Nonlinear Analysis: Modelling and Control, Vol. 20, No. 2, 249–262 ISSN 1392-5113
http://dx.doi.org/10.15388/NA.2015.2.7

Chaos and bifurcations in chaotic maps with parameter q: Numerical and analytical studies*

Hao Zhang, Xing-Yuan Wang, Xiao-Hui Lin

Faculty of Electronic Information and Electrical Engineering
Dalian University of Technology, Dalian 116024, China
zhangh545@126.com; wangxy@dlut.edu.cn

Received: May 7, 2013 / **Revised:** June 20, 2014 / **Published online:** January 15, 2015

Abstract. In this paper, a class of chaotic maps with parameter q are introduced and bifurcations and chaos in proposed maps are numerical and analytical studied. Euler method is employed to get the continuous systems corresponding to chaotic maps and the fractional styles in Caputo's definition. Based on that, we finally infer a class of chaotic maps with the Adams–Bashforth–Moulton predictor-corrector method. In the simulation and analysis, we discuss the Logistic map with q and Hénon map with q, observe the route from period to chaos and do tests to analyze properties of maps with parameter q.

Keywords: period-doubling bifurcation, period window, fractional calculus, predictor-corrector.

1 Introduction

Chaos is a class of outer pseudo random behavior which occurs in deterministic systems. Because of its potential use in physics, chemistry and industry engineering, chaos becomes a hot research field in recent years [4, 23, 25]. Chaos models are divided into chaotic systems and chaotic maps according to continuity of state variables. Many famous chaotic systems are discovered, such as the Lorenz system [16], the Chen system [5], the Rössler system [19], and the Lü system [17]. At the same time, a lot of classical chaotic maps are widely used in the study of generating chaos through a iterative model, such as the Logistic map, the Hénon map and the Tent map [12, 13]. In the study of chaos, fractional differential calculus is an effective tool. Throughout the paper, in order to solve fractional equation, the time-domain method [9] is chosen because it is more reliable than the frequency-domain method [21, 22] and it is more widely used by others [2, 7, 26, 28].

*This research is supported by the National Natural Science Foundation of China (Nos. 61370145, 61173183, and 60973152), the Doctoral Program Foundation of Institution of Higher Education of China (No. 20070141014), Program for Liaoning Excellent Talents in University (No. LR2012003), the National Natural Science Foundation of Liaoning province (No. 20082165) and the Fundamental Research Funds for the Central Universities (No. DUT12JB06).

With the help of fractional calculus, a class of fractional chaotic systems are found and dynamic properties versus fractional-order q are investigated. Recently, many fractional systems are proposed [3, 6, 8, 11, 14, 15]. Based on that, many researchers get the discrete model of a special fractional system and research its discrete changes and bifurcations with fractional-order q. For example, Sun and Sprott [20] study bifurcations of fractional diffusionless Lorenz system and simplified Lorenz system. Xu and his peers [27] investigate the control of fractional diffusionless Lorenz system. However, because fractional calculus is meaningless to discrete chaoticmaps, above discussions are limited in continuous systems. In fact, at least three state variables are needed to generate chaos in continuous systems while only one variable can generate chaotic series in discrete map, such as the Logistic map and the Tent map.

Motivated by above discussions, this paper studies a class of chaotic maps with q, it is different from previous work because such a map may contain less than three state variables and there is no corresponding fractional system in continuous chaotic systems. To the best of our knowledge, for this respect, there is no result in the literature so far. So it is still remains open and challenging.

The rest paper is organized as follows. Section 2 introduces the basic numerical algorithm. Section 3 is numerical simulations and theoretical analysis of one-dimensional Logistic map with q and 2-dimensional Hénon map with q. What is more, some stochasticity tests are done in this section. Finally, some concluding remarks are given in Section 4.

2 Numerical algorithm

2.1 Chaotic maps and the inverse process of discretization

In the numerical siumlation, discretization is an effective method. This method is also useful to deal with fractional dynamic systems. A general continuous system can be represent as

$$\dot{x} = f(x). \tag{1}$$

There are many methods to discretize system (1), such as Euler method, modified Euler method and Runge–Kutta method. For simplicity, the discretized system (1) with Euler method can be described as

$$x_{i+1} = x_i + hf(x_i), \quad i = 1, 2, \ldots, N-1, \tag{2}$$

where $h = T/N$, T is the time range and N is iteration times.

It can be concluded that there is an approximate corresponding relationship between a continuous system and a corresponding discrete system. In the numerical simulation, we used to discretize a continuous system but ignore the inverse process of discretization. Take Logistic map as an example

$$x_{i+1} = ux_i(1 - x_i), \quad i = 1, 2, \ldots, N-1, \tag{3}$$

where u is the system parameter. Suppose map (3) is a discretized system with Euler method and it can be translated into

$$x_{i+1} = ux_i(1 - x_i) = x_i + h\frac{ux_i(1 - x_i) - x_i}{h}, \quad i = 1, 2, \ldots, N - 1. \tag{4}$$

Then we can get a corresponding continuous system

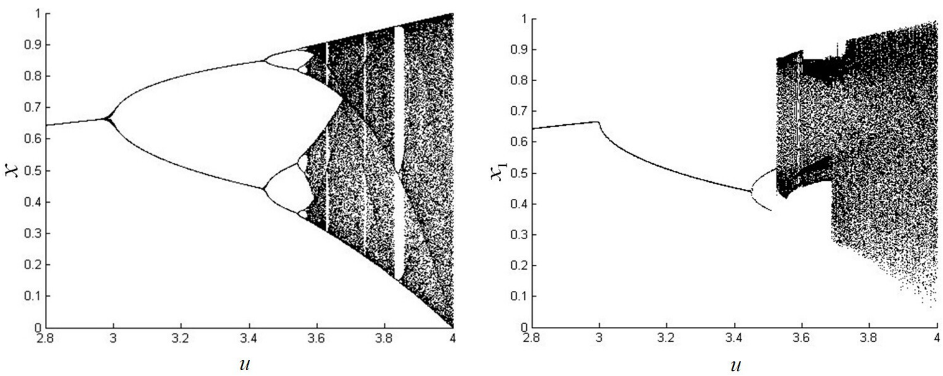
$$\dot{x} = \frac{ux(1 - x) - x}{h}. \tag{5}$$

However, system (5) contains only one variable and cannot generate chaos. On the other hand, we know that period-doubling bifurcation is a way to chaos and many famous systems such as the Lorenz system and the Chua system get into chaos with this method.

Remark 1. For a continuous system, at least 3 variables are need to generate chaos. Even though system (5) is not chaotic, an extended system (5) as follows

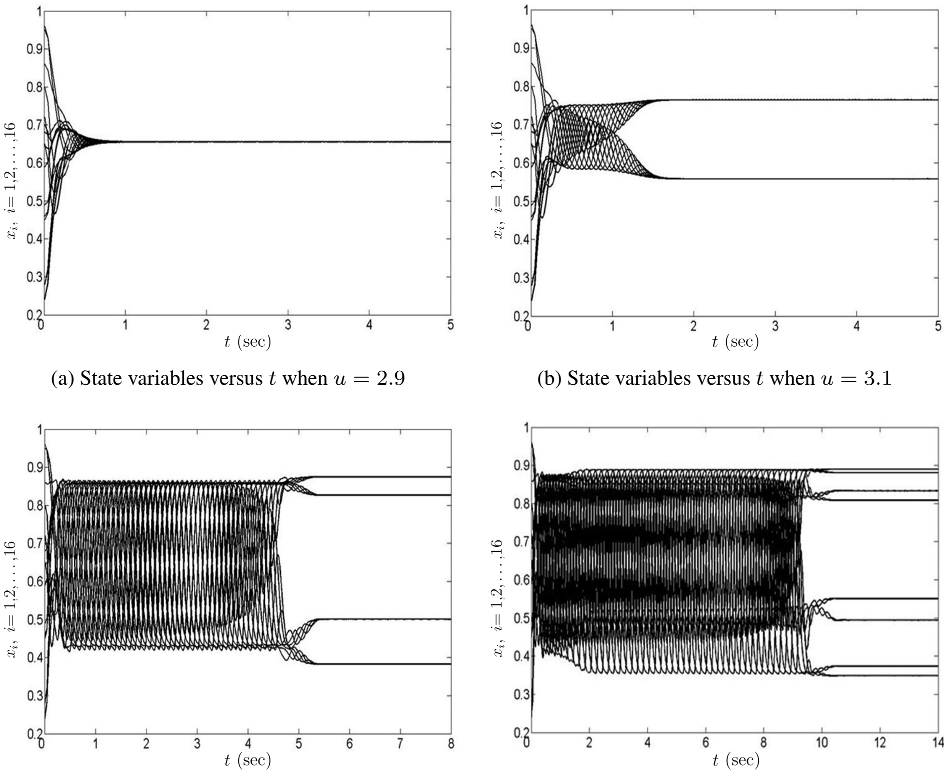
$$\begin{aligned}
\dot{x}_1 &= \frac{ux_2(1 - x_2) - x_1}{h}, \qquad \dot{x}_2 = \frac{ux_3(1 - x_3) - x_2}{h}, \\
&\vdots \\
\dot{x}_s &= \frac{ux_{s+1}(1 - x_{s+1}) - x_s}{h}, \quad s \neq n, \\
&\vdots \\
\dot{x}_n &= \frac{ux_1(1 - x_1) - x_n}{h}
\end{aligned} \tag{6}$$

can help us to understand the corresponding relationships of system (3) and system (5). Here n is the dimension of system (6). Choose $n = 2^4 = 16$, bifurcation diagrams of system (1) and period-doubling bifurcation process of system (6) are shown Figs. 1, 2.



(a) Bifurcation diagram of Logistic map (b) State variable x_1 versus u

Fig. 1. Bifurcation diagrams and period-doubling bifurcation process.



(a) State variables versus t when $u = 2.9$

(b) State variables versus t when $u = 3.1$

(c) State variables versus t when $u = 3.5$

(d) State variables versus t when $u = 3.55$

Fig. 2. Bifurcation diagrams and period-doubling bifurcation process.

2.2 Chaotic maps with q and the predictor-corrector method

From above, we find the corresponding continuous system and investigate the chaotic bifurcation of its deformation. Just as researchers get fractional systems from continuous systems, we can deal with system (5) with fractional differential calculus. At present, there are several definitions of a fractional-order differential system. In this paper, we will use the Caputo fractional derivative.

Definition 1. Caputo fractional derivative is defined as follows [1]:

$$D^{\theta}x(t) = J^{m-\theta}x^{(m)}(t) \quad (\theta > 0),$$

where m is the first integer which is not less than θ, $x^{(m)}$ is the mth-order derivative in the usual sense, and J^{θ} $(1 > \theta > 0)$ is the θth-order Reimann–Liouville integral operator with expression $J^{\theta}y(t) = 1/\Gamma(\theta)\int_0^t (t-\tau)^{\theta-1}y(\tau)\,\mathrm{d}\tau$. Here Γ stands for Gamma function, and the operator D^{θ} is generally called θth-order Caputo differential operator.

Based on Definition 1, a fractional system corresponding to system (5) is

$$\begin{aligned} &D^q x = f(t,x) = \frac{ux(1-x)-x}{h}, \\ &x^{(k)}(0) = x_0^{(k)}, \quad k = 0,1,\dots,n-1, \end{aligned} \tag{7}$$

where q is the fractional order. Eq. (7) is equivalent to Volterra integral equation

$$x(t) = \sum_{j=0}^{\lceil q\rceil-1} x_0^{(j)} \frac{t^j}{j!} + \frac{1}{\Gamma(q)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-q}}\, \mathrm{d}\tau. \tag{8}$$

With the rectangular formula, we can get the following approximate calculation:

$$\int_0^{t_{k+1}} \frac{f(\tau)}{(t_{k+1}-\tau)^{1-q}}\, \mathrm{d}\tau \approx \sum_{j=0}^{k} \frac{h^q}{q} \big[(k+1-j)^q - (k-j)^q\big] f(t_j), \tag{9}$$

where $t_i = ih$, substitute Eq. (9) into Eq. (7) and set

$$b_{j,k+1} = (k+1-j)^q - (k-j)^q,$$

we will get the predictor operator

$$x^p(t_{k+1}) = \sum_{j=0}^{\lceil q\rceil-1} x_0^{(j)} \frac{t_{k+1}^j}{j!} + \frac{h^q}{q\Gamma(q)} \sum_{j=0}^{k} b_{j,k+1} f(t_j). \tag{10}$$

On the other hand, with the trapezoidal formula, we can get

$$\int_0^{t_{k+1}} \frac{f(\tau)}{(t_{k+1}-\tau)^{1-q}}\, \mathrm{d}\tau \approx \sum_{j=0}^{k} \frac{h^q}{q(q+1)} a_{j,k+1} f(t_j), \tag{11}$$

where

$$a_{j,k+1} = \begin{cases} k^{q+1} - (k-q)(k+1)^q, & j = 0, \\ (k-j+2)^{q+1} + (k-j)^{q+1} - 2(k-j+1)^{q+1}, & 1 \leqslant j \leqslant k, \\ 1, & j = k+1. \end{cases}$$

Because $x(t_1)$, $x(t_2)$, …, $x(t_k)$ and predictor operator $x^p(t_{k+1})$ are known, we can get the corrected fractional formula as

$$\begin{aligned} x^p(t_{k+1}) = {}& \sum_{j=0}^{\lceil q\rceil-1} x_0^{(j)} \frac{t_{k+1}^j}{j!} + \frac{h^q}{\Gamma(q+2)} f\big(t_{k+1}, x^p(t_{k+1})\big) \\ &+ \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{k} a_{j,k+1} f\big(t_j, x(t_j)\big). \end{aligned} \tag{12}$$

This is an one order Adams–Bashforth–Moulton predictor-corrector method.

Remark 2. For a h which meets the requirement, error can be defined as

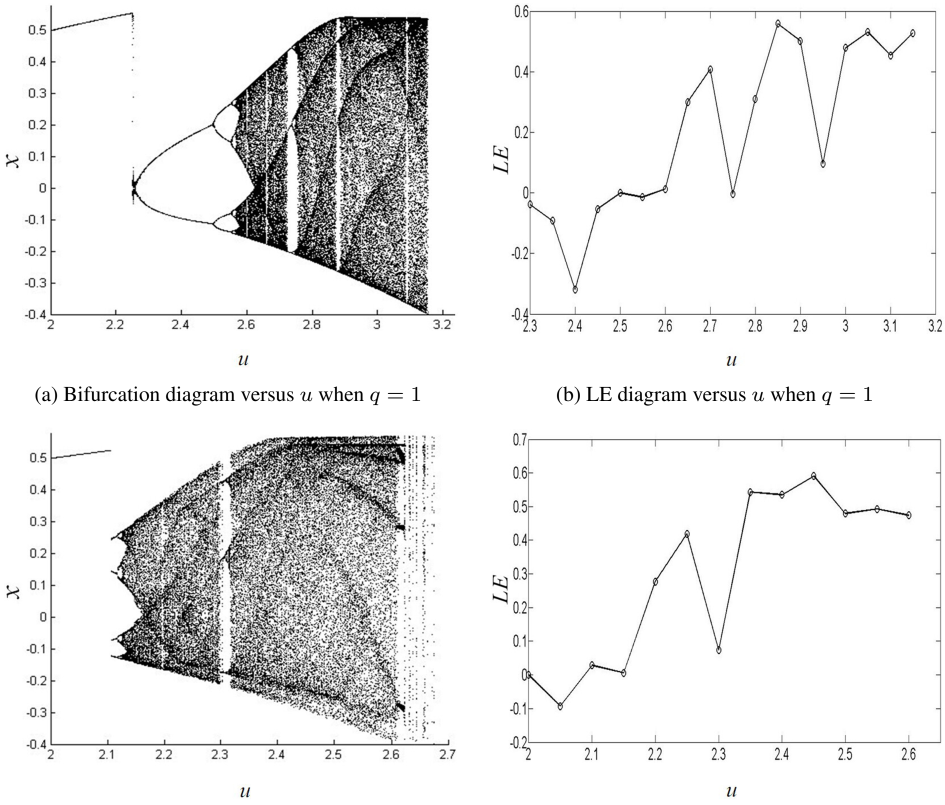
$$\max_{i=1,2,\ldots,N} \left|x(t_i) - x_i\right| = o\big(h^\alpha\big), \quad \alpha = \min(1+q, 2).$$

3 Numerical simulations and theoretical analysis

3.1 A class of chaotic maps with q

In the following, we will observe chaos and bifurcations visually and ensure series are chaotic by calculating largest Lyapunov exponents. Analysis in Section 2 is a universal method and it can be applied to one dimensional map, such as the Logistic map, and multidimensional map, such as the Hénon map. Bifurcation diagrams and LE diagrams of Logistic map with $q = 1$ and $q = 0.95$ are shown in Fig. 3.

Compare Fig. 1a with Fig. 3a, we find that even though the Logistic map with q comes from the Logistic map (3), chaos interval, period windows and bifurcation diagrams have



(a) Bifurcation diagram versus u when $q = 1$

(b) LE diagram versus u when $q = 1$

(c) Bifurcation diagram versus$u$when $q = 0.95$

(d) LE diagram versus u when $q = 0.95$

Fig. 3. Bifurcation diagrams and LE diagrams of Logistic map with q.

changed a lot. The Logistic map with q is totally different from the original one and becomes a new map. In Figs. 3a and 3c, different q lead to different bifurcations versus u.

Remark 3. Above numerical simulations show that a class of maps with q can be got after the inverse process of discretization in Section 2.1 and discretization in Section 2.2. Many famous chaotic maps have less than three variables and will not be chaotic after the inverse process of discretization just like system (5). So previous work ignores such kind of problem. In fact, from above we know, one-dimensional or two-dimensional map also has fractional style and worth being studied.

3.2 Chaotic bifurcation versus parameter *q*

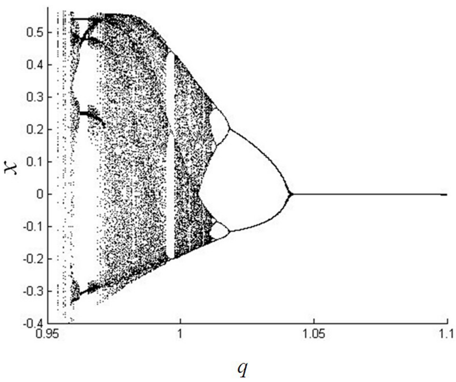
In order to observe bifurcations and chaos, for convenience, set $u = 2.7$ and $h = 0.01$ and $q \in [0.95, 1.1]$. Initial value $x_0 = 0.5$. The bifurcation diagram and the corresponding LE diagram are shown in Fig. 4. From Figs. 4a and 4b, we see that it is an inverse bifurcation process when parameter q increases. So it is an effective way to make system be more chaotic by reducing the value of parameter q within limit. System has the largest LE around $q = 0.975$. In the inverse bifurcation diagram, we observe two obvious period windows when $q \in [0.95, 0.98]$ and $q \in [0.99, 1]$. In Fig. 4c, there are also similar inverse bifurcations and the branch point is $q \cong 0.962$. Compared with Fig. 4c, inverse bifurcations in Fig. 4d are more obvious and clear. Chaos becomes period behaviors at $q \cong 0.9944$; Period three appears at $q \cong 0.9962$ and system gets back into chaos at $q \cong 0.9978$.

In 1978, Feigenbaum found Feigenbaum constant $\delta = 4.66920\ldots$ in the Logistic map with Renonnalization Group (RNG). It shows the regularity of period-doubling bifurcations [10]. In our simulations, we find the Logistic map with q is convergent in a fixed point when $1.0418 < q < 1.1$; when $1.0186 < q < 1.0418$, fixed point lose stability and period 2 appears; when $1.0134 < q < 1.0186$, period 2 becomes period 4 and when $1.0122 < q < 1.0134$ period 4 translates into period 8... the Logistic map with q keeps period-doubling bifurcating to chaos. Based on the idea of Feigenbaum, we get the values of period-doubling bifurcated points q_v ($v = 1, 2, \ldots$) and calculate the limit of interval ratios in the Logistic map with q as follows:

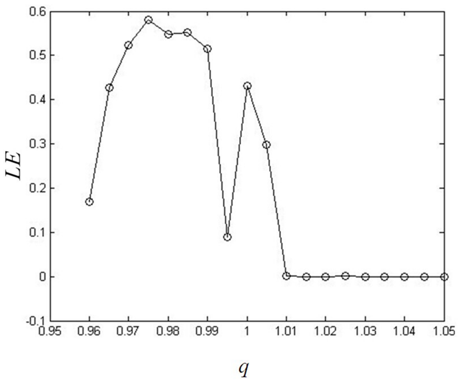
$$\delta' = \lim_{v\to\infty} \frac{q_v - q_{v-1}}{q_{v+1} - q_v} = 4.46153\ldots. \tag{13}$$

The calculated δ' is approximate to δ and it shows that there are common bifurcation laws in such kind of chaotic maps: In the process of chaotic period-doubling bifurcations, there are self-similarity and invariance of scale transformation in both parameter space and phase space.

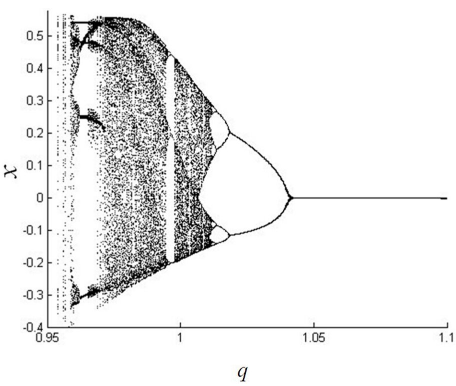
Remark 4. We study the effect of fractional order on the style of bifurcation and some phenomena are observed as follows. 1) The bifurcation versus fractional order q is an inverse bifuraction, this is different from the bifurcation versus system parameter u. 2) Feigenbaum law is satisfied and period window appears in the process of bifurcation.



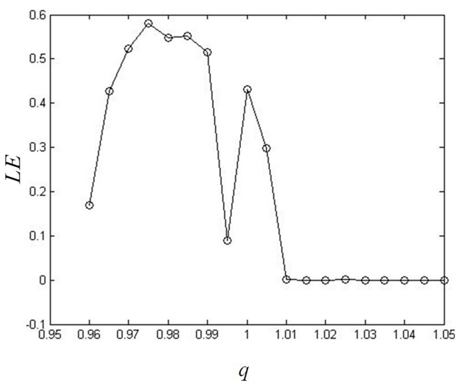
(a) Bifurcation diagram versus q when $u = 2.7$



(b) LE diagram versus q when $u = 2.7$



(c) Period window versus q when $u = 2.7$



(d) Period window versus q when $u = 2.7$

Fig. 4. Bifurcation diagram and LE diagram versus q.

This means the fractional order q plays an important role in the chaotic behavior of the proposed system. 3) The bifurcation diagram locates in different interval when parameter u ranges. This means the fractional order q and parameter u are interacted with each other and traditional one-dimensional system becomes a novel two-dimensional system.

3.3 Stochasticity tests

In order to further analyze maps with parameter q and study their stochasticity, we translate chaotic series (Logistic map and Logistic map with parameter q) into binary bit streams with the same method and do some tests in NIST 800-22 test suits [18, 24] to analyze their properties. Results of our tests are shown in Table 1.

From Table 1 and above analysis, we can see that maps with parameter q can pass many random tests and can replace the Logistic map in some areas. Even though in our simulation, we find that such chaotic maps have some disadvantages, such as bound-

Table 1. Results of stochasticity tests and comparisons.

Tests	P-value	
	Logistic map ($x_0 = 0.5285$)	Logistic map with $q = 0.99$, $u = 2.7$ ($x_0 = 0.08$)
Approximate Entropy test	0.994275 (Pass)	0.287588 (Pass)
Block frequency test	0.092452 (Pass)	0.997542 (Pass)
Cumulative Sums (Forward) test	0.575344 (Pass)	0.823133 (Pass)
Cumulative Sums (Reverse) test	0.731779 (Pass)	0.900487 (Pass)
Discrete Fourier Transform test	0.118754 (Pass)	0.383988 (Pass)
Frequency test	0.378859 (Pass)	0.612882 (Pass)
Non-overlapping Template Matching test $(001, 011, 100, 110)$	0.796794 (Pass), 0.772008 (Pass), 0.843604 (Pass), 0.720490 (Pass)	0.404853 (Pass), 0.534620 (Pass), 0.369769 (Pass), 0.493875 (Pass)

aries need to be hard to got theoretically. However, different dynamic behaviors related to parameter q provide researchers a novel way to improve the security and keep the stochasticity.

3.4 Hénon map with parameter *q*

As mentioned before, the method proposed in this paper is universal and can be applied not only in one-dimensional map but also multidimensional map. For multidimensional maps, predictors and correctors are translated into

$$x_s^p(t_{k+1}) = \sum_{j=0}^{\lceil q_s \rceil - 1} x_{s,0}^{(j)} \frac{t_{k+1}^j}{j!} + \frac{h^{q_s}}{q_s \Gamma(q_s)} \sum_{j=0}^{k} b_{s,j,k+1} f(t_j), \quad s = 1, 2, \dots, n, \tag{14}$$

and

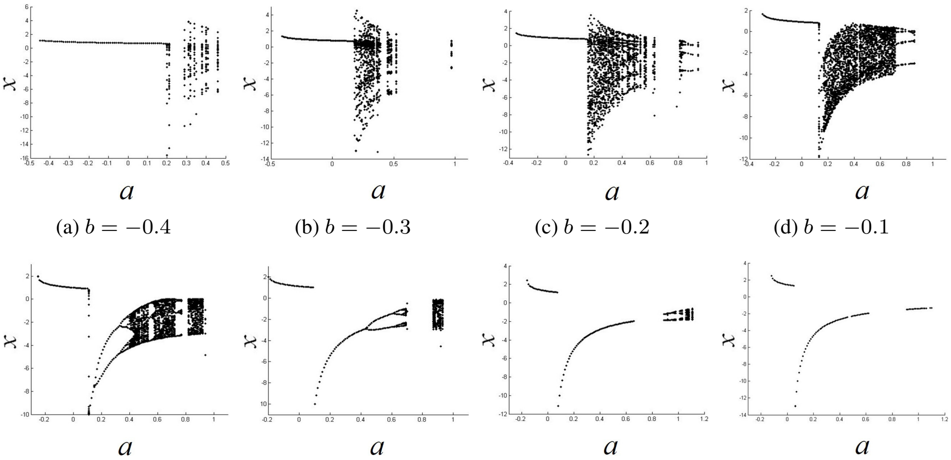
$$\begin{aligned} x_s^p(t_{k+1}) = {} & \sum_{j=0}^{\lceil q_s \rceil - 1} x_{s,0}^{(j)} \frac{t_{k+1}^j}{j!} + \frac{h^{q_s}}{\Gamma(q_s + 2)} f\bigl(t_{k+1}, x_s^p(t_{k+1})\bigr) \\ & + \frac{h^{q_s}}{\Gamma(q_s + 2)} \sum_{j=0}^{k} a_{s,j,k+1} f\bigl(t_j, x_s(t_j)\bigr), \quad s = 1, 2, \dots, n, \end{aligned} \tag{15}$$

where

$$b_{s,j,k+1} = (k + 1 - j)^{q_s} - (k - j)^{q_s}, \quad 1 \leqslant j \leqslant k,\ s = 1, 2, \dots, n,$$

and

$$a_{s,j,k+1} = \begin{cases} k^{q_s+1} - (k - q_s)(k+1)^{q_s}, & j = 0, \\ (k-j+2)^{q_s+1} + (k-j)^{q_s+1} - 2(k-j+1)^{q_s+1}, & 1 \leqslant j \leqslant k, \quad s = 1, 2, \dots, n. \\ 1, & j = k+1, \end{cases}$$



(a) $b = -0.4$ (b) $b = -0.3$ (c) $b = -0.2$ (d) $b = -0.1$

(e) $b = 0$ (f) $b = 0.1$ (g) $b = 0.2$ (h) $b = 0.3$

Fig. 5. Bifurcations corresponding to different system parameters.

$x_{s,0}$ is the initial value of x_s, n is the dimension of system. For Hénon map, $n = 2$. For fractional system (fractional Lorenz system, fractional Chen system and so on), $n = 3$. For hyperchaotic fractional system and high dimensional fractional style of systems like system (6), $n > 3$. So previous work in [8, 20] can be seen as a special case of this paper. In the following, we will take the Hénon map as an example.

A Hénon map can be represented as

$$x_{i+1} = 1 - ax_i^2 + y_i,$$
$$y_{i+1} = bx_i,$$

where $a = 1.4$, $b = 0.3$. Choose parameters $q = q_1 = q_2 = \cdots = q_n = 1$, we find that the corresponding system with q becomes divergence if we keep $a = 1.4$, $b = 0.3$. In order to find out appropriate system parameters and observe system with parameters variation, we set $q = q_1 = q_2 = 1$. The bifurcations of Hénon map with q are shown in Fig. 5. From Fig. 5, we can see the process of chaos variations with different system parameters. It is interesting to note that the inverse bifurcation gradually becomes bifurcation from $b = -0.1$ to $b = 0$. In Fig. 6, we can see bifurcation behaviors and inverse bifurcation behaviors. From Fig. 5, we find that when $a \in [0.2, 0.9]$ and $b \in [-0.2, 0]$, system is more chaotic and it is easier to observe bifurcation behaviors. Fix parameters at $a = 0.6$, $b = -0.1$ and $a = 0.9$, $b = 0$, bifurcation diagrams versus q are shown in Fig. 7.

From Fig. 7, we know that different parameters a and b make chaos and bifurcation appear in different intervals. Bifurcation in Hénon map with q is similar to one-dimensional case and reducing the value of parameter q within limit can make map be more chaotic. Besides that, it is worth noting that Hénon map with q is a two-dimensional map, so q_1 may be not equal to q_2 ($q_1 \neq q_2$). Fix parameters at $a = 0.6$, $b = -0.1$ and $q_1 = 1$, bifurcation diagram versus q_2 is shown in Fig. 8a; fix parameters at $a = 0.6$, $b = -0.1$

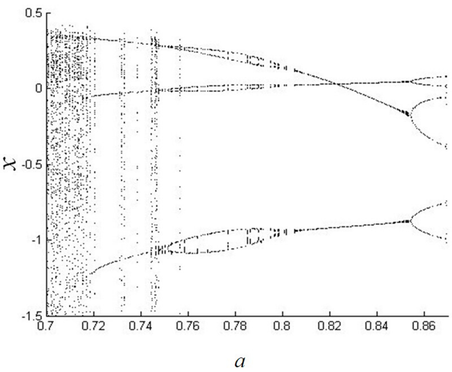
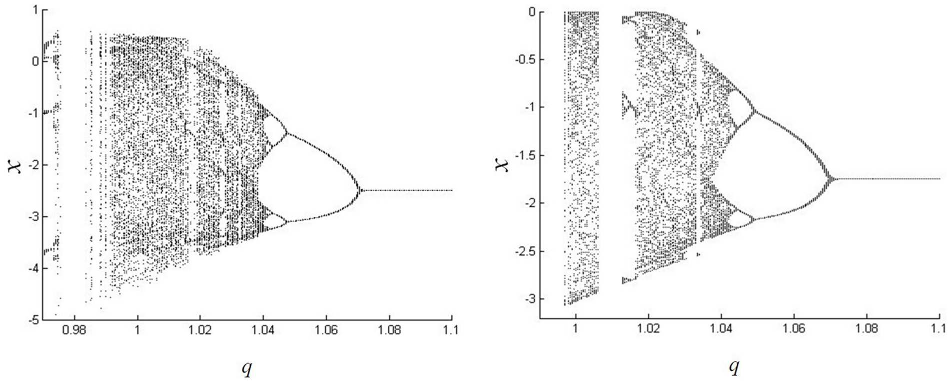
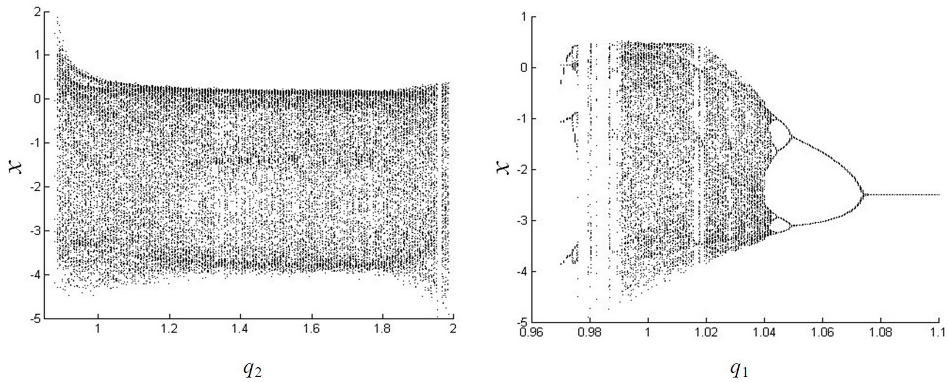


Fig. 6. Amplified period window when $b = -0.1$.



(a) $a = 0.6$, $b = -0.1$ (b) $a = 0.9$, $b = 0$

Fig. 7. Bifurcation diagrams versus q.



(a) $a = 0.6$, $b = -0.1$, $q_1 = 1$ (b) $a = 0.6$, $b = -0.1$, $q_2 = 1$

Fig. 8. Bifurcation versus q_1 and q_2.

and $q_2 = 1$, bifurcation diagram versus q_1 is shown in Fig. 8b. From Fig. 8, we can see that when q_1 is fixed, bifurcation doesn't appear and system keeps chaotic. However, when q_2 is fixed, bifurcation diagram versus q_1 is visible and inverse bifurcation appears. This shows that q_1 is the key factor of bifurcation and q_2 has little influence on chaotic bifurcation. It is different from previous work in fractional systems.

When $1.0746 < q_1 < 1.1$, $1.0498 < q_1 < 1.0746$, $1.0446 < q_1 < 1.0498\dots$, Hénon map with q_1 is period 1, period 2, period $4\dots$ in the bifurcation diagram. Based on numerical simulations, we get the values of branch points $q_{1,w}$ $(w = 1, 2, \dots)$ and calculate the limit of interval ratios in Hénon map with q_1 as follows:

$$\delta'' = \lim_{w\to\infty} \frac{q_{1,w} - q_{1,w-1}}{q_{1,w+1} - q_{1,w}} = 4.76923\dots. \tag{16}$$

The calculated δ'' is also approximate to δ which shows the self-similarity and invariance of scale transformation are generic in the chaotic systems and maps.

4 Conclusion

In this paper, with the Euler method and the Adams–Bashforth–Moulton predictor-corrector method, we get a class of chaotic maps with q. In numerical simulations, we investigate chaos and bifurcations of Logistic map with q and Hénon map with q. Stochasticity tests are done to further analyze the their properties. The chaotic maps with parameter q are totally different from original maps and become novel chaotic maps. They bifurcate not only with system parameters, but also inverse bifurcate versus parameter q. This provides us a new way to get chaos by reducing the value of q with limits. Especially, for Hénon map with q, bifurcation gradually becomes inverse bifurcation. In addition, bifurcation behaviors appear along with variation of q_1 and have little relationship with parameter q_2. This is different from fractional systems which bifurcate versus every fractional order. Finally, we find an available way to get a class of chaotic maps with q no matter whether system has three or more than three dimensions.

References

1. M. Caputo, Linear models of dissipation whose q is almost frequency independent, *Geophys. J. R. Astron. Soc.*, **13**:529–539, 1967.
2. L.P. Chen, Y. Chai, R.C. Wu, Linear matrix inequality criteria for robust synchronization of uncertain fractional-order chaotic systems, *Chaos*, **21**, 043107, 2011.
3. D.Y. Chen, C.F. Liu, C. Wu, H.H.C. Lu, Y.J. Liu, X.Y. Ma, Y.J. You, A new fractional-order chaotic system and its synchronization with circuit simulation, *Circuits Syst. Signal Process.*, **31**:1599–1613, 2012.
4. G.R. Chen, J.H. Lü, *Dynamical Analyses, Control and Synchronization of the Lorenz System Family*, Science Press, Beijing, 2003.
5. G.R. Chen, T. Ueta, Yet another chaotic attractor, *Int. J. Bifurcat. Chaos*, **19**:632–635, 2002.

6. D.Y. Chen, C. Wu, H.H.C. Lu, X.Y. Ma, Circuit simulation for synchronization of a fractional-order and integer-order chaotic system, *Nonlinear Dyn.*, **73**:1671–1686, 2013.
7. D.Y. Chen, R.F. Zhang, J.C. Sprott, H.T. Chen, X.Y. Ma, Synchronization between integer-order chaotic systems and a class of fractional-order chaotic systems via sliding mode control, *Chaos*, **22**, 023130, 2012.
8. W.H. Deng, C.P. Li, Chaos synchronization of the fractional Lü system, *Physica A*, **353**:61–72, 2005.
9. K. Diethelm, G. Walz, Numerical solution of fractional order differential equations by extrapolation, *Numer. Algorithms*, **16**:231–253, 1997.
10. M.J. Feigenbaum, Quantitative universality for a class of nonlinear transformations, *J. Stat. Phys.*, **19**:25–52, 1978.
11. I. Grigorenko, E. Grigorenko, Chaotic dynamics of the fractional Lorenz system, *Phys. Rev. Lett.*, **91**, 034101, 2003.
12. B.L. Hao, Symbolic dynamics and characterization of complexity, *Physica D*, **51**:161–176, 1991.
13. M. Hénon, 2-dimensional mapping with a strange attractor, *Commun. Math. Phys.*, **50**:69–77, 1976.
14. C.G. Li, G.R. Chen, Chaos in the fractional-order Chen system and its control, *Chaos Solitons Fractals*, **22**:549–554, 2004.
15. C.P. Li, G.J. Peng, Chaos in Chen's system with a fractional order, *Chaos Solitons Fractals*, **22**:443–450, 2004.
16. E.N. Lorenz, Deterministic nonperiodic flow, *J. Atmos. Sci.*, **20**:130–141, 1963.
17. J.H. Lü, G.R. Chen, A new chaotic attractor coined, *Int. J. Bifurcat. Chaos*, **12**:659–661, 2002.
18. NIST: A statistical test suite for random and pseudo random number generators for cryptographic applications, 2010, `http://csrc.nist.gov/publications/nistpubs/800-22-rev1a/SP800-22rev1a.pdf`.
19. O.E. Rössler, Equation for continuous chaos, *Phys. Lett. A*, **57**:397–398, 1976.
20. K.H. Sun, J.C. Sprott, Bifurcations of fractional-order diffusionless Lorenz system, *Int. J. Bifurcat. Chaos*, **20**:1209–1219, 2010.
21. M.S. Tavazoei, M. Haeri, Unreliability of frequency-domain approximation in recognising chaos in fractional-order systems, *IET Signal Process.*, **1**:171–181, 2007.
22. M.S. Tavazoei, M. Haeri, Limitations of frequency domain approximation for detecting chaos in fractional order systems, *Nonlinear Anal., Theory, Methods Appl.*, **69**:1299–1320, 2008.
23. X.Y. Wang, *Chaos in the Complex Nonlinearity System*, Electronics Industry Press, Beijing, 2003.
24. X.Y. Wang, X. Qin, A new pseudo-random number generator based on CML and chaotic iteration, *Nonlinear Dyn.*, **70**:1589–1592, 2012.

25. G.R. Wang, X.L. Yu, S.G. Chen, *Chaotic Control, Synchronization and Utilizing*, National Defence Industry Press, Beijing, 2001.

26. X.Y. Wang, H. Zhang, Chaotic synchronization of fractional-order spatiotemporal coupled Lorenz system, *Int. J. Mod. Phys. C*, **23**, 1250067, 2012.

27. Y. Xu, R.C. Gu, H.Q. Zhang, D.X. Li, Chaos in diffusionless lorenz system with a fractional order and its control, *Int. J. Bifurcat. Chaos*, **22**, 1250088, 2012.

28. R.X. Zhang, S.P. Yang, Robust chaos synchronization of fractional-order chaotic systems with unknown parameters and uncertain perturbations. *Nonlinear Dyn.*, **69**:983–992, 2012.