Be careful with variable separation solutions
via the extended tanh-function method
and periodic wave structures[*]

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Abstract. We analyze the extended tanh-function method to realize variable separation, however, we find that various "different" solutions obtained by this method are seriously equivalent to the general solution derived by the multilinear variable separation approach. In order to illustrate this point, we take a general $(2+1)$-dimensional Korteweg–de Vries system in water for example. Eight kind of variable separation solutions for a general $(2 + 1)$-dimensional Korteweg–de Vries system are derived by means of the extended tanh-function method and the improved tanh-function method. By detailed investigation, we find that these seemly independent variable separation solutions actually depend on each other. It is verified that many of so-called "new" solutions are equivalent to one another. Based on the uniform variable separation solution, abundant localized coherent structures can be constructed. However, we must pay our attention to the solution expression of all components to avoid the appearance of some un-physical related and divergent structures: seemly abundant structures for a special component are obtained while the divergence of the corresponding other component for the same equation appears.

Keywords: extended tanh-function method, uniform variable separation solutions, general $(2+1)$-dimensional Korteweg–de Vries system, periodic localized coherent structures.

1 Introduction

As the self-localized robust and long-lived nonlinear solitary wave object, the soliton provides a luxuriant source of inspiration in many areas of mathematics, fundamental physics, biology, chemistry and technology over the past 40 years.

The investigation of soliton solutions is an essential and important issue in nonlinear science. A vast variety of significant methods have been established such as the F-expansion method [10], the multilinear variable separation approach (MLVSA) [4], the Painlevé method [13], the mapping method [8], and the similarity transform method [5], and the like. Note that Navickas et al. [18] have remarked the Exp-function method and Zhao et al. [23] have also remarked the (G'/G)-expansion method. Many authors claim that they obtained large lists of so-called "new" exact solutions. However, many of these "new" solutions are equivalent to one another and can be reduced to some well-known solutions [18,23]. In this paper, we will verify again this conclusion.

As one of the effective tools in linear mathematical physics, the variable separation approach (VSA) has been successfully extended to nonlinear domain. Among all kind of "variable separation" procedures to solve NLEEs, the MLVSA is a crucial and powerful mean to obtain abundant and general solutions [20] since it was established firstly in 1996 for the Davey–Stewartsen equation [11]. Moreover, many direct methods [6, 7, 9, 16, 19, 21, 22, 26, 27], which used to obtain travelling wave solutions of NLEEs, have been successfully extended to derive variable separation solutions. As one of outstanding examples of the direct method, the extended $\tanh$-function method (ETM) was firstly extended to obtain variable separation solutions of $(2+1)$-dimensional Broer–Kaup system [25], and then was extensively applied to many other NLEEs [6,9,22,27]. Then, the projective Ricatti equation method (PREM) was also generalized to derive variable separation solutions for NLEEs [7]. Recently, some improved ETMs [16,19,21, 26] have been presented to derive variable separation solutions via "variable separation" procedures. Based on these variable separation solutions, abundant localized coherent structures such as the periodic, dromion, peakon, compacton and foldon solutions are discovered [6,7,9,11,16,19,20,21,22,25,26,27].

However, to the best of our knowledge, the study for the relation between different ETMs and the discussion and analysis for the rationality for the construction of abundant localized coherent structures based on variable separation solutions were less carried out. Motivated by these reasons, we consider the following general $(2+1)$-dimensional Korteweg–de Vries (GKdV) system [14]

$$u_t - u_{xxy} - auu_y - bu_x\partial_x^{-1}u_y = 0, \tag{1}$$

where a and b are two arbitrary constants. The KdV equation was initially used to describe competition between weak nonlinearity and weak dispersion in a shallow liquid. Now the KdV system has been applied in many apparently unrelated phenomena in various physical systems such as plasmas and lattice vibrations of a crystal at low temperatures. Many researchers have investigated the above GKdV system. For example, Calogero [2] proved the integrability of Eq. (1). Clarkson and Manshield [3] studied the Painlevé

property test and proved the completely integrable for the equation only when $a = 2b$. Lou et al. [12] and Ma et al. [15] obtained some variable separation solutions for the special model with $a = b$, respectively. Zheng et al. [4] discussed some semifolded localized coherent structures via MLVSA. Ma et al. [14] investigated also complex wave excitations and chaotic patterns for Eq. (1) with $a = 2b$.

In this paper, we firstly use the ETM and an improved ETM to obtain variable separation solutions of the $(2+1)$-dimensional GKdV system. Then, by further studying, we find that these variable separation solutions obtained by the ETM and improved ETM, which seem independent, actually depend on each other. These "different" solutions obtained by this method are seriously equivalent to the general solution derived by the MLVSA. Finally, based on the variable separation solutions, we discuss and analyze the rationality for the construction of abundant localized coherent structures.

2   Remark on ETM

In this section, we consider the integrable GKdV system with $a = 2b$. To solve this system, first, let us make a transformation for Eq. (1): $v = \partial_x^{-1} u_y$. Substituting this transformation into Eq. (1) yields a set of two coupled nonlinear partial differential equations

$$u_t - u_{xxy} - 2buv_x - bvu_x = 0, \qquad u_y - v_x = 0. \tag{2}$$

Introducing the transformation

$$u = G_x, \qquad v = G_y \tag{3}$$

into Eq. (2), one obtains

$$G_{xt} - G_{xxxy} - 2bG_x G_{xy} - bG_{xx} G_y = 0. \tag{4}$$

It is possible to use the generalized operator of differentiation to produce a formal structure of the solution for Eq. (4). In [17], authors used an analytical criterion based on the concept of H-ranks and determine if a solution of the KdV equation can be expressed in an analytical form comprising standard functions. The employment of this criterion gave the structure of the solution so that one does not have to guess what the form of the solution is. Here we focus on the formal structure of the solution for Eq. (4) via the ETM.

Along with the ETM [6,9,22,25,27], we assume that Eq. (4) has the following ansätz:

$$G = a_0 + \sum_{i=1}^{m} \left\{ a_i \phi^i \big[w(x,y,t) \big] + a_{-i} \phi^{-i} \big[w(x,y,t) \big] \right\}, \tag{5}$$

where $a_i = a_i(x,y,t)$ $(i = -m, \ldots, m)$, $w = w(x,y,t)$ are all arbitrary functions of indicated variables, and $\phi(w)$ satisfies

$$\frac{\mathrm{d}\phi}{\mathrm{d}w} = l_0 + \phi^2 \tag{6}$$

with its solutions

$$\phi = \begin{cases} -\sqrt{-l_0}\tanh(\sqrt{-l_0}w), & l_0 < 0, \\ -\sqrt{-l_0}\coth(\sqrt{-l_0}w), & l_0 < 0, \\ \sqrt{l_0}\tan(\sqrt{l_0}w), & l_0 > 0, \\ -\sqrt{l_0}\cot(\sqrt{l_0}w), & l_0 > 0, \\ -1/w, & l_0 = 0, \end{cases} \tag{7}$$

where l_0 is constant.

The homogeneous balance principle determines $m = 1$ in (4). Inserting ansätz (5) with (6) into Eq. (4), selecting the variable separation form $w = p(x) + q(y - ct)$ [14] with an arbitrary constant c and eliminating all the coefficients of polynomials of ϕ^i, we derive the following special solutions:

$$a_0 = -\int \frac{4l_0 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2}\,\mathrm{d}x,$$

$$a_1 = -\frac{4p_x}{b}, \qquad a_{-1} = 0 \tag{8}$$

and

$$a_0 = -\int \frac{16l_0 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2}\,\mathrm{d}x,$$

$$a_1 = -\frac{4p_x}{b}, \qquad a_{-1} = \frac{4l_0 p_x}{b}, \tag{9}$$

where $p \equiv p(x)$ and $q \equiv q(y - ct)$.

Therefore, we can derive two families of the variable separation solutions for the $(2+1)$-dimensional GKdV:

Family 1. From (3), (5), (7) and (8), one has:

Case 1. For $l_0 < 0$,

$$u_1 = \frac{4l_0 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2}$$
$$+ \frac{4\sqrt{-l_0}}{b}p_{xx}\tanh\big[\sqrt{-l_0}(p+q)\big] - \frac{4l_0}{b}p_x^2\operatorname{sech}^2\big[\sqrt{-l_0}(p+q)\big], \tag{10}$$

$$v_1 = -\frac{4l_0}{b}p_x q_y \operatorname{sech}^2\big[\sqrt{-l_0}(p+q)\big], \tag{11}$$

$$u_2 = \frac{4l_0 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2} + \frac{4\sqrt{-l_0}}{b}p_{xx}\coth\big[\sqrt{-l_0}(p+q)\big]$$
$$+ \frac{4l_0}{b}p_x^2\operatorname{csch}^2\big[\sqrt{-l_0}(p+q)\big], \tag{12}$$

$$v_2 = \frac{4l_0}{b}p_x q_y \operatorname{csch}^2\big[\sqrt{-l_0}(p+q)\big]. \tag{13}$$

Case 2. For $l_0 > 0$,

$$u_3 = \frac{4l_0 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2}$$
$$- \frac{4\sqrt{-l_0}}{b} p_{xx} \tan\left[\sqrt{l_0}(p+q)\right] - \frac{4l_0}{b} p_x^2 \sec^2\left[\sqrt{l_0}(p+q)\right], \tag{14}$$

$$v_3 = -\frac{4l_0}{b} p_x q_y \sec^2\left[\sqrt{l_0}(p+q)\right], \tag{15}$$

$$u_4 = \frac{4l_0 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2} + \frac{4\sqrt{-l_0}}{b} p_{xx} \cot\left[\sqrt{l_0}(p+q)\right]$$
$$- \frac{4l_0}{b} p_x^2 \csc^2\left[\sqrt{l_0}(p+q)\right], \tag{16}$$

$$v_4 = -\frac{4l_0}{b} p_x q_y \csc^2\left[\sqrt{l_0}(p+q)\right]. \tag{17}$$

Case 3. For $l_0 = 0$,

$$u_5 = \frac{p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2} + \frac{4p_{xx}}{b(p+q)} - \frac{4p_x^2}{b(p+q)^2}, \tag{18}$$

$$v_5 = -\frac{4p_x q_y}{b(p+q)^2}, \tag{19}$$

where $p \equiv p(x)$ and $q \equiv q(y - ct)$.

Family 2. From (3), (5), (7) and (9), we gets:

Case 4. For $l_0 < 0$,

$$u_6 = \frac{16l_0 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2}$$
$$+ \frac{4\sqrt{-l_0}}{b} p_{xx} \left\{ \tanh\left[\sqrt{-l_0}(p+q)\right] + \coth\left[\sqrt{-l_0}(p+q)\right] \right\}$$
$$+ \frac{4l_0}{b} p_x^2 \operatorname{sech}^2\left[\sqrt{-l_0}(p+q)\right] \operatorname{csch}^2\left[\sqrt{-l_0}(p+q)\right], \tag{20}$$

$$v_6 = \frac{4l_0}{b} p_x q_y \operatorname{sech}^2\left[\sqrt{-l_0}(p+q)\right] \operatorname{csch}^2\left[\sqrt{-l_0}(p+q)\right]. \tag{21}$$

Case 5. For $l_0 > 0$,

$$u_7 = \frac{16l_0 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2}$$
$$- \frac{4l_0}{b} p_{xx} \left\{ \tan\left[\sqrt{-l_0}(p+q)\right] + \cot\left[\sqrt{-l_0}(p+q)\right] \right\}$$
$$+ \frac{4l_0}{b} p_x^2 \left\{ \csc^2\left[\sqrt{l_0}(p+q)\right] - \sec^2\left[\sqrt{l_0}(p+q)\right] \right\} \tag{22}$$

$$v_7 = \frac{4l_0}{b} p_x q_y \left\{ \csc^2\left[\sqrt{l_0}(p+q)\right] - \sec^2\left[\sqrt{l_0}(p+q)\right] \right\}. \tag{23}$$

Case 6. For $l_0 = 0$, we can derive solutions (18) and (19) again.

Remark 1. By means of this ETM, it seems that seven new variable separation solutions of the $(2+1)$-dimensional GKdV are obtained. However, by detailed investigation, we find that only variable separation solutions (18) and (19) are essentially effective. For Family 1, when being re-defined $p = \exp\{-2\sqrt{-l_0}p\}$, $q = \exp\{2\sqrt{-l_0}q\}$ and $p = \exp\{-2\mathrm{i}\sqrt{l_0}p\}$, $q = \exp\{2\mathrm{i}\sqrt{l_0}q\}$ in solutions (18) and (19), solutions (10), (11) and (14), (15) can be obtained, respectively. Similarly, if being taken as $p = -\exp\{-2 \times \sqrt{-l_0}p\}$, $q = \exp\{2\sqrt{-l_0}q\}$ and $p = -\exp\{-2\mathrm{i}\sqrt{l_0}p\}$, $q = \exp\{2\mathrm{i}\sqrt{l_0}q\}$ in solutions (18) and (19), solutions (12), (13) and (16), (17) can be recovered, respectively. For Family 2, when we re-define $p = \exp[-(4\sqrt{-l_0}p + \mathrm{i}\pi)]$, $q = \exp[4\sqrt{-l_0}q]$ in solutions (18) and (19), solutions (20) and (21) can be obtained. Similarly, if one takes $p = \exp[-\mathrm{i}(4\sqrt{l_0}p + \pi)]$, $q = \exp[4\mathrm{i}\sqrt{l_0}q]$ in solutions (18) and (19), solutions (22) and (23) can be recovered. Note that solutions (18) and (19) have the same form with the general solution derived by the MLVSA in [19].

3 Remark on the improved ETMs

The procedure of solving the $(2+1)$-dimensional GKdV via an improved ETM is similar to that via ETM except that in ansätz (5), $a_{-i} = 0$ and $\phi(w)$ satisfies [21]

$$\frac{\mathrm{d}\phi}{\mathrm{d}w} = (A\phi - \alpha)(B\phi - \beta) \tag{24}$$

with its solutions

$$\phi = \frac{\beta \exp[(\alpha B - A\beta)w] - \alpha \exp[C_1(A\beta - \alpha B)]}{B \exp[(\alpha B - A\beta)w] - A \exp[C_1(A\beta - \alpha B)]}, \tag{25}$$

where C_1 is an integration constant, further, A, B, α and β are four arbitrary constants.

Inserting ansätz (5) with $m = 1$, $a_{-i} = 0$ and (24) into Eq. (4), selecting the variable separation form $w = p(x) + q(y - ct)$ and eliminating all the coefficients of polynomials of ϕ^i, we get the following special solutions:

$$a_0 = -\int \frac{4\alpha^2 B^2 p_x^4 - p_{xx}^2 + cp_x^2 + 2p_x p_{xxx}}{2bp_x^2}\,\mathrm{d}x,$$
$$a_1 = -\frac{4ABp_x}{b}, \quad A\beta + \alpha B = 0. \tag{26}$$

Therefore, we can derive variable separation solutions of the $(2+1)$-dimensional GKdV, namely

$$u_8 = \frac{-4\alpha^2 B^2 p_x^4 + p_{xx}^2 - cp_x^2 - 2p_x p_{xxx}}{2bp_x^2}$$
$$+ \frac{4\alpha B p_{xx}(BJ + AK)}{b(BJ - AK)} + \frac{16\alpha^2 AB^3 p_x^2 JK}{b(BJ - AK)^2}, \tag{27}$$

$$v_8 = \frac{16\alpha^2 AB^3 p_x q_y JK}{b(BJ - AK)^2}, \tag{28}$$

where $J = \exp[2\alpha B(p + q)]$ and $K = \exp[-2\alpha BC_1]$.

Remark 2. It seems that the mapping Ricatti equation (24) is a new equation. However, when we re-define $\phi \equiv \phi - (A\beta + \alpha B)/(2AB)$ and $l_0 = -(A^2\beta^2 + \alpha^2 B^2)/(A^2 B^2)$ in Eq. (6), Eq. (24) can be transformed to the known Ricatti equation (6). When choosing $C_1 = 0$, $\alpha = \beta = -\sqrt{-l_0}$, $-A = B = -1$ in solutions (27) and (28), we can obtain solutions (10) and (11). If taking $C_1 = 0$, $A = B = 1$, $\alpha = -\beta = \sqrt{-l_0}$ in solutions (27) and (28), one can derive solutions (12) and (13). When one selects $C_1 = 0$, $\alpha = \beta = -\mathrm{i}\sqrt{l_0}$, $-A = B = -1$ in solutions (27) and (28), solutions (14) and (15) can be recovered. Moreover, if we set $C_1 = 0$, $A = B = 1$, $\alpha = -\beta = \mathrm{i}\sqrt{l_0}$ in solutions (27) and (28), solutions (16) and (17) can be obtained. Therefore, solutions (27) and (28) are essentially equivalent to solutions (18) and (19).

Remark 3. The main idea of different ETMs is based on some Ricatti equations to obtain variable separation solutions of NLEEs. Therefore, based on different Ricatti equations, many improved ETMs were developed. In [16], Ma et al. obtained variable separation solutions of $(2+1)$-dimensional dispersive long-water wave system based on the Riccati equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}w} = l_0\phi + \phi^2 \tag{29}$$

with a constant l_0. In [26], Zhu obtained variable separation solutions of $(2+1)$-dimensional Boiti–Leon–Pempinelle equation based on the Riccati equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}w} = l_0 + l_1\phi + l_2\phi^2, \tag{30}$$

with three constants l_0, l_1 and l_2. Actually, one may readily find that Eqs. (29) and (30) are essentially equivalent to Eq. (6). In Eq. (29), we take $\phi + l_0/2 = \phi$, $-l_0^2/4 = l_0$, Eq. (29) is transformed into Eq. (6). Similarly, setting $\phi + l_1/2l_2 = \phi$, $(4l_0l_2 - l_1^2)/(4l_2^2) = l_0$, $l_2 w = w$ in Eq. (30) yields Eq. (6). Therefore, different improved ETMs based on different Ricatti equations [16, 26] are essentially equivalent to ETM based on Ricatti equation (6).

4 Remark on some periodic wave structures

Based on the uniform variable separation solution (19), abundant localized coherent structures such as the periodic, dromion, peakon, compacton and foldon solutions are discovered [6, 7, 9, 11, 16, 19, 20, 21, 22, 25, 26, 27]. However, to the best of our knowledge, the discussion and analysis for the rationality for the construction of abundant localized coherent structures based on variable separation solutions were hardly carried out. In this section, we take periodic waves for example to illustrate that we must pay our attention to the solution expressions of all components to avoid the appearance of some un-physical related and divergent structures.

The dynamical behaviors of periodic waves were extensively discussed [1], and here we re-study these periodic waves in terms of sn- and cn-functions based on variable separation solutions (18) and (19). There are several cases to be considered.
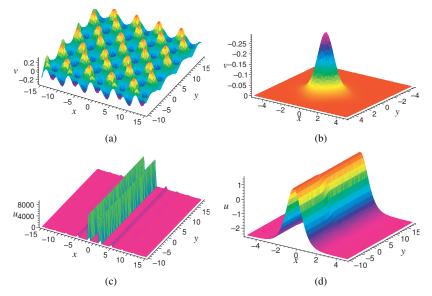
Fig. 1. (a), (c) A typical spatial periodic wave structure for v expressed by Eq. (32) and the corresponding dromion structure for v expressed by Eq. (34) at time $t = 1$, respectively. (b), (d) The divergent structure for u expressed by Eq. (31) and line soliton structure for u expressed by Eq. (33). The parameters are chosen as $C = D = 2, k = c = b = 1, m_1 = 0.3$ and $m_2 = 0.8$.

Case 1. $p = C + \operatorname{sn}(kx, m_1) \equiv C + \operatorname{sn}(\xi), \ q = D + \operatorname{sn}(y - ct, m_2) \equiv D + \operatorname{sn}(\eta)$.
It follows from Eqs. (18) and (19) that

$$u = \frac{k^2 \operatorname{sn}^2(\xi) \operatorname{dn}^2(\xi)}{2 \operatorname{cn}^2(\xi)} + \left[k^2 m_1^2 + \frac{k^2 m_1^4 \operatorname{sn}^2(\xi)}{2 \operatorname{dn}^2(\xi)} \right] \operatorname{cn}^2(\xi)$$
$$+ k^2 \left[\operatorname{dn}^2(\xi) - 3m_1^2 \operatorname{sn}^2(\xi) \right]$$
$$- \frac{c}{2} - \frac{4k^2 \operatorname{sn}(\xi)[\operatorname{dn}^2(\xi) + m_1^2 \operatorname{cn}^2(\xi)]}{C + D + \operatorname{sn}(\xi) + \operatorname{sn}(\eta)} - \frac{4k^2 \operatorname{cn}^2(\xi) \operatorname{dn}^2(\xi)}{[C + D + \operatorname{sn}(\xi) + \operatorname{sn}(\eta)]^2}, \quad (31)$$

$$v = -\frac{4k \operatorname{cn}(\xi) \operatorname{dn}(\xi) \operatorname{cn}(\eta) \operatorname{dn}(\eta)}{b[C + D + \operatorname{sn}(\xi) + \operatorname{sn}(\eta)]^2}, \quad (32)$$

where k, C and D are arbitrary constants, and m_1 and m_2 denote the moduli of the elliptic function.

A typical spatial periodic wave structure for v expressed by Eq. (32) is shown in Fig. 1a with the parameters $C = D = 2, k = c = b = 1, m_1 = 0.3, m_2 = 0.8$ at time $t = 1$, and they are valid throughout this section, unless otherwise stated. However, the other field component u is divergent because the first term in Eq. (31) has zeros for cn-function. The plots of the corresponding field component u is shown in Fig. 1b, from which the divergent phenomenon can be obviously observed.

When we take the long wave limit (m_1 and $m_2 \to 1$), Eqs. (31) and (32) degenerate to

$$u = 2k^2 \left[\operatorname{sech}^2(\xi) - \tanh^2(\xi) \right] - \frac{c}{2} - \frac{8k^2 \tanh(\xi) \operatorname{sech}(\xi)}{C + D + \tanh(\xi) + \tanh(\eta)}$$
$$- \frac{4k^2 \operatorname{sech}^4(\xi)}{[C + D + \tanh(\xi) + \tanh(\eta)]^2}, \tag{33}$$

$$v = -\frac{4k \operatorname{sech}^2(\xi) \operatorname{sech}^2(\eta)}{b[C + D + \tanh(\xi) + \tanh(\eta)]^2}. \tag{34}$$

Figures 1c and 1d exhibit these localized coherent structures expressed by these expressions. Obviously, Fig. 1c depicts a dromion structure for v which decays exponentially in all directions. Figure 1d describes a localized line soliton along y-axis for u. Although the field component u is divergent and un-physical for periodic wave case (c.f. Fig. 1b), it is localized for the limit case m_1 and $m_2 \to 1$ (c.f. Fig. 1d).

Case 2. $p = C + \operatorname{sn}(kx, m_1) \equiv C + \operatorname{sn}(\xi)$, $q = D + \operatorname{cn}(y - ct, m_2) \equiv D + \operatorname{cn}(\eta)$. In this case, we have

$$u = \frac{k^2 \operatorname{sn}^2(\xi) \operatorname{dn}^2(\xi)}{2 \operatorname{cn}^2(\xi)} + \left[k^2 m_1^2 + \frac{k^2 m_1^4 \operatorname{sn}^2(\xi)}{2 \operatorname{dn}^2(\xi)} \right] \operatorname{cn}^2(\xi)$$
$$+ k^2 \left[\operatorname{dn}^2(\xi) - 3m_1^2 \operatorname{sn}^2(\xi) \right]$$
$$- \frac{c}{2} - \frac{4k^2 \operatorname{sn}(\xi)[\operatorname{dn}^2(\xi) + m_1^2 \operatorname{cn}^2(\xi)]}{C + D + \operatorname{sn}(\xi) + \operatorname{cn}(\eta)} - \frac{4k^2 \operatorname{cn}^2(\xi) \operatorname{dn}^2(\xi)}{[C + D + \operatorname{sn}(\xi) + \operatorname{cn}(\eta)]^2}, \tag{35}$$

$$v = -\frac{4k \operatorname{cn}(\xi) \operatorname{dn}(\xi) \operatorname{sn}(\eta) \operatorname{dn}(\eta)}{b[C + D + \operatorname{sn}(\xi) + \operatorname{cn}(\eta)]^2}. \tag{36}$$

The long wave limit (m_1 and $m_2 \to 1$) gives

$$u = 2k^2 \left[\operatorname{sech}^2(\xi) - \tanh^2(\xi) \right] - \frac{c}{2} - \frac{8k^2 \tanh(\xi) \operatorname{sech}(\xi)}{C + D + \tanh(\xi) + \operatorname{sech}(\eta)}$$
$$- \frac{4k^2 \operatorname{sech}^4(\xi)}{[C + D + \tanh(\xi) + \operatorname{sech}(\eta)]^2}, \tag{37}$$

$$v = -\frac{4k \operatorname{sech}^2(\xi) \tanh(\eta) \operatorname{sech}(\eta)}{b[C + D + \tanh(\xi) + \operatorname{sech}(\eta)]^2}. \tag{38}$$

The typical spatial periodic wave structure for v expressed by Eq. (36) is shown in Fig. 2a, and the corresponding field component u expressed by Eq. (35) is shown in Fig. 2b, from which one can also observe the divergent phenomenon because the first term in Eq. (35) has zeros for cn-function. Figures 2c and 2d exhibit the corresponding localized coherent structures expressed by Eqs. (37) and (38). Figure 2c depicts a dromion-pair structure for v with one up and another down bounded peaks. Figure 2d also describes a localized line soliton along y-axis for u. Similar to Case 1, it is localized for
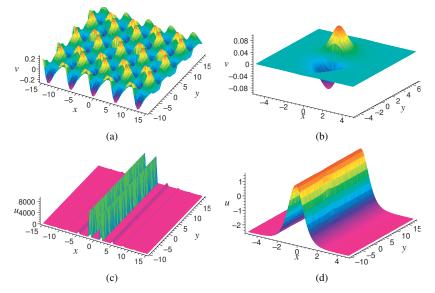
Fig. 2. (a), (c) A typical spatial periodic wave structure for v expressed by Eq. (36) and the corresponding dromion-pair structure for v expressed by Eq. (38) at time $t = 1$, respectively. (b), (d) The divergent structure for u expressed by Eq. (35) and line soliton structure for u expressed by Eq. (37). The parameters are chosen as $C = D = 2$, $k = c = b = 1$, $m_1 = 0.3$ and $m_2 = 0.8$.

the limit case m_1 and $m_2 \to 1$ (c.f. Fig. 2d) although the field component u is divergent and un-physical for periodic wave case (c.f. Fig. 2b).

Case 3. $p = C + \mathrm{cn}(kx, m_1) \equiv C + \mathrm{cn}(\xi)$, $q = D + \mathrm{cn}(y - ct, m_2) \equiv D + \mathrm{cn}(\eta)$. From Eqs. (18) and (19), one has

$$
\begin{aligned}
u = {} & \left[\frac{k^2 \, \mathrm{dn}^2(\xi)}{2 \, \mathrm{sn}^2(\xi)} + \frac{k^2 m_1^4 \, \mathrm{sn}^2(\xi)}{2 \, \mathrm{dn}^2(\xi)} + 3k^2 m_1^2 \right] \mathrm{cn}^2(\xi) + k^2 \left[\mathrm{dn}^2(\xi) - m_1^2 \, \mathrm{sn}^2(\xi) \right] \\
& - \frac{c}{2} - \frac{4k^2 \, \mathrm{cn}(\xi)[\mathrm{dn}^2(\xi) - m_1^2 \, \mathrm{sn}^2(\xi)]}{C + D + \mathrm{cn}(\xi) + \mathrm{cn}(\eta)} - \frac{4k^2 \, \mathrm{sn}^2(\xi) \, \mathrm{dn}^2(\xi)}{[C + D + \mathrm{cn}(\xi) + \mathrm{cn}(\eta)]^2},
\end{aligned}
\tag{39}
$$

$$
v = -\frac{4k \, \mathrm{sn}(\xi) \, \mathrm{dn}(\xi) \, \mathrm{sn}(\eta) \, \mathrm{dn}(\eta)}{b[C + D + \mathrm{cn}(\xi) + \mathrm{cn}(\eta)]^2}.
\tag{40}
$$

The long wave limit (m_1 and $m_2 \to 1$) yields

$$
\begin{aligned}
u = {} & k^2 \left[8 \, \mathrm{sech}^2(\xi) - \tanh^2(\xi) + \mathrm{sech}^2(\xi) \, \mathrm{csch}^2(\xi) \right] - \frac{c}{2} \\
& - \frac{4k^2 \, \mathrm{sech}(\xi)[\mathrm{sech}^2(\xi) - \tanh^2(\xi)]}{C + D + \mathrm{sech}(\xi) + \mathrm{sech}(\eta)} - \frac{4k^2 \tanh^2(\xi) \, \mathrm{sech}^2(\xi)}{[C + D + \mathrm{sech}(\xi) + \mathrm{sech}(\eta)]^2},
\end{aligned}
\tag{41}
$$

$$
v = -\frac{4k \tanh(\xi) \, \mathrm{sech}(\xi) \tanh(\eta) \, \mathrm{sech}(\eta)}{b[C + D + \mathrm{sech}(\xi) + \mathrm{sech}(\eta)]^2}.
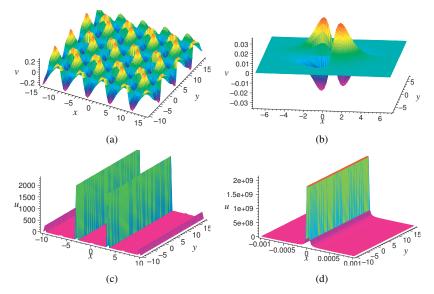\tag{42}
$$

Fig. 3. (a), (c) A typical spatial periodic wave structure for v expressed by Eq. (40) and the corresponding dromion-antidromion pair structure for v expressed by Eq. (42) at time $t = 1$, respectively. (b), (d) The divergent structures for u expressed by Eqs. (39) and (41). The parameters are chosen as $C = D = 2$, $k = c = b = 1$, $m_1 = 0.3$ and $m_2 = 0.8$.

Figures 3a and 3b present the typical spatial periodic wave structure for v expressed by Eq. (40) and the divergent phenomenon for u expressed by Eq. (39) due to the zeros for sn-function in the first term of Eq. (39). Figure 3c plots a dromion-antidromion pair structure for v expressed by Eq. (42) with two up and two down bounded peaks. Figure 3d exhibits the divergent phenomenon for u expressed by Eq. (41) due to the divergent for csch-function in the third term in Eq. (41). Different from Cases 1 and 2, it is also divergent for the limit case m_1 and $m_2 \to 1$ (see Fig. 3d).

Remark 4. It is obvious that the field u in Figs. 1b, 2b, 3b and 3d are un-physical related structures due to their divergence. Therefore, the corresponding localized periodic structures for the field v in Figs. 1a, 2a, 3a and 3c are false and can not been realized due to the futility of the other component u for the same equation. All examples indicate that although abundant localized coherent structures can be constructed for a special component, we must pay our attention to the solution expression of the corresponding other component for the same equation in order to avoid the appearance of some false, un-physical related and divergent structures.

5   Summary and discussion

In conclusion, our interest has been focused on two issues proposed in the introduction. Here we review the main points offered in this paper.

- *The relations between different ETMs are presented, and be careful with these methods.*

One can obtain abundant variable separation solutions for NLEEs via the ETM and different improved ETMs. This result can be illustrated by the solutions of the $(2 + 1)$-dimensional GKdV system. However, by detailed investigation, we find that these seemly independent variable separation solutions actually depend on each other. It is verified again that many of so-called "new" solutions are equivalent to one another. Various "different" solutions obtained by this method are seriously equivalent to the general solution derived by the MLVSA.

This conclusion for the equivalence of abundant variable separation solutions is also true for $(1 + 1)$-dimensional, $(2 + 1)$-dimensional and $(3 + 1)$-dimensional NLEEs, such as $(1 + 1)$-dimensional negative KdV hierarchy, $(2 + 1)$-dimensional dispersive long wave system, Broer–Kaup-Kupershmidt system, (asymmetric) Nizhnik–Novikov–Veselov system, breaking soliton model, and (3+1)-dimensional Burgers equation, and so on. Therefore, we should check carefully solutions obtained, and avoid casually asserting some so-called "new" solutions.

- *The rationality for the construction of abundant localized coherent structures is uncovered.*

In previous literatures, authors constructed abundant localized coherent structures for a special component, however, they ignore the divergence for the corresponding other component for the same equation. Although abundant localized coherent structures can be constructed for a special component, we must pay our attention to the solution expression of all components to avoid the appearance of some false, un-physical related and divergent structures: seemly abundant structures for a special component are obtained while divergence of the corresponding other component for the same equation appears.

Note that it is possible to use the generalized operator of differentiation to produce a formal structure of the solution, while the main contribution in this present paper is the discussion of different kinds of variable separation solutions through the generalization of the ETM. The detailed discussion about more general techniques based on the generalized operator of differentiation will be addressed in another separated paper.

We hope these remarks discussed here are helpful for the deep study of variable separation approach and the construction of localized coherent structures in higher-dimensional nonlinear models.

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