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Random convolution of $\mathcal{O}$-exponential distributions*

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Abstract. Assume that $\xi_1, \xi_2, \ldots$ are independent and identically distributed non-negative random variables having the $\mathcal{O}$-exponential distribution. Suppose that η is a nonnegative non-degenerate at zero integer-valued random variable independent of $\xi_1, \xi_2, \ldots$. In this paper, we consider the conditions for η under which the distribution of random sum $\xi_1 + \xi_2 + \cdots + \xi_\eta$ remains in the class of $\mathcal{O}$-exponential distributions.

Keywords: long tail, random sum, closure property, $\mathcal{O}$-exponential distribution.

1 Introduction

Let $\xi_1, \xi_2, \ldots$ be independent copies of a random variable (r.v.) ξ with distribution function (d.f.) F_ξ. Let η be a nonnegative non-degenerate at zero integer-valued r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. We suppose that F_ξ is $\mathcal{O}$-exponential and we find minimal conditions under which the d.f.

$$\begin{aligned} F_{S_\eta}(x) &:= \mathbf{P}(\xi_1 + \xi_2 + \cdots + \xi_\eta \leqslant x) \\ &= \sum_{n=0}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}(\xi_1 + \xi_2 + \cdots + \xi_n \leqslant x) \\ &= \sum_{n=0}^{\infty} \mathbf{P}(\eta = n) F_\xi^{*n}(x) \end{aligned}$$

belongs to the class of $\mathcal{O}$-exponential distributions as well. Here and elsewhere in this paper, F^{*n} denotes the n-fold convolution of d.f. F. Theorem 1 below is the main result of this paper. Before the exact formulation of this theorem, we recall the definition of $\mathcal{O}$-exponential and some related d.f.'s classes. In all definitions below, we assume that $\overline{F}(x) = 1 - F(x) > 0$ for all $x \in \mathbb{R}$.

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Definition 1. For $\gamma > 0$, by $\mathcal{L}(\gamma)$ we denote the class of exponential d.f.s, i.e. $F \in \mathcal{L}(\gamma)$ if for any fixed real y,

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\gamma y}.$$

In the case $\gamma = 0$, class $\mathcal{L}(0)$ is called the long-tailed distribution class and is denoted by $\mathcal{L}$.

Definition 2. A d.f. F belongs to the dominated varying-tailed class ($F \in \mathcal{D}$) if for any fixed $y \in (0, 1)$,

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Definition 3. A d.f. F is $\mathcal{O}$-exponential ($F \in \mathcal{OL}$) if for any fixed $y \in \mathbb{R}$,

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} < \infty.$$

It is easy to see that the following inclusions hold:

$$\mathcal{D} \subset \mathcal{OL}, \qquad \mathcal{L} \subset \mathcal{OL}, \qquad \bigcup_{\gamma\geqslant 0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

In [2, 3], Cline claimed that d.f. F_{S_η} remains in the class $\mathcal{L}(\gamma)$ if $F_\xi \in \mathcal{L}(\gamma)$ and η is any nonnegative non-degenerate at zero integer-valued r.v. Albin [1] observed that Cline's result is false in general. He obtained that d.f. F_{S_η} remains in the class $\mathcal{L}(\gamma)$ if F_ξ belongs to the class $\mathcal{L}(\gamma)$ and $\mathbf{E}\mathrm{e}^{\delta\eta} < \infty$ for each $\delta > 0$. In order to prove this claim, author used the upper estimate

$$\frac{\overline{F^{*n}}(x-t)}{\overline{F^{*n}}(x)} \leqslant (1+\varepsilon)\mathrm{e}^{\gamma t}, \tag{1}$$

provided that $\varepsilon > 0$, $t \in \mathbb{R}$, $F \in \mathcal{L}(\gamma)$, $x \geqslant n(c_1 - t) + t$ and $c_1 = c_1(\varepsilon, t)$ is sufficiently large such that

$$\frac{\overline{F}(x-t)}{\overline{F}(x)} \leqslant (1+\varepsilon)\mathrm{e}^{\gamma \mathrm{t}}$$

for $x \geqslant c_1$ (see [1, Lemma 1]). Unfortunately, the obtained estimate holds for positive t only. If t is negative, then the above estimate is incorrect in general. This fact was shown by Watanabe and Yamamuro (see [8, Remark 6.1]). Thus, the Cline proposition that $\mathbf{P}(\xi_1 + \xi_2 + \cdots + \xi_\eta \leqslant x)$ belongs to the class $\mathcal{L}(\gamma)$ remains not proved.

In this paper, we investigate a wider class, $\mathcal{OL}$, instead of the class $\mathcal{L}(\gamma)$. We show that the d.f. of the sum $\xi_1 + \xi_2 + \cdots + \xi_\eta$ remains in the class $\mathcal{OL}$, if r.v. η satisfies the conditions similar to that in [1]. The following theorem is the main statement in this paper.

Theorem 1. *Let $\xi_1, \xi_2, \ldots$ be independent copies of a nonnegative r.v. ξ with d.f. F_ξ. Let η be a nonnegative, non-degenerate at zero, integer-valued and independent of $\{\xi_1, \xi_2, \ldots\}$ r.v. with d.f. F_η. If F_ξ belongs to the class $\mathcal{OL}$ and $\overline{F_\eta}(\delta x) = O(\sqrt{x}\overline{F_\xi}(x))$ for each $\delta \in (0, 1)$, then $F_{S_\eta} \in \mathcal{OL}$.*

A detailed proof of Theorem 1 is presented in Section 3. Note that the proof is similar to that of Theorem 6 in [5].

The following assertion actually shows that Albin's conditions for the counting r.v. η are sufficient for d.f. F_{S_η} to remain in the class $\mathcal{OL}$. The proof of the following corollary is also presented in Section 3.

Corollary 1. *Let $\xi_1, \xi_2, \ldots$ be a sequence of independent nonnegative r.v.s with common d.f. $F_\xi \in \mathcal{OL}$.*

(i) *D.f. $\mathbf{P}(\xi_1 + \cdots + \xi_n \leqslant x)$ belongs to the class $\mathcal{OL}$ for each fixed $n \in \mathbb{N}$.*
(ii) *Let η be a r.v. which is nonnegative, non-degenerate at zero, integer-valued and independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbf{E}\mathrm{e}^{\varepsilon\eta} < \infty$ for each $\varepsilon > 0$, then $F_{S_\eta} \in \mathcal{OL}$.*

2 Auxiliary lemmas

Before proving our main results, we give three auxiliary lemmas. The first lemma is well known classical estimate for the concentration function of a sum of independent and identically distributed r.v.s. The proof of Lemma 1 can be found in [6] (see Theorem 2.22), for instance.

Lemma 1. *Let $X_1, X_2, \ldots,$ be a sequence of independent r.v.s with a common non-degenerate d.f. Then there exists a constant c_2, independent of λ and n, such that*

$$\sup_{x\in\mathbb{R}} \mathbf{P}(x \leqslant X_1 + X_2 + \cdots + X_n \leqslant x + \lambda) \leqslant c_2(\lambda + 1)n^{-1/2}$$

for all $\lambda \geqslant 0$ and all $n \in \mathbb{N}$.

The second auxiliary lemma is due to Shimura and Watanabe (see [7, Prop. 2.2]). The lemma describes an important property of a d.f. from the class $\mathcal{OL}$.

Lemma 2. *Let F be a d.f. from the class $\mathcal{OL}$. Then there exists positive Δ such that*

$$\lim_{x\to\infty} \mathrm{e}^{\Delta x}\overline{F}(x) = \infty.$$

The last auxiliary lemma is crucial in the proof of Theorem 1. The elements of the statement below can be found in [4] (see the proof of Theorem 3(b)). Inequality (1), which is a particular case of the statement below, is proved in [1] (see Lemma 2.1). Leipus and Šiaulys [5] generalized Albin's inequality (1) for an arbitrary d.f. with unbounded support. The analytical proof of Lemma 3 is given in [5] (see proof of Lemma 4). In this paper, we present another, completely probabilistic proof of the lemma below having in mind the importance of the statement.

Lemma 3. *Let d.f. F be such that $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$. Suppose that*

$$\sup_{x\geqslant d_2} \frac{\overline{F}(x-t)}{\overline{F}(x)} \leqslant d_1$$

for some positive constants t, d_1 *and* $d_2 > t$. *Then, for all* $n = 1, 2, \ldots$, *we have*:

$$\sup_{x \geqslant n(d_2 - t) + t} \frac{\overline{F^{*n}}(x-t)}{\overline{F^{*n}}(x)} \leqslant d_1.$$

Proof of Lemma 3. Let X be a r.v. with d.f. F. Then the condition of Lemma 3 says that

$$\sup_{x \geqslant d_2} \frac{\mathbf{P}(X > x - t)}{\mathbf{P}(X > x)} \leqslant d_1 \tag{2}$$

for some positive $t, d_1, d_2 > t$, and we need to prove that

$$\sup_{x \geqslant (nd_2 - t) + t} \frac{\mathbf{P}(S_n^X > x - t)}{\mathbf{P}(S_n^X > x)} \leqslant d_1 \tag{3}$$

for all $n \in \mathbb{N}$, where $S_n^X = X_1 + \cdots + X_n$, and $X_1, X_2, \ldots$ are independent copies of X.

The proof is proceeded by induction on n. According to condition (2), inequality (3) holds for $n = 1$. Suppose now that $N \geqslant 1$. For arbitrary real x, z and $t > 0$, we obtain

$$\begin{aligned}
\mathbf{P}\big(S_{N+1}^X > x\big) &= \mathbf{P}\big(S_N^X + X_{N+1} > x,\, X_{N+1} \leqslant x - z\big) \\
&\quad + \mathbf{P}\big(S_N^X + X_{N+1} > x,\, S_N^X \leqslant z\big) \\
&\quad + \mathbf{P}(X_{N+1} > x - z)\,\mathbf{P}\big(S_N^X > z\big) \\
&\geqslant \mathbf{P}\big(S_N^X > x - X_{N+1},\, x - X_{N+1} \geqslant z\big) \\
&\quad + \mathbf{P}\big(X_{N+1} > x - S_N^X,\, x - S_N^X \geqslant x - z + t\big) \\
&\quad + \mathbf{P}(X_{N+1} > x - z)\mathbf{P}\big(S_N^X > z\big).
\end{aligned} \tag{4}$$

If we replace x by $x - t$ and z by $z - t$ then we get

$$\begin{aligned}
\mathbf{P}\big(S_{N+1}^X > x - t\big) &= \mathbf{P}\big(S_N^X + X_{N+1} > x - t,\, X_{N+1} \leqslant x - z\big) \\
&\quad + \mathbf{P}\big(S_N^X + X_{N+1} > x - t,\, S_N^X \leqslant z - t\big) \\
&\quad + \mathbf{P}(X_{N+1} > x - z)\mathbf{P}\big(S_N^X > z - t\big) \\
&= \mathbf{P}\big(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z\big) \\
&\quad + \mathbf{P}\big(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t\big) \\
&\quad + \mathbf{P}(X_{N+1} > x - z)\mathbf{P}\big(S_N^X > z - t\big).
\end{aligned} \tag{5}$$

R.v.s $X_1, X_2, \ldots$ are independent. Therefore,

$$\begin{aligned}
&\mathbf{P}\big(S_N^X > x - X_{N+1} - t,\, x - X_{N+1} \geqslant z\big) \\
&\quad = \mathbf{E}\big(\mathbf{E}\big(\mathbf{1}_{\{S_N^X > x - X_{N+1} - t\}} \mathbf{1}_{\{x - X_{N+1} \geqslant z\}} \bigm| x - X_{N+1} = y\big)\big) \\
&\quad = \mathbf{E}\big(\mathbf{1}_{\{y \geqslant z\}} \mathbf{E}\big(\mathbf{1}_{\{S_N^X > y - t\}} \bigm| x - X_{N+1} = y\big)\big) \\
&\quad = \mathbf{E}\big(\mathbf{1}_{\{y \geqslant z\}} \mathbf{P}\big(S_N^X > y - t\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sup_{y \geqslant z} \frac{\mathbf{P}(S_N^X > y - t)}{\mathbf{P}(S_N^X > y)} \mathbf{E}\big(\mathbf{1}_{\{y \geqslant z\}} \mathbf{P}\big(S_N^X > y\big)\big) \\
&= \sup_{y \geqslant z} \frac{\mathbf{P}(S_N^X > y - t)}{\mathbf{P}(S_N^X > y)} \mathbf{P}\big(S_N^X > x - X_{N+1},\, x - X_{N+1} \geqslant z\big), \qquad (6)
\end{aligned}
$$

where $\mathbf{1}_A$ denotes the indicator function of an event A. Similarly,

$$
\begin{aligned}
&\mathbf{P}\big(X_{N+1} > x - S_N^X - t,\, x - S_N^X \geqslant x - z + t\big) \\
&\quad \leqslant \sup_{y \geqslant x - z + t} \frac{\mathbf{P}(X_{N+1} > y - t)}{\mathbf{P}(X_{N+1} > y)} \mathbf{P}\big(X_{N+1} > x - S_N^X,\, x - S_N^X \geqslant x - z + t\big). \qquad (7)
\end{aligned}
$$

Using estimates (4)–(7), we obtain

$$
\frac{\mathbf{P}(S_{N+1}^X > x - t)}{\mathbf{P}(S_{N+1}^X > x)} \leqslant \max\left\{ \sup_{y \geqslant z} \frac{\mathbf{P}(S_N^X > y - t)}{\mathbf{P}(S_N^X > y)},\, \sup_{y \geqslant x - z + t} \frac{\mathbf{P}(X > y - t)}{\mathbf{P}(X > y)} \right\} \qquad (8)
$$

if $x, z \in \mathbb{R}$, $t > 0$ and $N \geqslant 1$.

Suppose now that (3) is satisfied for $n = N$. We will show that (3) holds for $n = N + 1$.

Condition (2) and estimate (8) imply, taking $z = z_N = Nx/(N+1) + t/(N+1)$ and $w_N = x - z_N + t = x/(N+1) + Nt/(N+1)$, that

$$
\frac{\mathbf{P}(S_{N+1}^X > x - t)}{\mathbf{P}(S_{N+1}^X > x)} \leqslant \max\left\{ \sup_{y \geqslant z_N} \frac{\mathbf{P}(S_N^X > y - t)}{\mathbf{P}(S_N^X > y)},\, \sup_{y \geqslant w_N} \frac{\mathbf{P}(X > y - t)}{\mathbf{P}(X > y)} \right\} \leqslant d_1
$$

if $x \geqslant (N+1)(d_2 - t) + t$, because, in this case,

$$
z_N \geqslant N(d_2 - t) + t \quad \text{and} \quad w_N \geqslant d_2.
$$

So, estimate (3) holds for $n = N + 1$ and the validity of (3) for all n follows by induction. □

3 Proofs of main results

In this section, we present detailed proofs of our main results.

Proof of Theorem 1. First, we show that

$$
\limsup_{x \to \infty} \frac{\overline{F_{S_\eta}}(x - a)}{\overline{F_{S_\eta}}(x)} = \limsup_{x \to \infty} \frac{\mathbf{P}(S_\eta > x - a)}{\mathbf{P}(S_\eta > x)} < \infty \qquad (9)
$$

for each $a \in \mathbb{R}$.

If $a \leqslant 0$, then $\mathbf{P}(S_\eta > x - a) \leqslant \mathbf{P}(S_\eta > x)$ for all $x \in \mathbb{R}$, and estimate (9) is obvious.

Suppose now that $a > 0$. Since $F_\xi \in \mathcal{OL}$, we derive that

$$\limsup_{x\to\infty} \frac{\overline{F_\xi}(x-a)}{\overline{F_\xi}(x)} = c_3 \tag{10}$$

for some finite positive quantity c_3 maybe depending on a. So, there exists some $K = K_a > a+1$ such that

$$\sup_{x\geqslant K} \frac{\overline{F_\xi}(x-a)}{\overline{F_\xi}(x)} \leqslant 2c_3. \tag{11}$$

Applying Lemma 3, we obtain that

$$\sup_{x\geqslant n(K-a)+a} \frac{\mathbf{P}(S_n > x-a)}{\mathbf{P}(S_n > x)} = \sup_{x\geqslant n\,(K-a)+a} \frac{\overline{F_\xi^{*n}}(x-a)}{\overline{F_\xi^{*n}}(x)} \leqslant 2c_3, \tag{12}$$

where and below $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ if $n \in \mathbb{N}$.

For an arbitrarily chosen positive x, we have

$$\begin{aligned}\mathbf{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n) \geqslant \sum_{n=1}^{\infty} \mathbf{P}(\xi_1 > x)\mathbf{P}(\eta = n)\\ &= \overline{F}_\xi(x)\mathbf{P}(\eta \geqslant 1).\end{aligned} \tag{13}$$

If $x \geqslant K$, then, using (12), we get:

$$\begin{aligned}\mathbf{P}(S_\eta > x-a) &= \mathbf{P}\left(S_\eta > x-a,\, \eta \leqslant \frac{x-a}{K-a}\right) + \mathbf{P}\left(S_\eta > x-a,\, \eta > \frac{x-a}{K-a}\right)\\ &= \sum_{n\leqslant (x-a)/(K-a)} \mathbf{P}(S_n > x-a)\,\mathbf{P}(\eta = n)\\ &\quad + \sum_{n>(x-a)/(K-a)} \mathbf{P}(S_n > x-a)\mathbf{P}(\eta = n)\\ &\leqslant 2c_3 \sum_{n\leqslant (x-a)/(K-a)} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n)\\ &\quad + \sum_{n>(x-a)/(K-a)} \mathbf{P}(S_n > x-a)\mathbf{P}(\eta = n)\\ &\quad + \sum_{n>(x-a)/(K-a)} \mathbf{P}(S_n > x)\,\mathbf{P}(\eta = n)\\ &\quad - \sum_{n>(x-a)/(K-a)} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n)\end{aligned}$$

$$\leqslant c_4 \sum_{n=1}^{\infty} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n) + \sum_{n>(x-a)/(K-a)} \mathbf{P}(x - a < S_n \leqslant x)\,\mathbf{P}(\eta = n) \tag{14}$$

with $c_4 = \max\{2c_3, 1\}$.

According to Lemma 1, we obtain

$$\sup_{x\in\mathbb{R}} \mathbf{P}(x - a < S_n \leqslant x) \leqslant c_5(a + 1)\frac{1}{\sqrt{n}},$$

where the constant c_5 is independent of a and n. Thus, inequality (14) implies

$$\begin{aligned}\mathbf{P}(S_\eta > x - a) &\leqslant c_4\mathbf{P}(S_\eta > x) + c_5(a + 1) \sum_{n>(x-a)/(K-a)} \frac{\mathbf{P}(\eta = n)}{\sqrt{n}} \\ &\leqslant c_4\mathbf{P}(S_\eta > x) + c_5\sqrt{\frac{K - a}{x - a}}(a + 1)\mathbf{P}\left(\eta > \frac{x - a}{K - a}\right)\end{aligned} \tag{15}$$

provided that $x \geqslant K$.

Inequalities (13) and (15) imply that, for $x \geqslant K$, it holds

$$\frac{\mathbf{P}(S_\eta > x - a)}{\mathbf{P}(S_\eta > x)} \leqslant c_4 + \frac{c_5\sqrt{K - a}(a + 1)}{\sqrt{x - a}\,\mathbf{P}(\eta \geqslant 1)\overline{F_\xi}(x)}\overline{F_\eta}\left(\frac{x - a}{K - a}\right).$$

Consequently,

$$\begin{aligned}&\limsup_{x\to\infty} \frac{\mathbf{P}(S_\eta > x - a)}{\mathbf{P}(S_\eta > x)} \\ &\quad\leqslant c_4 + c_5\frac{(a + 1)\sqrt{K - a}}{\mathbf{P}(\eta \geqslant 1)} \limsup_{x\to\infty} \frac{\overline{F_\eta}((x - a)/(K - a))}{\sqrt{x - a}\,\overline{F_\xi}(x - a)} \limsup_{x\to\infty} \frac{\overline{F_\xi}(x - a)}{\overline{F_\xi}(x)} \\ &\quad= c_4 + c_3c_5\frac{(a + 1)\sqrt{K - a}}{\mathbf{P}(\eta \geqslant 1)} \limsup_{x\to\infty} \frac{\overline{F_\eta}(x/(K - a))}{\sqrt{x}\,\overline{F_\xi}(x)} < \infty\end{aligned}$$

due to equality (10) and requirement $\overline{F_\eta}(\delta x) = O(\sqrt{x}\,\overline{F_\xi}(x))$ which holds for arbitrary $\delta \in (0, 1)$. Therefore, relation (9) is satisfied for for all $a \in \mathbb{R}$.

It remains to prove that

$$\liminf_{x\to\infty} \frac{\overline{F_{S_\eta}}(x - a)}{\overline{F_{S_\eta}}(x)} = \liminf_{x\to\infty} \frac{\mathbf{P}(S_\eta > x - a)}{\mathbf{P}(S_\eta > x)} > 0,$$

where a is an arbitrarily chosen real number. But this relation follows from the proved estimate (9), because

$$\mathbf{P}(S_\eta > x) \geqslant \overline{F_\xi}(x)\mathbf{P}(\eta \geqslant 1) > 0$$

for each positive number x, and so

$$\liminf_{x\to\infty}\frac{\mathbf{P}(S_\eta > x-a)}{\mathbf{P}(S_\eta > x)} = \left(\limsup_{x\to\infty}\frac{\mathbf{P}(S_\eta > x+a)}{\mathbf{P}(S_\eta > x)}\right)^{-1} > 0.$$

The last inequality, together with estimate (9), implies that d.f. F_{S_η} belongs to the class $\mathcal{OL}$. Theorem 1 is proved. □

Proof of Corollary 1. Part (i) of Corollary 1 is evident. So we only prove part (ii). Let $\delta \in (0,1)$. According to the Markov inequality, we have

$$\overline{F_\eta}(\delta x) = \mathbf{P}(\eta > \delta x) = \mathbf{P}\big(\mathrm{e}^{y\eta} > \mathrm{e}^{y\delta x}\big) \leqslant \mathrm{e}^{-\delta y x}\mathbf{E}\mathrm{e}^{y\eta} \tag{16}$$

for each $y > 0$. The d.f. F_ξ belongs to the class $\mathcal{OL}$. Therefore, Lemma 2 implies that $\mathrm{e}^{\Delta x}\overline{F_\xi}(x) \to \infty$ as $x \to \infty$. for some positive Δ.

Choosing $y = \Delta/\delta > 0$ in (16), we obtain:

$$\frac{\overline{F_\eta}(\delta x)}{\sqrt{x}\,\overline{F_\xi}(x)} \leqslant \frac{\mathbf{E}\mathrm{e}^{y\eta}}{\mathrm{e}^{\delta y x}\sqrt{x}\,\overline{F_\xi}(x)} = \frac{1}{\sqrt{x}}\frac{1}{\mathrm{e}^{\Delta x}\overline{F_\xi}(x)}\mathbf{E}\mathrm{e}^{(\Delta/\delta)\eta} \underset{x\to\infty}{\to} 0$$

because $\mathbf{E}\mathrm{e}^{\varepsilon\eta}$ is finite for an arbitrarily positive ε according to the main condition of Corollary 1. The statement of Corollary 1 follows now from Theorem 1. □

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