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Pullback attractors for the non-autonomous complex Ginzburg–Landau type equation with p-Laplacian*

Fang Li[a], **Bo You**[b,1]

[a]Department of Mathematics, Nanjing University
Nanjing, 210093, China
lifang101216@126.com

[b]School of Mathematics and Statistics, Xi'an Jiaotong University
Xi'an, 710049, China
youb2013@mail.xjtu.edu.cn

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Abstract. In this paper, we are concerned with the long-time behavior of the non-autonomous complex Ginzburg–Landau type equation with p-Laplacian. We first prove the existence of pullback absorbing sets in $L^2(\Omega)\cap W_0^{1,p}(\Omega)\cap L^q(\Omega)$ for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ corresponding to the non-autonomous complex Ginzburg–Landau type equation with p-Laplacian. Next, the existence of a pullback attractor in $L^2(\Omega)$ is established by the Sobolev compactness embedding theorem. Finally, we prove the existence of a pullback attractor in $W_0^{1,p}(\Omega)$ for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ by asymptotic a priori estimates.

Keywords: pullback attractor, non-autonomous, p-laplacian, complex Ginzburg–Landau type equations, Sobolev compactness embedding theorem, asymptotic a priori estimates.

1 Introduction

In this paper, we consider the existence of pullback attractors in $L^2(\Omega)$ and $W_0^{1,p}(\Omega)$ of the following non-autonomous complex Ginzburg–Landau type equation with p-Laplacian:

$$\frac{\partial u}{\partial t}-(\lambda+\mathrm{i}\alpha)\Delta_p u+\kappa|u|^{q-2}u+\mathrm{i}\beta|u|^{r-2}u-\gamma u=g(x,t),\quad (x,t)\in\Omega\times\mathbb{R}_\tau, \tag{1}$$

$$u=0,\quad (x,t)\in\partial\Omega\times\mathbb{R}_\tau, \tag{2}$$

$$u(x,\tau)=u_\tau(x),\quad x\in\Omega, \tag{3}$$

where $\Omega\subset\mathbb{R}^n(n\geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathrm{i}=\sqrt{-1}$, $\lambda>0$, $\kappa>0$, $\gamma>0$, α, $\beta\in\mathbb{R}$, $\mathbb{R}_\tau=[\tau,+\infty)$, the exponent $p\geqslant 2$, $q>r\geqslant 2$ are constants and u is a complex-valued unknown function.

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[1]Corresponding author.

The complex Ginzburg–Landau equation is known as an important model describing spatial pattern formation or the amplitude evolution of instability in non-equilibrium fluid dynamical systems as well in the theory of phase transitions and superconductivity (see [11, 26, 27]). In its special cases, the equation meets the nonlinear Schrödinger equation which is recently studied as various type equations with generalized nonlinear term. Therefore, more and more mathematicians have paid attention to the complex Ginzburg–Landau type equation in both theoretical physics and mathematics.

The case that $p = 2$ is the usual complex Ginzburg–Landau equation and many authors have studied it extensively by different methods in the recent years (see [2, 7, 13, 15, 16, 17, 28, 29, 30, 31, 37, 41]). In [13], the authors proved the existence of weak and strong solutions of the complex Ginzburg–Landau equation. The global existence of unique strong solutions was established in [30] for the complex Ginzburg–Landau equation under the assumption $|\beta|/\kappa \leqslant 1/c_p$ by a monotonicity method. In [31], the uniqueness of strong solutions for the complex Ginzburg–Landau equation in a bounded domain $\varOmega \subset \mathbb{R}^2$ was obtained. The global existence and smoothing effect was established in [41] by a monotonicity method for the complex Ginzburg–Landau type equation with the nonlinearity $\kappa|u|^{p-2}u + \mathrm{i}\beta|u|^{r-2}u$, where $q > r \geqslant 2$. In [7], the authors proved the global existence of strong solutions for the complex Ginzburg–Landau equation in $\mathbb{R}^n$ with initial date $u_0 \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ by a compactness method without any upper restriction on $p \geqslant 2$ but with the following restriction on $(\alpha/\lambda, \beta/\kappa)$:

$$\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{CGL}\left(\frac{1}{c_p}\right),$$

where

$$\mathrm{CGL}\left(\frac{1}{c_p}\right) := \left\{(x, y) \in \mathbb{R}^2\colon\ xy \geqslant 0 \text{ or } \frac{|xy| - 1}{|x| + |y|} \leqslant \frac{1}{c_p} = \frac{2\sqrt{p-1}}{p-2}\right\}.$$

Furthermore, if $2 \leqslant p < 2^* = 2n/(n-2)$, the strong solutions for the complex Ginzburg–Landau equation is unique. However, most of the methods used for $p = 2$ cannot be applied to (1)–(3) with $p > 2$, but there are many mathematicians who are still devote to the existence and uniqueness of strong solutions for the quasi-linear complex Ginzburg–Landau equation with p-Laplacian. For example, the authors proved the global existence and uniqueness of strong solutions and the continuous dependence of the initial datum with respect to the $W_0^{1,p}(\varOmega) \cap L^q(\varOmega)$-topology for the quasi-linear complex Ginzburg–Landau equation with p-Laplacian for different kinds of regular initial datum under some assumptions on the ratio $(\alpha/\lambda, \beta/\kappa)$ of the coefficients of (1)–(3) in [28]. In [29], the global existence, uniqueness and smoothing effect was proved for the quasi-linear complex Ginzburg–Landau equation with p-Laplacian.

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a dissipative system is to analyze the existence and structure of its attractor. Generally speaking, the attractor has a very complicated geometry which reflects the complexity of the long-time behavior of the system. Therefore, it is necessary to study the existence of attractors for the quasi-linear complex Ginzburg–Landau equation with p-Laplacian in the case of $n \geqslant 3$ to explore the complexity of its geometric structure. There have been many results for the usual complex Ginzburg–Landau equation in one- or two-dimensional space. For example, the author obtained the upper semi-continuity of approximations of attractors of the equation in one-dimensional space with $p = 4$ in [25]. The existence of global attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$ for the complex Ginzburg–Landau equation in the two-dimensional spaces was proved in [36]. In [17], the authors proved the existence of a global attractor in $L^2(\Omega)$ for the degenerate Ginzburg–Landau and parabolic equations by the semi-flow method. The authors paid more attention to the long-time behavior of the complex Ginzburg–Landau equation in the one- or two- dimensional spaces with nonlinearity $p = 2$ or $p = 6$ and obtained the existence of global attractors for the complex Ginzburg–Landau equation in the one- or two- dimensional spaces with different nonlinearity in [2, 12, 14, 32]. The existence of global attractors for the quasi-linear complex Ginzburg–Landau equation with p-Laplacian was obtained for $n \geqslant 3$ under assumption (4) in [42]. Many mathematicians have considered the long-time behavior of p-Laplacian equation with different kinds of boundary conditions, such as Dirichlet boundary conditions, dynamic flux boundary conditions and so on (see [1, 6, 39, 43]).

Non-autonomous equations appear in many applications in the natural sciences, so they are of great importance and interest. The long-time behavior of solutions of non-autonomous equations have been studied extensively in recent years (e.g., see [4, 5, 9, 10, 18, 19, 22, 33, 38, 40]). For instance, the existence of a pullback attractor in $L^2(\Omega)$ was studied in [3] when the external forcing is allowed to be unbounded in the norm of $L^2(\Omega)$ and the existence of a pullback attractor in $H_0^1(\Omega)$ was obtained in [35] under the condition of translation boundedness of the external forcing. Later, the existence of a pullback attractor in $H_0^1(\Omega)$ was considered in [21], while the existence of a pullback attractor in $L^p(\Omega)$ was obtained in [20] for the external forcing satisfies the exponential growth bound

$$\|g(s)\|_2^2 \leqslant M\mathrm{e}^{\alpha|s|}$$

for all $s \in \mathbb{R}$ and $0 \leqslant \alpha < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. This condition was recently considerably weakened to

$$\int_{-\infty}^{t} \mathrm{e}^{\lambda_1 s}\|g(s)\|_2^2 \,\mathrm{d}s < \infty$$

for all $t \in \mathbb{R}$, under which the existence of a pullback attractor in $L^p(\Omega)$, $L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ was obtained in [24, 40], respectively, and the existence of a pullback attractor in $H_0^1(\Omega)$ was proved in [23, 34].

The study of non-autonomous dynamical systems is an important subject, it is necessary to study the existence of pullback attractors for the non-autonomous complex Ginzburg–Landau type equation with p-Laplacian in the case of $n \geqslant 3$. Nevertheless, there are few results about the existence of pullback attractors for the non-autonomous complex Ginzburg–Landau type equation with p-Laplacian in the case of $n \geqslant 3$. There are three main reasons: Firstly, compared with the non-autonomous quasi-linear real Ginzburg–Landau equation with p- Laplacian, due to

$$(\lambda+\mathrm{i}\alpha)\int\limits_{\Omega}|\nabla u|^{p-2}\nabla u\cdot\nabla\big(|u|^{q-2}\overline{u}\big)\,\mathrm{d}x$$

and

$$\kappa\int\limits_{\Omega}|\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\big(|u|^{q-2}u\big)\,\mathrm{d}x+\mathrm{i}\beta)\int\limits_{\Omega}|\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\big(|u|^{r-2}u\big)\,\mathrm{d}x$$

are indefinite, it is difficult to obtain the existence of pullback absorbing set in $W_0^{1,p}(\Omega)\cap L^q(\Omega)$. Secondly, $u\geqslant 0$ is not meaningful for any $u\in C\setminus\mathbb{R}$, therefore, we cannot obtain the existence of pullback attractor in $L^q(\Omega)$ by estimating

$$\int\limits_{\Omega(|U(t,\tau)u_\tau|\geqslant M)}\big|U(t,\tau)u_\tau\big|^q\,\mathrm{d}x<\epsilon^q$$

to verify the ω-limit compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$. Thirdly, in our case of the non-autonomous quasi-linear complex Ginzburg–Landau equation with p-Laplacian, the growth order of nonlinear term $|u|^{q-2}u$ has no other restriction so that we cannot use $-\Delta\overline{u}$ as the test function to obtain higher regular pullback absorbing set as in [44], which increase the difficulty in verifying the compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$ associated with (1)–(3). Furthermore, some a priori estimates obtained for $n=1,2$ or the autonomous complex Ginzburg–Landau equation with p-Laplacian will be lost for $n\geqslant 3$ and $-\Delta_p$ is nonlinear operator for $p>2$ so that it is difficult to obtain the existence of pullback absorbing sets and get an appropriate form of compactness by verifying the pullback $\mathcal{D}$ condition. Therefore, it is necessary to make a restriction (4) on the ratio $(\alpha/\lambda,\beta/\kappa)$ of the coefficients of the nonlinear term, give a new Lemma 6 and combine the idea of norm-to-weak continuous with asymptotic a priori estimates to overcome these difficulties.

The main purpose of this paper is to study the long-time behavior for the non-autonomous complex Ginzburg–Landau type equation (1)–(3) with p-Laplacian under the assumptions

$$\left(\frac{\alpha}{\lambda},\frac{\beta}{\kappa}\right)\in S_1\left(\frac{1}{c_p}\right)\cap S_1\left(\frac{1}{c_q}\right)\tag{4}$$

and

$$\int\limits_{-\infty}^{t}\mathrm{e}^{\theta s}\big\|g(s)\big\|_2^2\,\mathrm{d}s<\infty\tag{5}$$

for any $t\in\mathbb{R}$, where

$$S_1(x_0)=\big\{(x,y)\in\mathbb{R}^2\colon\ |x|\leqslant x_0\big\},\qquad\theta=\min\{\lambda,\kappa\}.$$

We first prove the existence of pullback absorbing sets in $L^2(\Omega) \cap W_0^{1,p}(\Omega) \cap L^q(\Omega)$. Next, the existence of a pullback attractor in $L^2(\Omega)$ is obtained by the Sobolev compactness embedding theorem. Finally, we obtain the existence of a pullback attractor in $W_0^{1,p}(\Omega)$ by asymptotic a priori estimates. Here, we state our main theorem as follows.

Theorem 1. *Under the assumptions* (4)–(5) *with* $(|\alpha|/\lambda)c_p < 1 - \delta$ *for some* $\delta \in (0,1)$ *and* $g \in H^1_{\mathrm{loc}}(\mathbb{R}_\tau; L^2(\Omega))$, *let* $\{U(t,\tau)\}_{t \geqslant \tau}$ *be a process associated with the non-autonomous complex Ginzburg–Landau type equation* (1)–(3) *with* p*-Laplacian. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *has a pullback* $\mathcal{D}$*-attractor* $\mathcal{A}$ *in* $W_0^{1,p}(\Omega)$.

This paper is organized as follows. In the next section, we first recall some definitions and lemmas of pullback attractor, and then we give the definition of weak solutions and the well-posedness of weak solutions for the non-autonomous complex Ginzburg–Landau type equation (1)–(3) with p- Laplacian. Section 3 is devoted to proving the existence of pullback attractors in $L^2(\Omega)$ and $W_0^{1,p}(\Omega)$ for the non-autonomous complex Ginzburg–Landau type equation (1)–(3) with p-Laplacian under the assumptions (4)–(5).

Throughout this paper, we denote the conjugate of u by $\overline{u}$, the real part and imaginary part of u by $\mathrm{Re}[u]$ and $\mathrm{Im}[u]$, respectively. For the sake of simplicity, we denote the norm in $L^p(\Omega)$ by $\|\cdot\|_p$. We shall denote by C the genetic constants depending on λ, α, κ, β, p, q, which may be different from line to line (and even in the same line).

2 Preliminaries

In this section, we first recall some basic definitions and abstract results about pullback attractor.

Definition 1. (See [8, 21].) Let X be a complete metric space. A two-parameter family of mappings $\{U(t,\tau)\}_{t\geqslant\tau}$ is said to be a norm-to-weak continuous process in X if:

(i) $U(\tau,\tau) = Id$ for any $\tau \in \mathbb{R}$;
(ii) $U(t,r)U(r,\tau) = U(t,\tau)$ for any $t \geqslant r \geqslant \tau$;
(iii) $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$, if $x_n \to x$ in X.

Lemma 1. (*See* [21, 40].) *Let* X, Y *be two Banach spaces, and let* X^*, Y^* *be the dual spaces of* X, Y, *respectively. If* X *is dense in* Y, *the injection* $i : X \to Y$ *is continuous and its adjoint* $i^* : Y^* \to X^*$ *is dense. In addition, assume that* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is a norm-to-weak continuous process on* Y. *Then* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is a norm-to-weak continuous process on* X *if and only if* $\{U(t,\tau)\}_{t\geqslant\tau}$ *maps compact sets of* X *into bounded sets of* X *for any* t, $\tau \in \mathbb{R}$, $t \geqslant \tau$.

Let $\mathcal{D}$ be a nonempty class of families $\hat{D} = \{D(t)\colon t \in \mathbb{R}\}$ of nonempty subsets of X.

Definition 2. (See [40].) A family $\hat{\mathcal{A}} = \{A(t)\colon t \in \mathbb{R}\}$ of nonempty subsets of X is said to be a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in X if:

(i) $A(t)$ is compact in X for any $t \in \mathbb{R}$;

(ii) $\hat{\mathcal{A}}$ is invariant, i.e., $U(t,\tau)A(\tau) = A(t)$ for any $\tau \leqslant t$;

(iii) $\hat{\mathcal{A}}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}\big(U(t,\tau)D(\tau), A(t)\big) = 0$$

for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$.

Such a family $\hat{\mathcal{A}}$ is called minimal $A(t) \subset C(t)$ if for any family $\hat{C} = \{C(t)\colon t \in \mathbb{R}\}$ of closed subsets of X, $\lim_{\tau \to -\infty} \operatorname{dist}(U(t,\tau)B(\tau), C(t)) = 0$.

Definition 3. (See [4, 40].) It is said that $\hat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}_{t \geqslant \tau}$ if for any $\hat{D} \in \mathcal{D}$ and any $t \in \mathbb{R}$, there exists a $\tau_0(t,\hat{D}) \leqslant t$ such that $U(t,\tau)D(\tau) \subset B(t)$ for any $\tau \leqslant \tau_0(t,\hat{D})$.

Definition 4. (See [4].) The process $\{U(t,\tau)\}_{t \geqslant \tau}$ is said to be pullback $\mathcal{D}$-asymptotically compact, if for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$ and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ is relatively compact in X.

Lemma 2. (*See* [4, 21, 40].) *Let $\{U(t,\tau)\}_{t \geqslant \tau}$ be a process in X satisfying the following conditions*:

(i) *$\{U(t,\tau)\}_{t \geqslant \tau}$ be norm-to-weak continuous in X*;

(ii) *there exists a family $\hat{B}$ of pullback $\mathcal{D}$-absorbing sets $\{B(t)\colon t \in \mathbb{R}\}$ in X*;

(iii) *$\{U(t,\tau)\}_{t \geqslant \tau}$ is pullback $\mathcal{D}$-asymptotically compact.*

Then there exists a minimal pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t)\colon t \in \mathbb{R}\}$ in X given by

$$A(t) = \bigcap_{s \leqslant t} \overline{\bigcup_{\tau \leqslant s} U(t,\tau)B(\tau)}.$$

Lemma 3. (*See* [28].) *Let $p \in (1,\infty)$. Then for non-zero z, $w \in \mathbb{C}$ with $z \neq w$,*

$$\big|\operatorname{Im}\big[\big(|z|^{p-2} - |w|^{p-2}, z - w\big)\big]\big| \leqslant c_p \operatorname{Re}\big[\big(|z|^{p-2} - |w|^{p-2}, z - w\big)\big].$$

Lemma 4. (*See* [28].)

(i) *Let $p \geqslant 2$. Then for z, $w \in \mathbb{C}$,*

$$\operatorname{Re}\big[\big(|z|^{p-2} - |w|^{p-2}, z - w\big)\big] \geqslant 2^{2-p}|z - w|^p.$$

(ii) *Let $p \in (1,2)$. Then for non-zero z, $w \in \mathbb{C}$,*

$$\operatorname{Re}\big[\big(|z|^{p-2} - |w|^{p-2}, z - w\big)\big] \geqslant \frac{(p-1)|z-w|^2}{\max\{|z|^{2-p}, |w|^{2-p}\}}.$$

Lemma 5. *Let $q \in (2,+\infty)$. Then for any $u \in C_0^\infty(\Omega)$, we have*

$$\left|\operatorname{Im}\left[\int_\Omega |\nabla u|^{p-2}\nabla \overline{u} \cdot \nabla\big(|u|^{q-2}u\big)\,\mathrm{d}x\right]\right| \leqslant c_q \operatorname{Re}\left[\int_\Omega |\nabla u|^{p-2}\nabla \overline{u} \cdot \nabla\big(|u|^{q-2}u\big)\,\mathrm{d}x\right].$$

Lemma 6. *Let $q > r \geqslant 2$. Then for every $\epsilon > 0$, there exists a positive constant C_ϵ such that for any $u \in C_0^\infty(\Omega)$, we have*

$$\left|\mathrm{Im}\left[\int_\Omega |\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\big(|u|^{r-2}u\big)\,\mathrm{d}x\right]\right| \leqslant \epsilon\,\mathrm{Re}\left[\int_\Omega |\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\big(|u|^{q-2}u\big)\,\mathrm{d}x\right] + C_\epsilon\|\nabla u\|_p^p.$$

Lemma 7. (See [23, 40].) *Suppose that*

$$y'(s) + \delta y(s) \leqslant b(s)$$

for some $\delta > 0$, $t_0 \in \mathbb{R}$ and for any $s \geqslant t_0$, where the functions y, y', b are assumed to be locally integrable and y, b are nonnegative on the interval $t < s < t + r$ for some $t \geqslant t_0$. Then

$$y(t+r) \leqslant \mathrm{e}^{-\delta r/2}\frac{2}{r}\int_t^{t+r/2} y(s)\,\mathrm{d}s + \mathrm{e}^{-\delta(t+r)}\int_t^{t+r}\mathrm{e}^{\delta s}b(s)\,\mathrm{d}s$$

for all $t \geqslant t_0$.

Next, we recall the definition of weak solutions for the non-autonomous complex Ginzburg–Landau type equation (1)–(3) with p-Laplacian.

Definition 5. (See [36].) Assume that $u_\tau \in L^2(\Omega)$, $g \in L^2_{\mathrm{loc}}(\mathbb{R}_\tau; L^2(\Omega))$. A complex-valued function $u(x,t)$ is called a weak solution for the non-autonomous complex Ginzburg–Landau type equation (1)–(3) with p-Laplacian if:

(i) $u(x,t) \in C(\mathbb{R}_\tau; L^2(\Omega)) \cap L^p(\mathbb{R}_\tau; W_0^{1,p}(\Omega)) \cap L^q(\mathbb{R}_\tau; L^q(\Omega))$;
(ii) $u(x,t)$ satisfies equation (1)–(3) in the sense of distribution and $u(x,\tau) = u_\tau \in L^2(\Omega)$.

Finally, we give the well-posedness of weak solution $u(x,t)$ for the non-autonomous complex Ginzburg–Landau type equation (1)-(3) with p-Laplacian, which can be obtained by the Faedo–Galerkin method (see [36]). Here we only state it as follows.

Theorem 2. *Assume that $u_\tau \in L^2(\Omega)$, $g \in L^2_{\mathrm{loc}}(\mathbb{R}_\tau; L^2(\Omega))$ and $(\alpha/\lambda, \beta/\kappa) \in S(1/c_p, 1/c_q)$. Then there exists a unique weak solution $u(x,t) \in C(\mathbb{R}_\tau; L^2(\Omega))$ for the non-autonomous complex Ginzburg–Landau type equation* (1)–(3) *with p-Laplacian and $u_\tau \to u(t)$ is continuous on $L^2(\Omega)$.*

By Theorem 2, we can define the operator process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $L^2(\Omega)$ as

$$U(.,\tau)u_\tau : \mathbb{R}^+ \times L^2(\Omega) \to L^2(\Omega),$$

which is $(L^2(\Omega), L^2(\Omega))$-continuous.

3 The existence of pullback attractors

In this section, we prove the existence of pullback attractors in $L^2(\Omega)$ and $W_0^{1,p}(\Omega)$ for the non-autonomous complex Ginzburg–Landau type equation (1)–(3) with p-Laplacian under assumptions (4)–(5).

3.1 The existence of a pullback attractor in $L^2(\Omega)$

In the following, let $\mathcal{D}$ be the class of all families $\{D(t)\colon\ t \in \mathbb{R}\}$ of nonempty subsets of $L^2(\Omega)$ such that

$$\lim_{t\to-\infty} \mathrm{e}^{\theta t}\bigl[D(t)\bigr] = 0,$$

where $[D(t)] = \sup\{\|u\|_2\colon\ u \in D(t)\}$. We first prove the existence of pullback absorbing sets in $L^2(\Omega)\cap W_0^{1,p}(\Omega)\cap L^q(\Omega)$ for the non-autonomous complex Ginzburg–Landau type equation (1)–(3) with p-Laplacian under assumptions (4)–(5).

Theorem 3. *Assume that the assumptions* (4)–(5) *hold and* $g \in L^2_{\mathrm{loc}}(\mathbb{R}_\tau; L^2(\Omega))$. *Let* $\{U(t,\tau)\}_{t\geqslant\tau}$ *be a process associated with the non-autonomous complex Ginzburg–Landau type equation* (1)–(3) *with p-Laplacian. Then there exists a pullback $\mathcal{D}$- absorbing set in* $L^2(\Omega) \cap L^q(\Omega) \cap W_0^{1,p}(\Omega)$.

Proof. Multiplying (1) by $\overline{u}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigl\|u(t)\bigr\|_2^2 + \lambda\|\nabla u\|_p^p + \kappa\|u\|_q^q - \gamma\|u\|_2^2 \leqslant \bigl\|g(t)\bigr\|_2\|u\|_2. \tag{6}$$

Taking the inner product of (1) with $-\Delta_p\overline{u}$, we have

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\bigl\|\nabla u(t)\bigr\|_p^p + \lambda\bigl\|\Delta_p u\|_2^2 + \kappa\,\mathrm{Re}\biggl[\int_\Omega |\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\bigl(|u|^{q-2}u\bigr)\,\mathrm{d}x\biggr]\\
&\quad - \beta\,\mathrm{Im}\biggl[\int_\Omega |\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\bigl(|u|^{r-2}u\bigr)\,\mathrm{d}x\biggr] - \gamma\|\nabla u\|_p^p\\
&\leqslant \bigl\|g(t)\bigr\|_2\|\Delta_p u\|_2.
\end{aligned} \tag{7}$$

Multiplying (1) by $|u|^{q-2}\overline{u}$ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\bigl\|u(t)\bigr\|_q^q + \lambda\,\mathrm{Re}\biggl[\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\bigl(|u|^{q-2}\overline{u}\bigr)\,\mathrm{d}x\biggr] + \kappa\|u\|_{2(q-1)}^{2(q-1)}\\
&\quad - \alpha\,\mathrm{Im}\biggl[\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\bigl(|u|^{q-2}\overline{u}\bigr)\,\mathrm{d}x\biggr] - \gamma\|u\|_q^q\\
&\leqslant \bigl\|g(t)\bigr\|_2\|u\|_{2(q-1)}^{q-1}.
\end{aligned} \tag{8}$$

Thanks to (4) and Lemmas 5–6, we deduce that

$$\kappa \operatorname{Re}\biggl[\int\limits_{\Omega} |\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\bigl(|u|^{q-2}u\bigr)\,\mathrm{d}x\biggr] - \beta \operatorname{Im}\biggl[\int\limits_{\Omega} |\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\bigl(|u|^{r-2}u\bigr)\,\mathrm{d}x\biggr]$$
$$\geqslant \frac{\kappa}{2}\operatorname{Re}\biggl[\int\limits_{\Omega} |\nabla u|^{p-2}\nabla\overline{u}\cdot\nabla\bigl(|u|^{q-2}u\bigr)\,\mathrm{d}x\biggr] - C\bigl(\kappa,|\beta|\bigr)\|\nabla u\|_p^p \tag{9}$$

and

$$\lambda \operatorname{Re}\biggl[\int\limits_{\Omega} |\nabla u|^{p-2}\nabla u\cdot\nabla\bigl(|u|^{q-2}\overline{u}\bigr)\,\mathrm{d}x\biggr] - \alpha \operatorname{Im}\biggl[\int\limits_{\Omega} |\nabla u|^{p-2}\nabla u\cdot\nabla\bigl(|u|^{q-2}\overline{u}\bigr)\,\mathrm{d}x\biggr]$$
$$\geqslant \lambda \operatorname{Re}\biggl[\int\limits_{\Omega} |\nabla u|^{p-2}\nabla u\cdot\nabla\bigl(|u|^{q-2}\overline{u}\bigr)\,\mathrm{d}x\biggr]\biggl(1-\frac{|\alpha|}{\lambda}c_p\biggr) \geqslant 0. \tag{10}$$

It follows from (6)–(10), Hölder inequality, interpolation inequality and Young inequality that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\|u(t)\|_2^2 + \|u(t)\|_q^q + \|\nabla u(t)\|_p^p\bigr)\\
&\quad + \theta\bigl(\|u(t)\|_2^2 + \|u(t)\|_q^q + \|\nabla u(t)\|_p^p + \|\Delta_p u\|_2^2 + \|u\|_{2(q-1)}^{2(q-1)}\bigr)\\
&\quad \leqslant C\bigl(\lambda,\gamma,\kappa,p,q,|\Omega|\bigr) + C(\lambda,\gamma,\kappa,p,q)\|g(t)\|_2^2 + C\bigl(\kappa,|\beta|,\gamma\bigr)\|\nabla u\|_p^p.
\end{aligned} \tag{11}$$

Let $H(u) = \|u(t)\|_2^2 + \|u(t)\|_q^q + \|\nabla u(t)\|_p^p$. From (10), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H(u) + \theta H(u)\\
&\quad \leqslant C\bigl(\lambda,\gamma,\kappa,p,q,|\Omega|\bigr) + C(\lambda,\gamma,\kappa,p,q)\|g(t)\|_2^2 + C\bigl(\kappa,|\beta|,\gamma\bigr)\|\nabla u\|_p^p.
\end{aligned} \tag{12}$$

Using Lemma 7, we obtain

$$\begin{aligned}
H\bigl(u(t+r)\bigr) &\leqslant \mathrm{e}^{-\theta r/2}\frac{2}{r}\int\limits_t^{t+r/2} H\bigl(u(s)\bigr)\,\mathrm{d}s + C\bigl(\lambda,\gamma,\kappa,p,q,|\Omega|\bigr)\mathrm{e}^{-\theta(t+r)}\int\limits_t^{t+r}\mathrm{e}^{\theta s}\,\mathrm{d}s\\
&\quad + C(\lambda,\gamma,\kappa,p,q)\mathrm{e}^{-\theta(t+r)}\int\limits_t^{t+r}\mathrm{e}^{\theta s}\|g(s)\|_2^2\,\mathrm{d}s\\
&\quad + C\bigl(\kappa,|\beta|,\gamma\bigr)\mathrm{e}^{-\theta(t+r)}\int\limits_t^{t+r}\mathrm{e}^{\theta s}\|\nabla u(s)\|_p^p\,\mathrm{d}s\\
&\leqslant \mathrm{e}^{-\theta r/2}\frac{2}{r}\int\limits_t^{t+r/2} H\bigl(u(s)\bigr)\,\mathrm{d}s + C\bigl(\lambda,\gamma,\kappa,p,q,|\Omega|\bigr)
\end{aligned}$$

$$+ C(\lambda,\gamma,\kappa,p,q)\mathrm{e}^{-\theta(t+r)}\int_t^{t+r}\mathrm{e}^{\theta s}\big\|g(s)\big\|_2^2\,\mathrm{d}s$$
$$+ C\big(\kappa,|\beta|,\gamma\big)\int_t^{t+r}\big\|\nabla u(s)\big\|_p^p\,\mathrm{d}s. \tag{13}$$

Next, we estimate the first term in the right hand side of (13).

Combining (6) with Young inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|u(t)\big\|_2^2 + \theta\big(\|\nabla u\|_p^p + \|u\|_q^q + \|u\|_2^2\big) \leqslant C(\gamma,\kappa)\big\|g(t)\big\|_2^2 + C\big(\kappa,\gamma,p,|\Omega|\big). \tag{14}$$

From the classical Gronwall inequality, we get

$$\big\|u(t)\big\|_2^2 \leqslant \|u_\tau\|_2^2\mathrm{e}^{\theta(\tau-t)} + C\big(|\Omega|,p,\kappa,\gamma,\theta\big) + C(\gamma,\kappa)\int_\tau^t \mathrm{e}^{-\theta(t-s)}\big\|g(s)\big\|_2^2\,\mathrm{d}s,$$

which implies

$$\big\|u(t)\big\|_2^2 \leqslant 2C\big(|\Omega|,p,\kappa,\gamma,\theta\big) + 2C(\gamma,\kappa)\mathrm{e}^{-\theta t}\int_{-\infty}^t \mathrm{e}^{\theta s}\big\|g(s)\big\|_2^2\,\mathrm{d}s \tag{15}$$

uniformly with respect to all initial conditions $u_\tau \in D(\tau)$ for $\tau \leqslant \tau_0(t,\hat{D})$.

Integrating (14) from t to $t+r/2$ and using (15), we get

$$\begin{aligned}\theta\int_t^{t+r/2} H\big(u(s)\big)\,\mathrm{d}s &\leqslant \big\|u(t)\big\|_2^2 + C(\gamma,\kappa)\int_t^{t+r/2}\big\|g(s)\big\|_2^2\,\mathrm{d}s + C\big(\kappa,\gamma,p,r,|\Omega|\big)\\ &\leqslant \big\|u(t)\big\|_2^2 + C(\gamma,\kappa)\mathrm{e}^{-\theta t}\int_t^{t+r/2}\mathrm{e}^{\theta s}\big\|g(s)\big\|_2^2\,\mathrm{d}s + C\big(\kappa,\gamma,p,r,|\Omega|\big)\\ &\leqslant C\big(\kappa,\gamma,p,r,|\Omega|\big)\Bigg(1+\mathrm{e}^{-\theta t}\int_{-\infty}^{t+r}\mathrm{e}^{\theta s}\big\|g(s)\big\|_2^2\,\mathrm{d}s\Bigg).\end{aligned} \tag{16}$$

By integrating (14) from t to $t+r$ and using (15), we find

$$\int_t^{t+r} H\big(u(s)\big)\,\mathrm{d}s \leqslant C\big(\kappa,\gamma,p,r,|\Omega|\big)\Bigg(1+\mathrm{e}^{-\theta t}\int_{-\infty}^{t+r}\mathrm{e}^{\theta s}\big\|g(s)\big\|_2^2\,\mathrm{d}s\Bigg) \tag{17}$$

uniformly with respect to all initial conditions $u_\tau \in D(\tau)$ for $\tau \leqslant \tau_0(t,\hat{D})$.

Combining (13) with (16)–(17), we conclude that

$$H\big(u(t+r)\big) \leqslant C\big(\kappa,\gamma,\lambda,p,q,r,|\Omega|\big)\left(1+\mathrm{e}^{-\theta t}\int\limits_{-\infty}^{t+r}\mathrm{e}^{\theta s}\big\|g(s)\big\|_2^2\,\mathrm{d}s\right) \tag{18}$$

uniformly with respect to all initial conditions $u_\tau \in D(\tau)$ for $\tau \leqslant \tau_0(t,\hat{D})$. □

Since $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ is compact, we have the following result.

Corollary 1. *Assume that assumptions* (4)–(5) *hold and* $g \in L^2_{\mathrm{loc}}(\mathbb{R}_\tau; L^2(\Omega))$. *Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *associated with the non-autonomous complex Ginzburg–Landau type equation* (1)–(3) *with* p*-Laplacian has a pullback* $\mathcal{D}$*-attractor* $\mathcal{A}_2$ *in* $L^2(\Omega)$*, which is compact, connected and invariant.*

3.2 The existence of a pullback attractor in $W_0^{1,p}(\Omega)$

From Theorem 3 and Lemma 1, we deduce that the process $\{U(t,\tau)\}_{t\geqslant\tau}$ associated with (1)-(3) is norm-to-weak continuous in $W_0^{1,p}(\Omega)$. In this subsection, we prove the existence of a pullback attractor in $W_0^{1,p}(\Omega)$ for the non-autonomous complex Ginzburg–Landau type equation (1)-(3) with p-Laplacian by asymptotic a priori estimates.

First, we give a auxiliary theorem to prove the asymptotical compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $W_0^{1,p}(\Omega)$.

Theorem 4. *Assume that* $(\alpha/\lambda, \beta/\kappa)$ *satisfies* (4) *and* $g \in H^1_{\mathrm{loc}}(\mathbb{R}_\tau; L^2(\Omega))$. *Let* $\{U(t,\tau)\}_{t\geqslant\tau}$ *be a process associated with the non-autonomous complex Ginzburg–Landau type equation* (1)–(3) *with* p*-Laplacian. Then for any* $\hat{D} \in \mathcal{D}$ *and* $t \in \mathbb{R}$*, there exist a family of positive constants* $\{\rho(t)\colon t \in \mathbb{R}\}$ *and* $\tau_1(t,\hat{D}) \leqslant t$ *such that*

$$\big\|u_t(s)\big\|^2_{L^2(\Omega)} \leqslant \rho(t)$$

for any $u_\tau \in D(\tau)$ *and* $\tau \leqslant \tau_1(t,\hat{D})$*, where* $u_t(s) = (\mathrm{d}/\mathrm{d}t)U(t,\tau)u_\tau|_{t=s}$ *and* $\rho(t)$ *is a positive constant which is independent of the initial data.*

Proof. Denote $v = u_t$. It is clear that v satisfies the following equation obtained by differentiating equation (1) with respect to t:

$$\frac{\partial v}{\partial t} - (\lambda+\mathrm{i}\alpha)\frac{\partial(\Delta_p u)}{\partial t} + \kappa\frac{\partial(|u|^{q-2}u)}{\partial t} + \mathrm{i}\beta\frac{\partial(|u|^{r-2}u)}{\partial t} - \gamma v = g_t(x,t). \tag{19}$$

Taking the inner product of (19) with $\overline{v}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_2^2 + \lambda\,\mathrm{Re}\bigg[\int\limits_\Omega \frac{\partial|\nabla u|^{p-2}\nabla u}{\partial t}\cdot\nabla\overline{v}\,\mathrm{d}x\bigg] - \alpha\,\mathrm{Im}\bigg[\int\limits_\Omega \frac{\partial|\nabla u|^{p-2}\nabla u}{\partial t}\cdot\nabla\overline{v}\,\mathrm{d}x\bigg]\\
&\quad + \kappa\,\mathrm{Re}\bigg[\int\limits_\Omega \frac{\partial|u|^{q-2}u}{\partial t}\overline{v}\,\mathrm{d}x\bigg] - \beta\,\mathrm{Im}\bigg[\int\limits_\Omega \frac{\partial|u|^{r-2}u}{\partial t}\overline{v}\,\mathrm{d}x\bigg]\\
&\quad \leqslant \gamma\|v\|_2^2 + \|g_t\|_2\|v\|_2.
\end{aligned} \tag{20}$$

By mean of the method in the proof of Lemmas 5–6 and combining (4) with Lemmas 3–4, we obtain

$$\lambda \operatorname{Re}\biggl[\int_\Omega \frac{\partial |\nabla u|^{p-2}\nabla u}{\partial t}\cdot \nabla \overline{v}\,\mathrm{d}x\biggr] - \alpha \operatorname{Im}\biggl[\int_\Omega \frac{\partial |\nabla u|^{p-2}\nabla u}{\partial t}\cdot \nabla \overline{v}\,\mathrm{d}x\biggr] \geqslant 0 \tag{21}$$

and

$$\begin{aligned}&\kappa \operatorname{Re}\biggl[\int_\Omega \frac{\partial |u|^{q-2}u}{\partial t}\overline{v}\,\mathrm{d}x\biggr] - \beta \operatorname{Im}\biggl[\int_\Omega \frac{\partial |u|^{r-2}u}{\partial t}\overline{v}\,\mathrm{d}x\biggr]\\ &\quad\geqslant \kappa \operatorname{Re}\biggl[\int_\Omega \frac{\partial |u|^{q-2}u}{\partial t}\overline{v}\,\mathrm{d}x\biggr] - C\|v\|_2^2.\end{aligned} \tag{22}$$

Thanks to (20)–(22), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_2^2 \leqslant 2(\gamma + C)\|v\|_2^2 + C\|g_t\|_2^2.$$

Integrating (11) from t to $t+r$ and using (17)–(18), we obtain

$$\begin{aligned}&\lambda \int_t^{t+1} \bigl\|\Delta_p u(s)\bigr\|_2^2\,\mathrm{d}s + \kappa \int_t^{t+1} \bigl\|u(s)\bigr\|_{2(q-1)}^{2(q-1)}\,\mathrm{d}s\\ &\quad\leqslant C\bigl(\kappa,\gamma,\lambda,p,q,r,|\Omega|\bigr)\Biggl(1 + \mathrm{e}^{-\theta t}\int_{-\infty}^{t+r} \mathrm{e}^{\theta s}\bigl\|g(s)\bigr\|_2^2\,\mathrm{d}s\Biggr)\end{aligned}$$

uniformly with respect to all initial conditions $u_\tau \in D(\tau)$ for $\tau \leqslant \tau_0(t,\hat{D})$.

Since

$$\|v\|_2 \leqslant \sqrt{\lambda^2 + |\alpha|^2}\|\Delta_p u\|_2 + \kappa\|u\|_{2(q-1)}^{q-1} + |\beta|\|u\|_{2(r-1)}^{r-1} + \gamma\|u\|_2 + \|g(t)\|_2,$$

we obtain that

$$\int_t^{t+1} \bigl\|u_t(s)\bigr\|_2^2\,\mathrm{d}s \leqslant C\bigl(\kappa,\gamma,\lambda,p,q,r,|\Omega|\bigr)\Biggl(1 + \mathrm{e}^{-\theta t}\int_{-\infty}^{t+r} \mathrm{e}^{\theta s}\bigl\|g(s)\bigr\|_2^2\,\mathrm{d}s\Biggr)$$

uniformly with respect to all initial conditions $u_\tau \in D(\tau)$ for $\tau \leqslant \tau_0(t,\hat{D})$.

Using the uniform Gronwall inequality, we get

$$\bigl\|u_t(t+2r)\bigr\|_2^2 \leqslant C\bigl(\kappa,\gamma,\lambda,p,q,r,|\Omega|\bigr)\Biggl(1 + \mathrm{e}^{-\theta t}\int_{-\infty}^{t+r} \mathrm{e}^{\theta s}\bigl(\bigl\|g(s)\bigr\|_2^2 + \bigl\|g_t(s)\bigr\|_2^2\bigr)\,\mathrm{d}s\Biggr)$$

uniformly with respect to all initial conditions $u_\tau \in D(\tau)$ for $\tau \leqslant \tau_1(t,\hat{D})$. □

Theorem 5. *Under the assumptions* (4)–(5) *with* $|\alpha|/\lambda c_p < 1-\delta$ *for some* $\delta \in (0,1)$ *and* $g \in H^1_{\mathrm{loc}}(\mathbb{R}_\tau; L^2(\Omega))$*, let* $\{U(t,\tau)\}_{t\geqslant\tau}$ *be a process associated with the non-autonomous complex Ginzburg–Landau type equation* (1)–(3) *with* p*-Laplacian. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is pullback* $\mathcal{D}$*-asymptotically compact in* $W_0^{1,p}(\Omega)$*.*

Proof. Let $B_0 = \{B(t)\colon t \in \mathbb{R}\}$ be a pullback $\mathcal{D}$-absorbing set in $W_0^{1,p}(\Omega) \cap L^q(\Omega) \cap L^2(\Omega)$ obtained in Theorem 3, then we need only to show that for any $t \in \mathbb{R}$, any $\tau_n \to -\infty$ and $u_{0n} \in B(\tau_n)$, $\{u_n(\tau_n)\}_{n=0}^\infty$ is pre-compact in $W_0^{1,p}(\Omega)$, where $u_n(\tau_n) = u(t;\tau_n,u_{0n}) = U(t,\tau_n)u_{0n}$.

In fact, from Corollary 1, we know that $\{u_n(\tau_n)\}_{n=0}^\infty$ is pre-compact in $L^2(\Omega)$. Without loss of generality, we assume that $\{u_n(\tau_n)\}_{n=0}^\infty\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(\Omega)$.

In the following, we prove that$\{u_n(\tau_n)\}_{n=0}^\infty$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$. By simply calculations, we deduce from Lemmas 3–4 that

$$\begin{aligned}
&\lambda\delta 2^{2-p}\|u_{n_k}(\tau_{n_k}) - u_{n_j}(\tau_{n_j})\|^p_{W_0^{1,p}(\Omega)}\\
&\quad+\kappa\,\mathrm{Re}\big[\big(\big|u_{n_k}(\tau_{n_k})\big|^{q-2}u_{n_k}(\tau_{n_k})-\big|u_{n_j}(\tau_{n_j})\big|^{q-2}u_{n_j}(\tau_{n_j}), u_{n_k}(\tau_{n_k})-u_{n_j}(\tau_{n_j})\big)\big]\\
&\quad-\beta\,\mathrm{Re}\big[\big(\big|u_{n_k}(\tau_{n_k})\big|^{r-2}u_{n_k}(\tau_{n_k})-\big|u_{n_j}(\tau_{n_j})\big|^{r-2}u_{n_j}(\tau_{n_j}), u_{n_k}(\tau_{n_k})-u_{n_j}(\tau_{n_j})\big)\big]\\
&\quad\leqslant \left(-\frac{\mathrm{d}}{\mathrm{d}t}u_{n_k}(\tau_{n_k})+\gamma u_{n_k}(\tau_{n_k})+\frac{\mathrm{d}}{\mathrm{d}t}u_{n_j}(\tau_{n_j})-\gamma u_{n_j}(\tau_{n_j}), u_{n_k}(\tau_{n_k})-u_{n_j}(\tau_{n_j})\right)\\
&\quad\leqslant \left\|\frac{\mathrm{d}}{\mathrm{d}t}u_{n_k}(\tau_{n_k}) - \frac{\mathrm{d}}{\mathrm{d}t}u_{n_j}(\tau_{n_j})\right\|_{L^2(\Omega)}\big\|u_{n_k}(\tau_{n_k}) - u_{n_j}(\tau_{n_j})\big\|_{L^2(\Omega)}\\
&\quad\quad+\gamma\big\|u_{n_k}(\tau_{n_k}) - u_{n_j}(\tau_{n_j})\big\|^2_{L^2(\Omega)}.
\end{aligned}$$

By mean of the method in the proof of Lemmas 5-6 and combining (4) with Lemmas 3-4, we obtain

$$\begin{aligned}
&\kappa\,\mathrm{Re}\big[\big(\big|u_{n_k}(\tau_{n_k})\big|^{q-2}u_{n_k}(\tau_{n_k})-\big|u_{n_j}(\tau_{n_j})\big|^{q-2}u_{n_j}(\tau_{n_j}), u_{n_k}(\tau_{n_k})-u_{n_j}(\tau_{n_j})\big)\big]\\
&\quad-\beta\,\mathrm{Re}\big[\big(\big|u_{n_k}(\tau_{n_k})\big|^{r-2}u_{n_k}(\tau_{n_k})-\big|u_{n_j}(\tau_{n_j})\big|^{r-2}u_{n_j}(\tau_{n_j}), u_{n_k}(\tau_{n_k})-u_{n_j}(\tau_{n_j})\big)\big]\\
&\quad\geqslant -C\big\|u_{n_k}(\tau_{n_k})-u_{n_j}(\tau_{n_j})\big\|^2_{L^2(\Omega)}.
\end{aligned}$$

Thanks to Theorem 1 and Theorem 4, Theorem 5 is proved immediately. □

From Theorems 3, 5 and Lemma 2, we obtain directly our main Theorem 1.

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