}$. Here we assume that the set U_{ad} of admissible controls has the non-empty interior with respect to $L^2(0,T)$ topology, i.e., $\mathrm{int}_{L^2(0,T)}\, U_{\mathrm{ad}} \neq \emptyset$. Of course, this is the normal assumption on the admissible control set, which is often used in literatures [20]. Moreover, subsequently, one will see that this assumption on the non-empty interior, even that of the convexity of control set, will be removed in free final horizon case for both optimal distributed and boundary control problems. That means, in free final horizon case, the admissible control set does neither need to be convex nor contains interior points, which is usually regarded as the most difficult situation in extremum problems.

The following assumptions for the cost functional are assumed:

(a) L is a functional defined on $V \times U_{\mathrm{ad}} \times [0,T] \times [0,1]$ and

$$\frac{\partial L(y(t,x),u(t),t,x)}{\partial y}, \qquad \frac{\partial L(y(t,x),u(t),t,x)}{\partial u}$$

exist for every $(y,u) \in V \times U_{\mathrm{ad}}$ and L is continuous in its variables.

(b) $$\int_0^1 \left|\frac{\partial L(y(t,x),u(t),t,x)}{\partial y}\right| \mathrm{d}x, \qquad \int_0^1 \left|\frac{\partial L(y(t,x),u(t),t,x)}{\partial u}\right| \mathrm{d}x$$

are bounded for $t \in [0,T]$. In addition, we assume the existence of optimal control here and keep this assumption in other three cases.

Define $X_T = W(0,T;V) \times L^2(0,T)$. Let (y^*,u^*) be the optimal solution to the control problem (3) subject to equation (1). Set

$$\begin{aligned}\varOmega_1 &= \bigl\{(y,u) \in X_T \bigm| u(t) \in U_{\mathrm{ad}},\ t \in [0,T] \text{ a.e.}\bigr\},\\ \varOmega_2 &= \bigl\{(y,u) \in X_T \bigm| y_t(t,x) - \nu y_{xx}(t,x) + y(t,x)y_x(t,x) = f(t,x) + u(t),\\ &\qquad y(t,0) = y(t,1) = 0,\ y(0,x) = y_0(x),\ y(T,x) = y^*(T,x)\bigr\}.\end{aligned}$$

Then problem (3) is equivalent to questing for $(y^*,u^*) \in \varOmega = \varOmega_1 \cap \varOmega_2$ such that

$$J(y^*,u^*) = \min_{(y,u)\in\varOmega} J(y,u). \tag{5}$$

It is seen that problem (5) is an extremum problem on the constraint $\varOmega_1$ and the equality constraint $\varOmega_2$. In this situation, the Dubovitskii and Milyutin functional analytical approach has been turned out to be very powerful to solve such kind of extremum problems (see, e.g., [3, 4, 16, 17]). The general Dubovitskii and Milyutin theorem for problem (5) can be stated as the following theorem.

Theorem 1 (Dubovitskii–Milyutin). *Suppose the functional $J(y,u)$ assumes a minimum at the point (y^*,u^*) in $\varOmega$. Assume that $J(y,u)$ is regularly decreasing at (y^*,u^*) with the cone of directions of decrease K_0 and the constraint $\varOmega_1$ is regular at (y^*,u^*) with the cone of feasible directions K_1; and that the equality constraint $\varOmega_2$ is also regular at (y^*,u^*) with the cone of tangent directions K_2. Then there exist continuous linear functionals f_0, f_1, f_2, not all identically zero, such that $f_i \in K_i^*$, the dual cone of K_i, $i = 0,1,2$, which satisfy the condition*

$$f_0 + f_1 + f_2 = 0. \tag{6}$$

3 The Pontryagin maximum principle

In this section, we are interested in optimal distributed control problem (3) in fixed final horizon case. To apply Theorem 1 and obtain the Pontryagin maximum principle, we proceed as follows: to determine all cones K_i, $i = 0, 1, 2$, and their dual cones one by one; by equation (6), to derive the final result step by step. First of all, let us find the cone of directions of decrease K_0.

3.1 The cone of directions of decrease K_0

By assumption, $J(y, u)$ is differentiable at any point (y^0, u^0) in any direction (y, u) and its directional derivative is

$$\begin{aligned} J'(y^0, u^0; y, u) &= \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\bigl[J\bigl(y^0+\varepsilon y, u^0+\varepsilon u\bigr) - J\bigl(y^0, u^0\bigr)\bigr] \\ &= \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\left\{\int_0^T\int_0^1 \bigl[L\bigl(y^0+\varepsilon y, u^0+\varepsilon u, t, x\bigr) - L\bigl(y^0, u^0, t, x\bigr)\bigr]\,\mathrm{d}x\,\mathrm{d}t\right\} \\ &= \int_0^T\int_0^1 \left[\frac{\partial L(y^0, u^0, t, x)}{\partial y}y + \frac{\partial L(y^0, u^0, t, x)}{\partial u}u\right]\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Hence the cone of directions of decrease of the functional $J(y, u)$ at point (y^*, u^*) is determined by

$$\begin{aligned} K_0 &= \bigl\{(y, u) \in X_T \bigm| J'\bigl(y^*, u^*; y, u\bigr) < 0\bigr\} \\ &= \left\{(y, u) \in X_T \;\middle|\; \int_0^T\int_0^1 \left[\frac{\partial L(y^*, u^*, t, x)}{\partial y}y + \frac{\partial L(y^*, u^*, t, x)}{\partial u}u\right]\mathrm{d}x\,\mathrm{d}t < 0\right\}. \end{aligned}$$

If $K_0 \neq \emptyset$, then, for any $f_0 \in K_0^*$, there exists a $\kappa_0 \geqslant 0$ such that

$$f_0(y, u) = -\kappa_0 \int_0^T\int_0^1 \left[\frac{\partial L(y^*, u^*, t, x)}{\partial y}y + \frac{\partial L(y^*, u^*, t, x)}{\partial u}u\right]\mathrm{d}x\,\mathrm{d}t.$$

3.2 The cone of feasible directions K_1

Since $\Omega_1 = W(0, T; V) \times U_{\mathrm{ad}}$, in which $\mathrm{int}_{L^2(0,T)}\, U_{\mathrm{ad}} \neq \emptyset$, so the interior of Ω_1 is not empty, i.e., $\mathring{\Omega}_1 \neq \emptyset$ and, at point (y^*, u^*), the cone of feasible directions K_1 of Ω_1 is determined by

$$\begin{aligned} K_1 &= \bigl\{\kappa\bigl(\mathring{\Omega}_1 - (y^*, u^*)\bigr) \bigm| \kappa > 0\bigr\} \\ &= \bigl\{h \bigm| h = \kappa(y - y^*, u - u^*),\ (y, u) \in \mathring{\Omega}_1,\ \kappa > 0\bigr\}. \end{aligned}$$

Therefore, for an arbitrary $f_1 \in K_1^*$, if there is an $\bar{a}(t) \in L^2(0, T)$ such that the linear functional defined by

$$f_1(y, u) = \int_0^T \bar{a}(t)u(t)\,\mathrm{d}t$$

is a support to $\tilde{\Omega}_1$ at point u^*, then, for all $u(t) \in U_{\mathrm{ad}}$, $t \in [0,T]$ a.e.,

$$\bar{a}(t)\big[u(t) - u^*(t)\big] \geqslant 0. \tag{7}$$

3.3 The cone of tangent directions K_2

Define the operator $G : X_T \to L^2(0,T;V^*) \times (L^2(0,T))^2 \times (H)^2$ by

$$G(y,u) = \big(\vartheta(t,x),\ y(t,0),\ y(t,1),\ y(0,x) - y_0(x),\ y(T,x) - y^*(T,x)\big),$$

in which $\vartheta(t,x) = y_t(t,x) - \nu y_{xx}(t,x) + y(t,x)y_x(t,x) - f(t,x) - u(t)$. Then

$$\Omega_2 = \big\{(y,u) \in X_T \;\big|\; G\big(y(t,x),u(t)\big) = 0\big\}.$$

The Fréchet derivative of the operator $G(y,u)$ is

$$G'(y,u)(\hat{y},\hat{u}) = \big(\hat{\vartheta}(t,x), \hat{y}(t,0), \hat{y}(t,1), \hat{y}(0,x), \hat{y}(T,x)\big),$$

in which $\hat{\vartheta}(t,x) = \hat{y}_t(t,x) - \nu\hat{y}_{xx}(t,x) + y(t,x)\hat{y}_x(t,x) + \hat{y}(t,x)y_x(t,x) - \hat{u}(t)$.

Since (y^*,u^*) is the solution to problem (3), it has $G(y^*,u^*) = 0$. Choosing arbitrary

$$(g_1,g_2,g_3,g_4,g_5) \in L^2(0,T;V^*) \times \big(L^2(0,T)\big)^2 \times (H)^2$$

and solving the equation

$$G'(y^*,u^*)(\hat{y},\hat{u}) = \big(g_1(t,x), g_2(t), g_3(t), g_4(x), g_5(x)\big),$$

we obtain

$$\begin{aligned} &\hat{y}_t(t,x) - \nu\hat{y}_{xx}(t,x) + y^*(t,x)\hat{y}_x(t,x) + y^*_x(t,x)\hat{y}(t,x) - \hat{u}(t) = g_1(t,x),\\ &\hat{y}(t,0) = g_2(t), \quad \hat{y}(t,1) = g_3(t), \qquad \hat{y}(0,x) = g_4(x), \quad \hat{y}(T,x) = g_5(x). \end{aligned} \tag{8}$$

Next, assume that the linearized system

$$\begin{aligned} &y_t(t,x) - \nu y_{xx}(t,x) + y^*(t,x)y_x(t,x) + y^*_x(t,x)y(t,x) = u(t),\\ &y(t,0) = y(t,1) = 0, \qquad y(0,x) = 0 \end{aligned} \tag{9}$$

is controllable. (People can refer to [3, 4] for the information of the linearization.) Then choose $u(t) = \hat{u}(t) \in L^2(0,T)$ such that $y(T,x) = g_5(x) - \xi(T,x)$ and let y be the solution to the linearized system (9). Choose $\hat{y}(t,x) = y(t,x) + \xi(t,x)$, where ξ satisfies the following equations:

$$\begin{aligned} &\xi_t(t,x) - \nu\xi_{xx}(t,x) + y^*(t,x)\xi_x(t,x) + y^*_x(t,x)\xi(t,x) = g_1(t,x),\\ &\xi(t,0) = g_2(t), \quad \xi(t,1) = g_3(t), \qquad \xi(0,x) = g_4(x). \end{aligned}$$

In this way, it suffices for $(\hat{y},\hat{u})$ satisfying (8). Therefore, $G'(y^*,u^*)$ maps the space X_T onto $L^2(0,T;V^*) \times (L^2(0,T))^2 \times (H)^2$. Moreover, the cone of the tangent directions

K_2 to the constraint Ω_2 at point (y^*, u^*) consists of the kernel of $G'(y^*, u^*)$, i.e., (y, u) satisfies the following equations in X_T:

$$\begin{aligned} &y_t(t,x) - \nu y_{xx}(t,x) + y^*(t,x) y_x(t,x) + y^*_x(t,x) y(t,x) = u(t), \\ &y(t,0) = y(t,1) = 0, \qquad y(0,x) = 0 \end{aligned} \tag{10}$$

and

$$y(T,x) = 0. \tag{11}$$

Define

$$\begin{aligned} K_{21} &= \{(y,u) \in X_T \mid (y(t,x), u(t)) \text{ satisfies (10)}\}, \\ K_{22} &= \{(y,u) \in X_T \mid (y(t,x), u(t)) \text{ satisfies (11)}\}. \end{aligned}$$

Then the cone of tangent directions $K_2 = K_{21} \bigcap K_{22}$. Hence

$$K_2^* = K_{21}^* + K_{22}^*.$$

For any $f_2 \in K_2^*$, decompose $f_2 = f_{21} + f_{22}$, $f_{2i} \in K_{2i}^*$, the dual cone of K_{2i}, $i = 1, 2$. Then $f_{21}(y, u) = 0$ and, for all $y(t, x) \in W(0, T; V)$ satisfying $y(T, x) = 0$, there exists $\phi(x) \in V^*$ such that

$$f_{22}(y(t,x), u(t)) = \int_0^1 y(T,x)\phi(x)\,\mathrm{d}x.$$

Then from Theorem 1 follows that there exist continuous linear functionals, not all identically zero, such that

$$f_0 + f_1 + f_{21} + f_{22} = 0.$$

Therefore, when selecting (y, u) satisfies (10), $f_{21}(y, u) = 0$. Moreover,

$$\begin{aligned} f_1(y(t,x), u(t)) &= -f_0(y(t,x), u(t)) - f_{22}(y(t,x), u(t)) \\ &= \kappa_0 \int_0^T \int_0^1 \left[\frac{\partial L(y^*, u^*, t, x)}{\partial y} y(t,x) + \frac{\partial L(y^*, u^*, t, x)}{\partial u} u(t) \right] \mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^1 y(T,x)\phi(x)\,\mathrm{d}x. \end{aligned}$$

3.4 Maximum principle of problem (3)

Define the adjoint system of (9) as

$$\begin{aligned} &z_t(t,x) + \nu z_{xx}(t,x) + y^*(t,x) z_x(t,x) = \kappa_0 \frac{\partial L(y^*(t,x), u^*(t), t, x)}{\partial y}, \\ &z(t,0) = z(t,1) = 0, \qquad z(T,x) = \phi(x). \end{aligned} \tag{12}$$

Theorem 2. *The solution of system* (9) *and that of its adjoint system* (12) *have the following relationship*:

$$\kappa_0 \int_0^T \int_0^1 \frac{\partial L(y^*, u^*, t, x)}{\partial y} y(t, x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^1 y(T, x)\phi(x)\,\mathrm{d}x = -\int_0^T \int_0^1 z(t, x)u(t)\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Multiply the first equation of (12) by $y(t, x)$ and integrate the product by parts over $[0, T] \times [0, 1]$ with respect to t and x, respectively. The proof then follows. □

Now, by virtue of Theorem 2, we can rewrite $f_1(y, u)$ as

$$f_1(y, u) = \int_0^T \left\{ \int_0^1 \left[\kappa_0 \frac{\partial L(y^*, u^*, t, x)}{\partial u} - z(t, x) \right] \mathrm{d}x \right\} u(t)\,\mathrm{d}t.$$

Therefore,

$$\bar{a}(t) = \int_0^1 \left[\kappa_0 \frac{\partial L(y^*, u^*, t, x)}{\partial u} - z(t, x) \right] \mathrm{d}x$$

and (7) then reads

$$\left\{ \int_0^1 \left[\kappa_0 \frac{\partial L(y^*, u^*, t, x)}{\partial u} - z(t, x) \right] \mathrm{d}x \right\} \left[u(t) - u^*(t) \right] \geqslant 0 \tag{13}$$

for all $u(t) \in U_{\mathrm{ad}}$, $t \in [0, T]$ a.e., where κ_0 and $z(t, x)$ are not identical to zero simultaneously. Since otherwise, there are definitely $f_0 = 0$, $f_1 = 0$, $f_{22} = 0$ and $f_{21} = 0$, which contradict with the fact in Theorem 1 that these continuous linear functionals are not all identically zero.

On the other hand, if K_0 is a null set, then, for all $(y, u) \in X_T$,

$$\int_0^T \int_0^1 \left[\frac{\partial L(y^*, u^*, t, x)}{\partial y} y(t, x) + \frac{\partial L(y^*, u^*, t, x)}{\partial u} u(t) \right] \mathrm{d}x\,\mathrm{d}t = 0.$$

In particular, if we choose $\kappa_0 = 1$ and $\phi(x) = 0$, then from Theorem 2 follows that

$$\int_0^T \int_0^1 \frac{\partial L(y^*, u^*, t, x)}{\partial y} y(t, x)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_0^1 z(t, x)u(t)\,\mathrm{d}x\,\mathrm{d}t.$$

Therefore, for all $u(t) \in L^2(0, T)$,

$$\int_0^T \left\{ \int_0^1 \left[\frac{\partial L(y^*, u^*, t, x)}{\partial u} - z(t, x) \right] \mathrm{d}x \right\} u(t)\,\mathrm{d}t = 0,$$

from which we obtain

$$\int_0^1 \left[\frac{\partial L(y^*, u^*, t, x)}{\partial u} - z(t,x)\right] \mathrm{d}x = 0$$

for all $t \in [0,T]$ a.e. Therefore, (13) still holds.

Finally, if there is a nonzero solution $\hat{z}(t,x)$ (in which case, $\hat{z}(T,x) \not\equiv 0$) to the adjoint system

$$\begin{aligned} &\hat{z}_t(t,x) + \nu \hat{z}_{xx}(t,x) + y^*(t,x)\hat{z}_x(t,x) = \kappa_0 \frac{\partial L(y^*(t,x), u^*(t), t, x)}{\partial y}, \\ &\hat{z}(t,0) = \hat{z}(t,1) = 0 \end{aligned} \tag{14}$$

such that, for all $t \in [0,T]$ a.e.,

$$\int_0^1 \hat{z}(t,x)\,\mathrm{d}x = 0,$$

then if we choose $\kappa_0 = 0$ and $\phi(x) = \hat{z}(T,x)$, (13) is still valid. Since otherwise, if for any nonzero solution $\hat{z}$ of (14), it has

$$\int_0^1 \hat{z}(t,x)\,\mathrm{d}x \not\equiv 0,$$

in this case, we say the situation is non-degenerate. Then the linearized system (9) is controllable. In fact, if (9) is not controllable, then there exists a $\phi(x) \in V^*$ such that

$$\int_0^1 y(T,x)\phi(x)\,\mathrm{d}x = 0, \quad \phi(x) \not\equiv 0.$$

Choose $\kappa_0 = 0$, $\hat{z}$ to be the solution of (14). Then it follows from Theorem 2 that, for all $u(t) \in L^2(0,T)$,

$$\int_0^T \left[\int_0^1 \hat{z}(t,x)\,\mathrm{d}x\right] u(t)\,\mathrm{d}t = 0.$$

Hence

$$\int_0^1 \hat{z}(t,x)\,\mathrm{d}x = 0$$

for all $t \in [0,T]$ a.e. This is a contradiction. Therefore, under the case of (14), system (9) is controllable.

Combining the results above, we have obtained the Pontryagin maximum principle for problem (3) subject to system (1).

Theorem 3. *Suppose* (y^*, u^*) *is a solution to the optimal control problem* (3)*. Then there exist* $\kappa_0 \geqslant 0$ *and* $z(t,x)$*, not identically zero, such that the following maximum principle holds true*:

$$\left\{ \int_0^1 \left[\kappa_0 \frac{\partial L(y^*, u^*, t, x)}{\partial u} - z(t,x) \right] \mathrm{d}x \right\} \big[u(t) - u^*(t) \big] \geqslant 0, \tag{15}$$

$u(t) \in U_{\mathrm{ad}}$, $t \in [0,T]$ *a.e., where the function* $z(t,x)$ *satisfies the adjoint equation* (12).

4 Free final horizon case

In the preceding section, we give the Pontryagin maximum principle for optimal distributed control problem of system (1) with fixed final horizon. Those results were derived under two additional conditions. The first one is that the admissible control set U_{ad} must be convex and contains interior points, i.e., $\mathrm{int}_{L^2(0,T)}\, U_{\mathrm{ad}} \neq \emptyset$, and the second requires the cost functional to be differentiable with respect to the control variable. In this section, we consider optimal distributed control of the system with free final time without these assumptions.

Consider the following control system defined in the fixed domain $[0, t_1] \times [0,1]$:

$$\begin{aligned} &y_t(t,x) - \nu y_{xx}(t,x) + y(t,x) y_x(t,x) = f(t,x) + u(t), \\ &y(t,0) = y(t,1) = 0, \qquad y(0,x) = y_0(x), \quad y(t_1,x) = y_1(x), \\ &(t,x) \in Q_{t_1} = (0,t_1) \times (0,1),\ t_1 > 0,\ u \in M \subset \mathbb{R}, \end{aligned} \tag{16}$$

and formulate the optimal control Problem I below. Surely it is worth emphasizing the cancellation of assumptions imposed on the preceding fixed final horizon problem. That is to say, in this section, the admissible control set M neither need be convex nor contains interior points as well as the cost functional $L(y,u)$ need not be differentiable with respect to the control variable u. Therefore, M can be any set. An interesting case is that M is allowed to contain only finite many points. The optimal distributed control problem with free final horizon t_1 is presented as follows.

Problem I. Minimize

$$J(y,u) = \int_0^{t_1} \int_0^1 L\big(y(t,x), u(t)\big)\, \mathrm{d}x\, \mathrm{d}t \tag{17}$$

for $y(t,x) \in W(0,t_1;V)$, $u(t) \in L^2(0,t_1)$ under constraints (16), where the functional L defined on $V \times \mathbb{R}$ satisfies:

(c) $L(y,u)$ is continuous in u;

(d) $|\partial L(y,u)/\partial y|$ is bounded for every bounded subset of $V \times \mathbb{R}$.

In this section, we will derive the Pontryagin maximum principle of Problem I with free final horizon. Introduce a time transformation $t \to s$, mapping $[0,t_1]$ onto $[0,1]$, defined by a certain function $v(\cdot) \geqslant 0$,

$$t(s) = \int_0^s v(\varsigma)\,\mathrm{d}\varsigma, \quad t(1) = t_1,$$

and let $y(s,x) = y(t(s),x)$,

$$u(s) = \begin{cases} u(t(s)), & s \in \Xi_1 = \{s \mid s \in [0,1],\ v(s) > 0\}, \\ \text{arbitrary}, & s \in \Xi_2 = \{s \mid s \in [0,1],\ v(s) = 0\}. \end{cases} \tag{18}$$

Then $(y(s,x), u(s))$ satisfies the following equations:

$$\begin{aligned} &y_s(s,x) - \nu y_{xx}(s,x)v(s) + y(s,x)y_x(s,x)v(s) = f(s,x)v(s) + g(s,x)u(s)v(s), \\ &y(s,0)v(s) = y(s,1)v(s) = 0, \qquad y(0,x) = y_0(x), \quad y(1,x) = y_1(x), \end{aligned} \tag{19}$$

where

$$f(s,x) = f\big(t(s),x\big), \qquad g(s,x) = g\big(t(s),x\big).$$

To make the definition of $s(t)$ one-to-one, we shall assume that

$$s(t) = \inf\big\{s \;\big|\; t(s) = t\big\}.$$

And then we can formulate a new problem.

Problem II. Minimize

$$J(y,u,v) = \int_0^1 \int_0^1 v(s) L\big(y(s,x), u(s)\big)\,\mathrm{d}x\,\mathrm{d}s$$

for $y(s,x) \in W(0,1;V)$, $u(s) \in L^2(0,1)$, $v(s) \in L^\infty(0,1)$ under constraints (19) with $v(s) \geqslant 0$, $u(s) \in M$ for almost all $0 \leqslant s \leqslant 1$.

If (y^*, u^*) is an optimal solution to the control problem (17) subject to equations (16), then, for any $v^*(s) \geqslant 0$ satisfying $\int_0^1 v^*(\varsigma)\,\mathrm{d}\varsigma = t_1$, $u^*(s)$ defined similar to (18), (y^*, u^*, v^*) solves Problem II [10]. Fixing $u = u^*$, another optimal control problem can be formulated as follows.

Problem III. Minimize

$$J(y,u^*,v) = \int_0^1 \int_0^1 v(s) L\big(y(s,x), u^*(s)\big)\,\mathrm{d}x\,\mathrm{d}s$$

for $(y(s,x), v(s)) \in X_1 = W(0,1;V) \times L^\infty(0,1)$ subject to

$$\begin{aligned} &y_s(s,x) - \nu y_{xx}(s,x)v(s) + y(s,x)y_x(s,x)v(s) = f(s,x)v(s) + g(s,x)u^*(s)v(s), \\ &y(s,0)v(s) = y(s,1)v(s) = 0, \qquad y(0,x) = y_0(x), \quad y(1,x) = y_1(x), \end{aligned}$$

in which $v(s)$ plays the role of control. Now we can observe that Problem III is an optimal control problem with fixed final horizon, which can be tackled by the same method adopted in the investigation of the preceding optimal control problem (3). As such, in this case, we can derive the corresponding theorem similar to Theorem 1. Those continuous linear functionals in that theorem can, respectively, be determined as

$$f_0(y,u^*,v) = -\kappa_0 \int_0^1 \int_0^1 \left[v^*(s) \frac{\partial L(y^*,u^*)}{\partial y} y(s,x) + L(y^*,u^*)v(s) \right] \mathrm{d}x\,\mathrm{d}s,$$
$$f_1(y,u^*,v) = \int_0^1 \bar{a}(s)v(s)\,\mathrm{d}s,$$
$$f_{21}(y,u^*,v) = 0, \qquad f_{22}(y,u^*,v) = \int_0^1 y(1,x)\psi(x)\,\mathrm{d}x,$$

in which there exist $\kappa_0 \geqslant 0$, $\bar{a}(s) \in L(0,1)$, and $\psi(x) \in V^*$ such that the expressions above hold. Here we still adopt the same symbols to denote these functionals in the case of no confusions caused. And the linearized system can be read as

$$\begin{aligned} &y_s(s,x) - \nu v^*(s) y_{xx}(s,x) + y^*(s,x)v^*(s)y_x(s,x) + y_x^*(s,x)v^*(s)y(s,x) \\ &\quad = \left[\nu y_{xx}^*(s,x) - y^*(s,x)y_x^*(s,x) + f(s,x) + g(s,x)u^*(s)\right]v(s), \\ &y(s,0)v^*(s) + y^*(s,0)v(s) = 0, \quad y(s,1)v^*(s) + y^*(s,1)v(s) = 0, \\ &y(0,x) = 0. \end{aligned} \tag{20}$$

Correspondingly, its adjoint system is

$$\begin{aligned} &z_s(s,x) + \nu v^*(s) z_{xx}(s,x) + y^*(s,x)v^*(s)z_x(s,x) = \kappa_0 v^*(s) \frac{\partial L(y^*,u^*)}{\partial y}, \\ &z(s,0) = z(s,1) = 0, \qquad z(1,x) = \psi(x). \end{aligned} \tag{21}$$

Moreover, the relationship between the solution of the linearized system (20) and that of its adjoint system (21) is

$$\begin{aligned} &\int_0^1 \int_0^1 \kappa_0 v^*(s) \frac{\partial L(y^*,u^*)}{\partial y} y(s,x)\,\mathrm{d}x\,\mathrm{d}s - \int_0^1 y(1,x)\psi(x)\,\mathrm{d}x \\ &\quad = -\int_0^1 \int_0^1 \left[\nu y_{xx}^*(s,x) - y^*(s,x)y_x^*(s,x) + f(s,x) + g(s,x)u^*(s)\right] z(s,x)v(s)\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad - \int_0^1 \nu\left[y^*(s,1)z_x(s,1) - y^*(s,0)z_x(s,0)\right]v(s)\,\mathrm{d}s. \end{aligned}$$

After we get the Pontryagin maximum principle of Problem III, the maximum principle of Problem I with free final horizon can be obtained easily. The obtained result can be stated as the following Theorem 4, which is none other than the Pontryagin maximum principle of Problem I with free final horizon.

Theorem 4. *Suppose* (y^*, u^*, t_1) *is a solution to Problem* I*, then there exist* $\kappa_0 \geqslant 0$ *and* $z(t,x)$*, not identically zero, such that, for all* $t \in [0, t_1]$ *a.e.,*

$$\begin{aligned}&\int_0^1 \bigl\{\kappa_0 L\bigl(y^*(t,x), u^*(t)\bigr) - \bigl[\nu y^*_{xx}(t,x) - y^*(t,x) y^*_x(t,x) + f(t,x) + u^*(t)\bigr] z(t,x)\bigr\}\,\mathrm{d}x \\ &\quad - \nu y^*(t,1) z_x(t,1) + \nu y^*(t,0) z_x(t,0) = 0, \\ &\int_0^1 \bigl\{\kappa_0 L\bigl(y^*(t,x), u^*(t)\bigr) - \bigl[\nu y^*_{xx}(t,x) - y^*(t,x) y^*_x(t,x) + f(t,x) + u\bigr] z(t,x)\bigr\}\,\mathrm{d}x \\ &\quad - \nu y^*(t,1) z_x(t,1) + \nu y^*(t,0) z_x(t,0) \geqslant 0 \quad \forall u \in M,\end{aligned}$$

where the function $z(t,x)$ *satisfies*

$$\begin{aligned}&z_t(t,x) + \nu z_{xx}(t,x) + y^*(t,x) z_x(t,x) = \kappa_0 \frac{\partial L(y^*, u^*)}{\partial y}, \quad (t,x) \in Q_{t_1}, \\ &z(t,0) = z(t,1) = 0, \quad z(t_1, x) = \psi(x).\end{aligned}$$

5 Optimal (Neumann) boundary control problems

In this section, we consider the optimal (Neumann) boundary control problems of the Burgers equation in both fixed and free final horizon cases. The investigated model is

$$\begin{aligned}&y_t(t,x) - \nu y_{xx}(t,x) + y(t,x) y_x(t,x) = f(t,x) \in L^2(\tilde{V}^*), \\ &y_x(t,0) = \alpha(t), \quad y_x(t,1) = \beta(t), \quad t \in [0,T] \text{ a.e.}, \\ &y(0,x) = y_0(x) \in H.\end{aligned} \tag{22}$$

Here $\tilde{V} = H^1(0,1)$ and its dual space $\tilde{V}^* = \mathrm{BMO}(0,1)$ that is the space of functions of bounded mean oscillation [8]. Two boundary control variables $\alpha(\cdot), \beta(\cdot) \in L^2(0,T)$. Introduce the definition of weak solution to (22) from [21] as follows.

A function $y(t,\cdot) \in W(0,T;\tilde{V})$ is called a weak solution to (22) if $y(0,\cdot) = y_0(\cdot) \in H$ and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\bigl\langle y(t,\cdot), \varphi(\cdot)\bigr\rangle_H + \nu\bigl\langle y(t,\cdot), \varphi(\cdot)\bigr\rangle_{\tilde{V}} - \nu\bigl\langle y(t,\cdot), \varphi(\cdot)\bigr\rangle_H + b\bigl(y(t,\cdot), y(t,\cdot), \varphi(\cdot)\bigr) \\ &\quad = \bigl\langle f(t,\cdot), \varphi(\cdot)\bigr\rangle_{\tilde{V}^*,\tilde{V}} + \nu\beta(t)\varphi(1) - \nu\alpha(t)\varphi(0)\end{aligned}$$

for all $\varphi(\cdot) \in \tilde{V}$, $t \in [0,T]$ a.e., in which $b(\varphi, \psi, \varpi)$ is defined as before. For the nonlinear partial differential equations (22), Volkwein in [21] proves the existence of the unique weak solution $y(t,\cdot) \in W(0,T;\tilde{V})$.

Take the state space be $W(0,T;\tilde{V})$ and the control space $L^2(0,T)$. Consider the following optimal boundary control problem for system (22) with the general cost functional:

$$\min_{\alpha(\cdot),\beta(\cdot)\in U_{\mathrm{ad}}} J(y,\alpha,\beta) = \min_{\alpha(\cdot),\beta(\cdot)\in U_{\mathrm{ad}}} \int_0^T \int_0^1 L\bigl(y(t,x),\alpha(t),\beta(t),t,x\bigr)\,\mathrm{d}x\,\mathrm{d}t, \tag{23}$$

in which the control functions satisfy the convex constraint $\alpha(\cdot),\beta(\cdot)\in U_{\mathrm{ad}}$ and L satisfies the similar conditions (a), (b) in problem (3). Here the assumption $\operatorname{int}_{L^2(0,T)} U_{\mathrm{ad}} \neq \emptyset$ is again assumed in this fixed final horizon case although it will be removed in investigation of the subsequent free final horizon case.

Let (y^*,α^*,β^*) be the solution of the optimal boundary control problem (23). Adopting the same approach, we give the following linearized system:

$$\begin{aligned} &y_t(t,x) - \nu y_{xx}(t,x) + y^*(t,x)y_x(t,x) + y_x^*(t,x)y(t,x) = 0,\\ &y_x(t,0) = \alpha(t), \quad y_x(t,1) = \beta(t), \qquad y(0,x) = 0. \end{aligned} \tag{24}$$

In this case, the continuous linear functionals in the general Dubovitskii and Milyutin theorem can be determined as

$$\begin{aligned} f_0(y,\alpha,\beta) &= -\kappa_0 \int_0^T \int_0^1 \biggl[\frac{\partial L(y^*,\alpha^*,\beta^*,t,x)}{\partial y} y(t,x) + \frac{\partial L(y^*,\alpha^*,\beta^*,t,x)}{\partial \alpha}\alpha(t)\\ &\qquad + \frac{\partial L(y^*,\alpha^*,\beta^*,t,x)}{\partial \beta}\beta(t)\biggr]\,\mathrm{d}x\,\mathrm{d}t,\\ f_1(y,\alpha,\beta) &= \int_0^T \bigl[a_\alpha(t)\alpha(t) + a_\beta(t)\beta(t)\bigr]\,\mathrm{d}t,\\ f_{21}(y,\alpha,\beta) &= 0, \qquad f_{22}(y,\alpha,\beta) = \int_0^1 y(T,x)\phi(x)\,\mathrm{d}x, \end{aligned}$$

in which $\kappa_0 \geqslant 0$, $a_\alpha(t), a_\beta(t) \in L^2(0,T)$, and $\phi(x) \in \tilde{V}^*$ are given and can be determined as in investigation of optimal distributed control problems before. As such, we adopt the same symbols to denote these functionals in the case of no confusions caused.

Moreover, the adjoint system of (24) can be given as

$$\begin{aligned} &z_t(t,x) + \nu z_{xx}(t,x) + y^*(t,x)z_x(t,x) = \kappa_0 \frac{\partial L(y^*(t,x),\alpha^*(t),\beta^*(t),t,x)}{\partial y},\\ &\nu z_x(t,0) + y^*(t,0)z(t,0) = 0, \quad \nu z_x(t,1) + y^*(t,1)z(t,1) = 0,\\ &z(T,x) = \phi(x). \end{aligned} \tag{25}$$

And the relationship between the solution of the linearized system (24) and that of its adjoint system (25) above is

$$\int_0^T \int_0^1 \kappa_0 \frac{\partial L(y^*, \alpha^*, \beta^*, t, x)}{\partial y} y(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T y(T,x)\phi(x)\,\mathrm{d}x$$
$$= \int_0^T \nu\big[z(t,0)\alpha(t) - z(t,1)\beta(t)\big]\,\mathrm{d}t. \tag{26}$$

By the general Dubovitskii–Milyutin theorem and relationship (26), there is

$$\begin{aligned}
f_1(y,\alpha,\beta) &= -f_0(y,\alpha,\beta) - f_{22}(y,\alpha,\beta) \\
&= \kappa_0 \int_0^T \int_0^1 \bigg[\frac{\partial L(y^*, \alpha^*, \beta^*, t, x)}{\partial y} y(t,x) + \frac{\partial L(y^*, \alpha^*, \beta^*, t, x)}{\partial \alpha}\alpha(t) \\
&\qquad + \frac{\partial L(y^*, \alpha^*, \beta^*, t, x)}{\partial \beta}\beta(t)\bigg]\,\mathrm{d}x\,\mathrm{d}t - \int_0^1 y(T,x)\phi(x)\,\mathrm{d}x \\
&= \kappa_0 \int_0^T \int_0^1 \bigg[\frac{\partial L(y^*, \alpha^*, \beta^*, t, x)}{\partial \alpha}\alpha(t) + \frac{\partial L(y^*, \alpha^*, \beta^*, t, x)}{\partial \beta}\beta(t)\bigg]\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \nu\big[z(t,0)\alpha(t) - z(t,1)\beta(t)\big]\,\mathrm{d}t,
\end{aligned}$$

so we have theorem that follows.

Theorem 5. *Suppose* (y^*, α^*, β^*) *is a solution to the optimal control problem* (23). *Then there exist* $\kappa_0 \geqslant 0$ *and* $z(t,x)$, *not identically zero, such that the following maximum principle holds true*:

$$\bigg[\int_0^1 \kappa_0 \frac{\partial L(y^*, \alpha^*, \beta^*, t, x)}{\partial \alpha}\,\mathrm{d}x + \nu z(t,0)\bigg]\big[\alpha(t) - \alpha^*(t)\big]$$
$$+ \bigg[\int_0^1 \kappa_0 \frac{\partial L(y^*, \alpha^*, \beta^*, t, x)}{\partial \beta}\,\mathrm{d}x - \nu z(t,1)\bigg]\big[\beta(t) - \beta^*(t)\big] \geqslant 0$$

for all $\alpha(t), \beta(t) \in U_{\mathrm{ad}}$, $t \in [0,T]$ *a.e., where the function* $z(t,x)$ *satisfies the adjoint equation* (25).

Similarly, we can consider the following free final horizon problem of system (22).

Problem IV. Minimize

$$J(y, \alpha, \beta) = \int_0^{t_1} \int_0^1 L\big(y(t,x), \alpha(t), \beta(t)\big)\,\mathrm{d}x\,\mathrm{d}t, \tag{27}$$

in which L satisfies the similar properties with (c), (d) and t_1 is defined as before.

Here we omit the proof and directly present the final result. The corresponding Pontryagin maximum principle for optimal boundary control problem of system (22) in free final horizon case can be stated as theorem below.

Theorem 6. *Suppose $(y^*, \alpha^*, \beta^*, t_1)$ is a solution to Problem* IV*, then there exist $\kappa_0 \geqslant 0$ and $z(t,x)$, not identically zero, such that, for all $t \in [0, t_1]$ a.e.,*

$$\begin{aligned}
&\int_0^1 \big\{\kappa_0 L\big(y^*(t,x), \alpha^*(t), \beta^*(t)\big) - \big[\nu y^*_{xx}(t,x) - y^*(t,x) y^*_x(t,x) + f(t,x)\big] z(t,x)\big\}\,\mathrm{d}x \\
&\quad + \nu\big[\alpha^*(t) - y^*_x(t,0)\big] z(t,0) - \nu\big[\beta^*(t) - y^*_x(t,1)\big] z(t,1) = 0, \\
&\int_0^1 \big\{\kappa_0 L\big(y^*(t,x), \alpha^*(t), \beta^*(t)\big) - \big[\nu y^*_{xx}(t,x) - y^*(t,x) y^*_x(t,x) + f(t,x)\big] z(t,x)\big\}\,\mathrm{d}x \\
&\quad + \nu\big[\alpha - y^*_x(t,0)\big] z(t,0) - \nu\big[\beta - y^*_x(t,1)\big] z(t,1) \geqslant 0, \quad \alpha, \beta \in M,
\end{aligned}$$

where the function $z(t,x)$ satisfies

$$\begin{aligned}
&z_t(t,x) + \nu z_{xx}(t,x) + y^*(t,x) z_x(t,x) = \kappa_0 \frac{\partial L(y^*, \alpha^*, \beta^*)}{\partial y}, \quad (t,x) \in Q_{t_1}, \\
&\nu z_x(t,0) + y^*(t,0) z(t,0) = 0, \quad \nu z_x(t,1) + y^*(t,1) z(t,1) = 0, \\
&z(t_1, x) = \psi(x),
\end{aligned}$$

and $\psi(x) \in \tilde{V}^*$ as well as the set M defined as before.

6 A remark

Generally speaking, there are three kinds of numerical methods for solving the optimal control problem. By the necessary condition of optimal control, such as the Pontryagin maximum principle, a two-point boundary-value problem solution is one of these three methods mentioned above. Though it is difficult to solve it, the Pontryagin maximum principle provides a possibility of seeking numerical solution for the optimal control at least in open-loop form. Basically, there are two ways for numerically solving optimal control problems through necessary conditions. It is generally believed that the indirect method that is mainly the multiple shooting method is the most powerful numerical method in seeking the optimal control of the lumped parameter systems through solving a two-point boundary-value problem obtained by the Pontryagin maximum principle. Of course, except for the complexity when the original problem involves inequality constraints of both state variables and controls, the difficulty for shooting method additionally includes the “guess" for the initial data to start the iterative numerical process. It demands that the user understands the essential of the problem well in physics, which is often not a trivial task. In addition, there are many other existing effective algorithms available, such as gradient method introduced for overcoming the difficulty of the initial guess, the “min-H” approach corrected from the gradient method for the higher convergence rate, and so on. Surely, both these algorithms based on the Pontryagin maximum principle to obtain the optimal control do give us the satisfying solutions.

In this section, we shall show, by the min-H iterative method, how to use the results obtained before for solving the extremum problems. Specifically, we discuss the optimal distributed control of the Burgers equation in fixed final horizon case, namely, problem (3).

To this end, rewrite the Pontryagin maximum principle (15) as follows:

$$u^*(t)H_u(y^*,u^*) = \max_{u(\cdot)\in U_{\mathrm{ad}}} u(t)H_u(y^*,u^*), \tag{28}$$

in which

$$H(y,u) = u(t)z(t,x) - \kappa_0 L(y,u,t,x).$$

Upon that, we may utilize the so-called “min-H” iterative algorithm [9, 22] to solve the extremum problem. The algorithm is formulated below:

(i) Guess $u^0(t)$ and solve the state equation (1) to get $y^0(t,x)$.

(ii) By $u^0(t)$, $y^0(t,x)$, solve the adjoint equation (12) to get $z^0(t,x)$.

(iii) In view of $y^0(t,x)$, $z^0(t,x)$ and the Pontryagin maximum principle (28), to determine $u^1(t)$.

(iv) Calculate $J(u^1(t))$. If it does not reach the minimum, replace $u^0(t)$ with $u^1(t)$ and redo the steps above until we get the proper $J(u^1(t))$.

After setting some parameters and functions such as $y_0(x)$, T, $y^\dagger(t,x)$, $u^\dagger(t)$, $f(t,x)$, $L(y,u,t,x)$, one can proceed the numerical simulation using the algorithm above. Moreover, people can choose the quadratic cost functional (4) for the convenience. Although it is definitely not easy, the concrete steps given by the algorithm make it possible for people to follow and finish this nontrivial work, to get the numerical solutions for optimal control problems of distributed parameter systems governed by nonlinear partial differential equations.

7 Conclusions

To sum up, in this paper we study optimal distributed and (Neumann) boundary control problems for the Burgers equation in both fixed and free final horizon cases, four optimal control problems in all. The Pontryagin maximum principles of optimal control systems are, respectively, investigated by the Dubovitskii and Milyutin functional analytical approach and the first-order necessary optimality conditions in these four cases are presented, successively. Then a remark is made for the illustration and the min-H iterative algorithm is expounded.

Overall, the paper provides a framework for using functional analysis and control methods to analyze and optimize the distributed parameter systems. People can adopt the similar techniques for solving much more and interesting extremum problems.

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