

Optimal control problem for the BCF model describing crystal surface growth*

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Abstract. In this paper, for the BCF model describing crystal surface growth, the optimal control problem is considered, the existence of optimal solution is proved and the optimality system is established.

Keywords: optimal control, BCF model, optimal solution, optimality condition.

1 Introduction

In past decades, the optimal control of distributed parameter system had been received much more attention in academic field. A wide spectrum of problems in applications can be solved by methods of optimal control, such as chemical engineering and vehicle dynamics. Modern optimal control theories and applied models are not only represented by ODE, but also by PDE, especially nonlinear parabolic equation. Many papers have already been published to study the control problems of nonlinear parabolic equations, for example, [1, 3, 8, 11, 13, 14, 16] and so on.

In this paper, we consider the optimal control problem for the BCF model

$$\frac{\partial u}{\partial t} + au_{xxxx} + \mu \left(\frac{u_x}{1 + |u_x|^2} \right)_x = 0, \quad (x, t) \in \Omega \times (0, T), \quad (1)$$

where a and μ are positive constants, $\Omega = (0, 1)$. On the basis of physical considerations, equation (1) is supplemented with the boundary value conditions

$$u_x(0, t) = u_x(1, t) = u_{xxx}(0, t) = u_{xxx}(1, t) = 0 \quad (2)$$

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and the initial value condition

$$u(x, 0) = u_0(x) \quad \forall x \in \Omega. \quad (3)$$

Equation (1) was presented by Johnson et al. [7] for describing the process of growing of a crystal surface on the basis of the BCF theory due to Burton et al. [2]. Here $u(x, t)$ denotes a displacement of height of surface from the standard level ($u = 0$) at a position $x \in \Omega$. The term au_{xxxx} in equation (1) denotes a surface diffusion of adatoms, which is caused by the difference of the chemical potential. In the mean time, $\mu u_x / (1 + |u_x|^2)_x$ denotes the effect of surface roughening.

During the past years, many authors have paid much attention to equation (1). It was Johnson et al. [7] who presented this equation for describing the process of growing of a crystal surface on the basis of the BCF theory. Rost and Krug [10] studied the unstable epitaxy on singular surfaces using equation (1) with a prescribed slopedependent surface current. In their papers, they derived scaling relations for the late stage of growth, where power law coarsening of the mound morphology is observed. In [9], in the limit of weak desorption, Pierre-Louis et al. derived equation (1) for a vicinal surface growing in the step flow mode. This limit turned out to be singular, and nonlinearities of arbitrary order need to be taken into account.

Recently, Fujimura and Yagi [4, 5] also considered equation (1). In their papers, the uniqueness of local solutions and the global solutions were obtained, a dynamical system determined from the initial-boundary value problem of the model equation was constructed, and the asymptotic behavior of trajectories of the dynamical system was also considered. In [6], Grasselli, Mola and Yagi showed that equation (1) endowed with no-flux boundary conditions generates a dissipative dynamical system under very general assumptions on $\partial\Omega$ on a phase-space of L^2 -type. They proved that the system possesses a global as well as an exponential attractor. In addition, if $\partial\Omega$ is smooth enough, they showed that every trajectory converges to a single equilibrium by means of a suitable Łojasiewicz–Simon inequality. In [15], based on the iteration technique for regularity estimates and the classical existence theorem of global attractors, Zhao and Liu proved the existence of global attractor for equation (1) on some affine space of H^k ($0 \leq k < +\infty$).

In this article, we are concerned with the distributed optimal control problem

$$\min J(u, w) = \frac{1}{2} \|Cu - z_d\|_S^2 + \frac{\delta}{2} \|w\|_{L^2(Q_0)}^2$$

subject to

$$\begin{aligned} \frac{\partial u}{\partial t} + au_{xxxx} + \mu \left(\frac{u_x}{1 + |u_x|^2} \right)_x &= Bw, \quad (x, t) \in \Omega \times (0, T), \\ u_x(0, t) = u_x(1, t) = u_{xxx}(0, t) = u_{xxx}(1, t) &= 0, \\ u(0) &= u_0, \end{aligned} \quad (4)$$

where the operator C is called the observer, S is a real Hilbert space of observations. The control target is to match the given desired state z_d in L^2 -sense by adjusting the body force w in a control volume $Q_0 \subseteq Q = \Omega \times (0, T)$ in the L^2 -sense.

Now, we introduce some notations that will be used throughout the paper. For fixed $T > 0$, let $V = H^2_E(0, 1) = \{y: y \in H^2(0, 1), y_x|_{x=0,1} = 0\}$, $U = H^1(0, 1)$ and $H = L^2(0, 1)$, let V^* , U^* and H^* are dual spaces of V , U and H . Then we obtain

$$V \hookrightarrow U \hookrightarrow H = H^* \hookrightarrow U^* \hookrightarrow V^*.$$

Clearly, each embedding being dense.

The extension operator $B \in \mathcal{L}(L^2(Q_0), L^2(0, T; H))$, which is called the controller introduced as

$$Bq = \begin{cases} q, & q \in Q_0, \\ 0, & q \in Q \setminus Q_0. \end{cases}$$

We supply H with the inner product (\cdot, \cdot) and the norm $\|\cdot\|$, and we define a space $W(0, T; V)$ as

$$W(0, T; V) = \{y: y \in L^2(0, T; V), y_t \in L^2(0, T; V^*)\},$$

which is a Hilbert space endowed with common inner product.

This paper is organized as follows. In Section 2, we prove the existence and uniqueness of the weak solution to problem (1)–(3) in a special space and discuss the relation among the norms of weak solution, initial value and control item. In Section 3, we consider the optimal control problem of BCF model and prove the existence of optimal solution. In Section 4, the optimality conditions for BCF model is showed and the optimality system is derived.

In the following, the letters c, c_i ($i = 1, 2, \dots$) will always denote positive constants different in various occurrences.

2 Existence and uniqueness of weak solutions

In this section, we prove the existence and uniqueness of weak solution for problem (4), where $Bw \in L^2(0, T; H)$ and a control $w \in L^2(Q_0)$.

In the following Definition 1, we give the definition of the weak solution for problem (4) in the space $W(0, T; V)$.

Definition 1. For all $\eta \in V, t \in (0, T)$, a function $u \in W(0, T; V)$ is called a weak solution to problem (4) if

$$\begin{aligned} \left(\frac{d}{dt}u, \eta\right)_{V^*, V} + a(u_{xx}, \eta_{xx}) - \mu\left(\frac{u_x}{1 + |u_x|^2}, \eta_x\right) &= (Bw, \eta) \\ (u(0), \eta) &= (u_0, \eta). \end{aligned}$$

Here we supply V and V^* with the inner product

$$(v, w)_{V^*, V} = \int_{\Omega} vw \, dx \quad \forall v \in V^*, w \in V.$$

We shall give Theorem 1 on the existence and uniqueness of weak solution to problem (4) and prove it.

Theorem 1. Assume that $u_0 \in V$, $Bw \in L^2(0, T; H)$, then problem (4) admits a unique weak solution $u \in W(0, T; V)$.

Proof. The Galerkin method is applied to the proof.

Denote $\mathbb{A} = (-\partial_x^2)^2$ as a differential operator, let $\{\psi_i\}_{i=1}^\infty$ denote the eigenfunctions of the operator \mathbb{A} . For $n \in N$, define the discrete ansatz space by

$$V_n = \text{span}\{\psi_1, \psi_2, \dots, \psi_n\} \subset V.$$

Performing the Galerkin procedure for equation (4). We find the approximation solutions $u_n(t) = u_n(x, t) = \sum_{i=1}^n u_i^n(t)\psi_i(x)$ satisfies

$$\begin{aligned} \frac{\partial u_n}{\partial t} + a u_{n,xxxx} + \mu \left(\frac{u_{n,x}}{1 + |u_{n,x}|^2} \right)_x &= Bw, \quad (x, t) \in \Omega \times (0, T), \\ u_{n,x}(0, t) = u_{n,x}(1, t) = u_{n,xxx}(0, t) = u_{n,xxx}(1, t) &= 0, \\ u_n(x, 0) &= u_{n,0}(x). \end{aligned} \quad (5)$$

It is easy to see that the equation of (5) is an ordinary differential equation and, according to ODE theory, there exists a unique solution to the equation of (5) in the interval $[0, t_n)$. What we should do is to show that the solution is uniformly bounded when $t_n \rightarrow T$. We need also to show that the times t_n there are not decaying to 0 as $n \rightarrow \infty$.

Therefore, we shall prove the existence of solution in the following steps.

Step 1. Multiplying the equation of (5) by u_n and integrating with respect to x over $(0, 1)$, we deduce that

$$\frac{1}{2} \frac{d}{dt} \|u_n\|^2 + a \|u_{n,xx}\|^2 = \mu \int_0^1 \frac{u_{n,x}^2}{1 + |u_{n,x}|^2} dx + \int_0^1 u_n Bw dx.$$

Note that

$$\mu \int_0^1 \frac{u_{n,x}^2}{1 + |u_{n,x}|^2} dx \leq \mu \int_0^1 1 dx \leq \mu,$$

and

$$\int_0^1 u_n Bw dx \leq \|u_n\| \|Bw\| \leq \frac{1}{2} \|u_n\|^2 + \frac{1}{2} \|Bw\|^2.$$

Summing up, we immediately get

$$\frac{d}{dt} \|u_n\|^2 + 2a \|u_{n,xx}\|^2 \leq \|u_n\|^2 + \|Bw\|^2 + 2\mu.$$

Since $Bw \in L^2(0, T; H)$ is the control item, we can assume that $\|Bw\| \leq M$, where M is a positive constant. Then we have

$$\frac{d}{dt} \|u_n\|^2 + 2a \|u_{n,xx}\|^2 \leq \|u_n\|^2 + M^2 + 2\mu.$$

Using Gronwall's inequality, for all $t \in [0, T]$, we obtain

$$\|u_n\|^2 \leq e^t (\|u_{n,0}\|^2 + M^2 + 2\mu) \leq e^T (\|u_{n,0}\|^2 + M^2 + 2\mu) = c_1^2 \quad (6)$$

Step 2. Multiplying the equation of (5) by $u_{n,xx}$ and integrating with respect to x over $(0, 1)$, we deduce that

$$\frac{1}{2} \frac{d}{dt} \|u_{n,x}\|^2 + a \|u_{n,xxx}\|^2 + \mu \int_0^1 \frac{u_{n,x}}{1 + |u_{n,x}|^2} u_{n,xxx} \, dx = -(Bw, u_{n,xx}). \quad (7)$$

Note that

$$\|u_{n,xx}\|^2 = - \int_0^1 u_{n,x} u_{n,xxx} \, dx \leq \frac{1}{a} \|u_{n,x}\|^2 + \frac{a}{4} \|u_{n,xxx}\|^2,$$

and

$$\int_0^1 \left(\frac{u_{n,x}}{1 + |u_{n,x}|^2} \right)^2 \, dx \leq \int_0^1 u_{n,x}^2 \, dx = \|u_{n,x}\|^2.$$

Then it follows from (7) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_{n,x}\|^2 + a \|u_{n,xxx}\|^2 \\ & \leq \frac{a}{4} \|u_{n,xxx}\|^2 + \frac{\mu^2}{a} \int_0^1 \left(\frac{u_{n,x}}{1 + |u_{n,x}|^2} \right)^2 \, dx + \|u_{n,xx}\|^2 + \frac{1}{4} \|Bw\|^2 \\ & \leq \frac{a}{4} \|u_{n,xxx}\|^2 + \frac{\mu^2}{a} \|u_{n,x}\|^2 + \frac{a}{4} \|u_{n,xxx}\|^2 + \frac{1}{a} \|u_{n,x}\|^2 + \frac{1}{4} \|Bw\|^2, \end{aligned}$$

that is,

$$\frac{d}{dt} \|u_{n,x}\|^2 + a \|u_{n,xxx}\|^2 \leq \frac{2(\mu^2 + 1)}{a} \|u_{n,x}\|^2 + \frac{1}{2} \|Bw\|^2.$$

Since $\|Bw\| \leq M$, using Gronwall's inequality, we obtain

$$\begin{aligned} \|u_{n,x}\|^2 & \leq e^{(2(\mu^2+1)/a)t} \left(\|u_{n,x}(0)\|^2 + \frac{aM^2}{4\mu^2 + 4} \right) \\ & \leq e^{(2(\mu^2+1)/a)T} \left(\|u_{n,x}(0)\|^2 + \frac{aM^2}{4\mu^2 + 4} \right) = c_2^2 \quad \forall t \in [0, T]. \end{aligned} \quad (8)$$

Step 3. Multiplying the equation of (5) by $u_{n,xxxx}$ and integrating with respect to x over $(0, 1)$, we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_{n,xx}\|^2 + a \|u_{n,xxxx}\|^2 + \mu \int_0^1 \left(\frac{u_{n,x}}{(1+|u_{n,x}|^2)^2} \right)_x u_{n,xxxx} dx \\ & = \int_0^1 u_{n,xxxx} Bw dx. \end{aligned}$$

Note that

$$\int_0^1 \left(\frac{u_{n,x}}{1+|u_{n,x}|^2} \right)_x u_{n,xxxx} dx = \int_0^1 \frac{u_{n,xx} u_{n,xxxx}}{1+|u_{n,xx}|^2} dx - \int_0^1 \frac{2|u_{n,x}|^2 u_{n,xx} u_{n,xxxx}}{(1+|u_{n,x}|^2)^2} dx.$$

Hence,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_{n,xx}\|^2 + a \|u_{n,xxxx}\|^2 \\ & = \mu \int_0^1 \frac{2|u_{n,x}|^2 u_{n,xx} u_{n,xxxx}}{(1+|u_{n,x}|^2)^2} dx - \mu \int_0^1 \frac{u_{n,xx} u_{n,xxxx}}{1+|u_{n,xx}|^2} dx + \int_0^1 u_{n,xxxx} Bw dx \\ & \leq \mu \sup_{x \in [0,1]} \frac{2|u_{n,x}|^2}{(1+|u_{n,x}|^2)^2} \|u_{n,xx}\| \|u_{n,xxxx}\| \\ & \quad + \mu \sup_{x \in [0,1]} \frac{1}{1+|u_{n,x}|^2} \|u_{n,xx}\| \|u_{n,xxxx}\| + \|Bw\| \|u_{n,xxxx}\| \\ & \leq 3\mu \|u_{n,xx}\| \|u_{n,xxxx}\| + \|Bw\| \|u_{n,xxxx}\| \\ & \leq \frac{a}{2} \|u_{n,xxxx}\|^2 + \frac{9\mu^2}{a} \|u_{n,xx}\|^2 + \frac{1}{a} \|Bw\|^2. \end{aligned}$$

Therefore, we have

$$\frac{d}{dt} \|u_{n,xx}\|^2 + a \|u_{n,xxxx}\|^2 \leq \frac{18\mu^2}{a} \|u_{n,xx}\|^2 + \frac{2}{a} \|Bw\|^2.$$

Since $\|Bw\| \leq M$, using Gronwall's inequality, we obtain

$$\begin{aligned} \|u_{n,xx}\|^2 & \leq e^{(18\mu^2/a)t} \left(\|u_{n,xx}(0)\|^2 + \frac{1}{9\mu^2} M^2 \right) \\ & \leq e^{(9\mu^2/(2a))T} \left(\|u_{n,xx}(0)\|^2 + \frac{1}{9\mu^2} M^2 \right) = c_3^2 \quad \forall t \in [0, T]. \end{aligned} \quad (9)$$

Adding (6), (8) and (9) together gives $\|u_n\|_{H^2} \leq c$, then we have

$$\|u_n\|_{L^2(0,T;V)}^2 = \int_0^T \|u_n\|_{H^2}^2 dt \leq c_4.$$

Then the uniform $L^2(0, T; V)$ bound on a sequence $\{u_n\}$ is proved.

Step 4. We prove a uniform $L^2(0, T; V^*)$ bound on a sequence $\{u_{n,t}\}$. Note that for all $\eta \in V$, we have

$$\begin{aligned} |(u_{n,xxxx}, \eta)_{V^*, V}| &\leq |(u_{n,xx}, \eta_{xx})| \leq \|u_{n,xx}\| \|\eta_{xx}\| \leq \|u_{n,xx}\| \|\eta\|_V, \\ \left| \left(\left[\frac{u_{n,x}}{1 + |u_{n,x}|^2} \right]_x, \eta \right)_{V^*, V} \right| &\leq \left| \left(\frac{u_{n,x}}{1 + |u_{n,x}|^2}, \eta_x \right) \right| \leq \|u_{n,x}\| \|\eta_x\| \leq \|u_{n,x}\| \|\eta\|_V, \\ |(Bw, \eta)_{V^*, V}| &\leq \|Bw\| \|\eta\| \leq \|Bw\| \|\eta\|_V. \end{aligned}$$

Therefore, by (8), we have

$$\begin{aligned} &\|u_{n,t}\|_{V^*} \\ &\leq a \|u_{n,xxxx}\|_{V^*} + \mu \left\| \left(\frac{u_{n,x}}{1 + |u_{n,x}|^2} \right)_x \right\|_{V^*} + \|Bw\|_{V^*} \\ &\leq c \left(\sup \frac{|(u_{n,xxxx}, \eta)_{V^*, V}|}{\|\eta\|_V} + \sup \frac{|(\left[\frac{u_{n,x}}{1 + |u_{n,x}|^2} \right]_x, \eta)_{V^*, V}|}{\|\eta\|_V} + \sup \frac{|(Bw, \eta)_{V^*, V}|}{\|\eta\|_V} \right) \\ &\leq c (\|u_{n,xx}\| + \|u_{n,x}\| + \|Bw\|) \leq c(c_2 + c_3 + M). \end{aligned}$$

Hence, we get

$$\|u_{n,t}\|_{L^2(0, T; V^*)} \leq c(c_2 + c_3 + M)T = c_5.$$

Collecting the previous, we get:

- (i) For every $t \in [0, T]$, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T; V)$, which is independent of the dimension of ansatz space n .
- (ii) For every $t \in [0, T]$, the sequence $\{u_{n,t}\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T; V^*)$, which is independent of the dimension of ansatz space n .

By the above discussion, we obtain $u \in W(0, T; V)$. It is easy to check that $W(0, T; V)$ is continuously embedded into $C([0, T]; H)$, which denote the space of continuous functions. We concludes convergence of a subsequences, again denoted by $\{u_n\}$ weak into $W(0, T; V)$, weak-star in $L^\infty(0, T; H)$ and strong in $L^2(0, T; H)$ to functions $u \in W(0, T; V)$.

Since the proof of uniqueness is easy, we omit it.

Then Theorem 1 is proved. □

Now, we shall discuss the relation among the norm of weak solution and initial value and control item.

Theorem 2. Assume that $Bw \in L^2(0, T; H)$, $u_0 \in V$, then there exist positive constants C' and C'' such that

$$\|u\|_{W(0, T; V)}^2 \leq C' (\|u_0\|_V^2 + \|w\|_{L^2(Q_0)}^2) + C''. \tag{10}$$

Proof. Clearly, (10) means

$$\|u\|_{L^2(0,T;V)}^2 + \|u_t\|_{L^2(0,T;V^*)}^2 \leq C'(\|u_0\|_V^2 + \|Bw\|_{L^2(0,T;H)}^2) + C''. \quad (11)$$

Multiplying the equation of (4) by u , integrating the resulting on $(0, 1)$ and using the same argument as in the proof of Theorem 1, we obtain

$$\frac{d}{dt}\|u\|^2 + 2a\|u_{xx}\|^2 \leq \|Bw\|^2 + \|u\|^2 + 2\mu. \quad (12)$$

Using Gronwall's inequality, we have

$$\|u\|^2 \leq e^t(\|u_0\|^2 + 2\mu + \|Bw\|^2).$$

Then

$$\int_0^T \|u\|^2 dt \leq Te^T(\|u_0\|^2 + 2\mu) + e^T\|Bw\|_{L^2(0,T;H)}^2. \quad (13)$$

Integrating (12) with respect to t on $[0, T]$, we thus derive that

$$\|u(t)\|^2 - \|u(0)\|^2 + 2a \int_0^T \|u_{xx}\|^2 dt \leq \int_0^T \|Bw\|^2 dt + \int_0^T \|u\|^2 dt + 2\mu T.$$

Therefore, by (13) and above inequality, we deduce that

$$\begin{aligned} \int_0^T \|u_{xx}\|^2 dt &\leq \frac{1}{2a} \left(\int_0^T \|Bw\|^2 dt + \int_0^T \|u\|^2 dt + 2\mu T + \|u(0)\|^2 \right) \\ &\leq \frac{1}{2a} \left((1 + e^T)\|Bw\|_{L^2(0,T;H)}^2 + (1 + Te^T)\|u(0)\|^2 + 4\mu T \right). \end{aligned} \quad (14)$$

We also have

$$\int_0^T \|u_x\|^2 dt = - \int_0^T (u, u_{xx}) dt \leq \frac{1}{2} \int_0^T \|u\|^2 dt + \frac{1}{2} \int_0^T \|u_{xx}\|^2 dt. \quad (15)$$

On the other hand, for all $\eta \in V$, we have

$$\begin{aligned} |(u_{xxxx}, \eta)_{V^*,V}| &\leq |(u_{xx}, \eta_{xx})| \leq \|u_{xx}\| \|\eta_{xx}\| \leq \|u_{xx}\| \|\eta\|_V, \\ \left| \left(\left[\frac{u_x}{1 + |u_x|^2} \right]_x, \eta \right)_{V^*,V} \right| &\leq \left| \left(\frac{u_x}{1 + |u_x|^2}, \eta_x \right) \right| \leq \|u_x\| \|\eta_x\| \leq \|u_x\| \|\eta\|_V, \\ |(Bw, \eta)_{V^*,V}| &\leq \|Bw\| \|\eta\| \leq \|Bw\| \|\eta\|_V. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u_t\|_{V^*} &\leq a\|u_{xxxx}\|_{V^*} + \mu \left\| \left(\frac{u_x}{1 + |u_x|^2} \right)_x \right\|_{V^*} + \|Bw\|_{V^*} \\ &\leq c \left(\sup \frac{|(u_{xxxx}, \eta)_{V^*, V}|}{\|\eta\|_V} + \sup \frac{|([\frac{u_x}{1 + |u_x|^2}]_x, \eta)_{V^*, V}|}{\|\eta\|_V} + \sup \frac{|(Bw, \eta)_{V^*, V}|}{\|\eta\|_V} \right) \\ &\leq c(\|u_{xx}\| + \|u_x\| + \|Bw\|). \end{aligned}$$

Hence, we get

$$\begin{aligned} \int_0^T \|u_t\|_{V^*}^2 dt &\leq C \int_0^T (\|u_{xx}\|^2 + \|u_x\|^2 + \|Bw\|^2) dt \\ &\leq C \left(\|Bw\|_{L^2(0, T; H)}^2 + \int_0^T \|u_{xx}\|^2 dt + \int_0^T \|u_x\|^2 dt \right). \end{aligned} \tag{16}$$

By (13)–(16) and the definition of extension operator B , we obtain (11). Hence, Theorem 2 is proved. \square

3 Optimal control problem

In this section, we consider the optimal control problem associated with problem (4) and prove the existence of optimal solution.

In the following, we suppose $L^2(Q_0)$ is a Hilbert space of control variables, we also suppose $B \in \mathcal{L}(L^2(Q_0), L^2(0, T; H))$ is the controller and a control $w \in L^2(Q_0)$. Consider the following control system:

$$\begin{aligned} \frac{\partial u}{\partial t} + au_{xxxx} + \mu \left(\frac{u_x}{1 + |u_x|^2} \right)_x &= Bw, \quad (x, t) \in \Omega \times (0, T), \\ u_x(0, t) = u_x(1, t) = u_{xxx}(0, t) = u_{xxx}(1, t) &= 0, \\ u(0) &= u_0. \end{aligned} \tag{17}$$

Here in (17), it is assumed that $u_0 \in V$. By virtue of Theorem 1, we can define the solution map $w \rightarrow u(w)$ of $L^2(Q_0)$ into $W(0, T; V)$. The solution u is called the state of the control system (17). The observation of the state is assumed to be given by Cu . Here $C \in \mathcal{L}(W(0, T; V), S)$ is an operator, which is called the observer, S is a real Hilbert space of observations. The cost function associated with the control system (17) is given by

$$J(u, w) = \frac{1}{2} \|Cu - z_d\|_S^2 + \frac{\delta}{2} \|w\|_{L^2(Q_0)}^2, \tag{18}$$

where $z_d \in S$ is a desired state and $\delta > 0$ is fixed.

An optimal control problem about problem (17) is

$$\min J(u, w), \tag{19}$$

where (u, w) satisfies (17).

Let $X = W(0, T; V) \times L^2(Q_0)$ and $Y = L^2(0, T; V) \times H$. We define an operator $e = e(e_1, e_2): X \rightarrow Y$, where

$$e_1 = G = (\Delta^2)^{-1} \left[\frac{\partial u}{\partial t} + au_{xxxx} + \mu \left(\frac{u_x}{1 + |u_x|^2} \right)_x - Bw \right],$$

and

$$e_2 = u(x, 0) - u_0.$$

Here Δ^2 is an operator from V to V^* . Then we write (19) in the following form:

$$\min J(u, w) \quad \text{subject to } e(u, w) = 0.$$

Theorem 3. Assume that $Bw \in L^2(0, T; H)$, $u_0 \in V$, then there exists an optimal control solution (u^*, w^*) to problem (17).

Proof. Suppose (u, w) satisfy the equation $e(u, w) = 0$. In view of (18), we deduce that

$$J(u, w) \geq \frac{\delta}{2} \|w\|_{L^2(Q_0)}^2.$$

By Theorem 2, we obtain

$$\|u\|_{W(0, T; V)} \rightarrow \infty \quad \text{yields} \quad \|w\|_{L^2(Q_0)} \rightarrow \infty.$$

Therefore,

$$J(u, w) \rightarrow \infty \quad \text{when} \quad \|(u, w)\|_X \rightarrow \infty. \tag{20}$$

As the norm is weakly lower semi-continuous, we achieve that J is weakly lower semi-continuous. Since for all $(u, w) \in X$, $J(u, w) \geq 0$, there exists $\lambda \geq 0$ defined by

$$\lambda = \inf \{ J(u, w) : (u, w) \in X, e(u, w) = 0 \},$$

which means the existence of a minimizing sequence $\{(u^n, w^n)\}_{n \in \mathbb{N}}$ in X such that

$$\lambda = \lim_{n \rightarrow \infty} J(u^n, w^n) \quad \text{and} \quad e(u^n, w^n) = 0 \quad \forall n \in \mathbb{N}.$$

From (20), there exists an element $(u^*, w^*) \in X$ such that when $n \rightarrow \infty$,

$$\begin{aligned} u^n &\rightharpoonup u^* \quad \text{weakly in } W(0, T; V), \\ w^n &\rightharpoonup w^* \quad \text{weakly in } L^2(Q_0). \end{aligned} \tag{21}$$

Using (21), we get

$$\lim_{n \rightarrow \infty} \int_0^T (u_t^n(x, t) - u_t^*, \psi(t))_{V^*, V} dt = 0 \quad \forall \psi \in L^2(0, T; V),$$

where $(\cdot, \cdot)_{V^*, V}$ denotes the scalar product between V^* and V .

Since $W(0, T; V)$ is compactly embedded into $L^2(0, T; L^\infty)$, we have $u^n \rightarrow u^*$ strongly in $L^2(0, T; L^\infty)$. On the other hand, we know that $u_n \in L^\infty(0, T; V)$ and $u_{n,t} \in L^2(0, T; V^*)$. Hence, by [12, Lemma 4] we have $u^n \rightarrow u^*$ strongly in $C([0, T]; L^\infty)$, $u_x^n \rightarrow u_x^*$ strongly in $C([0, T]; H)$ as $n \rightarrow \infty$.

As the sequence $\{u^n\}_{n \in \mathbb{N}}$ converges weakly, then $\|u^n\|_{W(0, T; V)}$ is bounded. And $\|u^n\|_{L^2(0, T; L^\infty)}$ is also bounded based on the embedding theorem.

Because $u_x^n \rightarrow u_x^*$ in $L^2(0, T; L^\infty)$ as $n \rightarrow \infty$, we know that $\|u_x^*\|_{L^2(0, T; L^\infty)}$ is bounded too.

Using (21), we deduce that

$$\begin{aligned} & \left| \int_0^T \int_0^1 \left[\left(\frac{u_x^n}{1 + |u_x^n|^2} \right)_x - \left(\frac{u_x^*}{1 + |u_x^*|^2} \right)_x \right] \eta \, dx \, dt \right| \\ &= \left| \int_0^T \int_0^1 \left(\frac{u_x^n}{1 + |u_x^n|^2} - \frac{u_x^*}{1 + |u_x^*|^2} \right) \eta_x \, dx \, dt \right| \\ &= \left| \int_0^T \int_0^1 \frac{(1 - u_x^n u_x^*)(u_x^n - u_x^*)}{(1 + |u_x^n|^2)(1 + |u_x^*|^2)} \eta_x \, dx \, dt \right| \\ &\leq \left| \int_0^T \int_0^1 (1 - u_x^n u_x^*)(u_x^n - u_x^*) \eta_x \, dx \, dt \right| \\ &\leq \int_0^T \|1 - u_x^* u_x^n\|_{L^\infty} \|u_x^n - u_x^*\|_H \|\eta_x\|_H \, dt \\ &\leq \|1 - u_x^* u_x^n\|_{L^2(0, T; L^\infty)} \|u_x^n - u_x^*\|_{C([0, T]; H)} \|\eta_x\|_{L^2(0, T; H)} \\ &\rightarrow 0, \quad n \rightarrow \infty, \quad \forall \eta \in L^2(0, T; V). \end{aligned}$$

In view of the above discussion, we get

$$e_1(u^*, w^*) = 0.$$

From $u^* \in W(0, T; V)$, we derive that $u^*(0) \in H$. Since $u^n \rightarrow u^*$ weakly in $L^2(0, T; H)$, we can infer that $u^n(0) \rightarrow u^*(0)$ weakly when $n \rightarrow \infty$. Thus, we obtain

$$(u^n(0) - u^*(0), \eta) \rightarrow 0, \quad n \rightarrow \infty, \quad \forall \eta \in H,$$

that is,

$$e_2(u^*, w^*) = 0.$$

Therefore, we obtain

$$e(u^*, w^*) = 0 \quad \text{in } Y.$$

So, there exists an optimal solution (u^*, w^*) to problem (17), Theorem 3 is proved. \square

4 Optimality conditions

It is well known that the optimality conditions for w is given by the variational inequality

$$J'(u, w)(v - w) \geq 0 \quad \forall v \in L^2(Q_0), \tag{22}$$

where $J'(u, w)$ denotes the Gateaux derivative of $J(u, v)$ at $v = w$.

The following lemma is essential in deriving necessary optimality conditions.

Lemma 1. *The map $v \rightarrow u(v)$ of $L^2(Q_0)$ into $W(0, T; V)$ is weakly Gateaux differentiable at $v = w$ and such the Gateaux derivative of $u(v)$ at $v = w$ in the direction $v - w \in L^2(Q_0)$, say $z = \mathcal{D}u(w)(v - w)$, is a unique weak solution of the following problem:*

$$\begin{aligned} & (z_t, \eta)_{V^*, V} + a(z_{xx}, \eta_{xx}) - \mu \left(\frac{(1 - |u_x|^2)z_x}{(1 + |u_x|^2)^2}, \eta_x \right) \\ & = (B(v - w), \eta) \quad \forall \eta \in V, \quad 0 < t \leq T, \\ & z(0) = 0, \quad x \in (0, 1). \end{aligned} \tag{23}$$

Proof. Let $0 \leq h \leq 1$, u_h and u be the solutions of (17) corresponding to $w + h(v - w)$ and w , respectively. We prove the lemma in the following two steps.

Step 1. We prove $u_h \rightarrow u$ strongly in $C([0, T]; H)$ as $h \rightarrow 0$. Let $q = u_h - u$, then

$$\begin{aligned} & \left(\frac{dq}{dt}, \eta \right)_{V^*, V} + a(q_{xx}, \eta_{xx}) - \mu \left(\frac{u_{hx}}{1 + |u_{hx}|^2} - \frac{u_x}{1 + |u_x|^2}, \eta_x \right) \\ & = (hB(v - w), \eta) \quad \forall \eta \in V, \quad 0 < t \leq T, \\ & q(0) = 0, \quad x \in (0, 1). \end{aligned} \tag{24}$$

Note that

$$\|q_x\|^2 \leq \varepsilon \|q_{xx}\|^2 + \frac{1}{4\varepsilon} \|q\|^2,$$

where ε is a positive constant. By Theorem 2, we get

$$\begin{aligned} \|u\|_{W^{1, \infty}} &= \sup_{x \in [0, 1]} (|u| + |u_x|) \leq c, \\ \|u_h(x, t)\|_{W^{1, \infty}} &= \sup_{x \in [0, 1]} (|u_h| + |u_{hx}|) \leq c. \end{aligned}$$

Then, setting $\eta = q$ in (24), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|q\|^2 + a \|q_{xx}\|^2 \\ &= \mu \left(\frac{u_{hx}}{1 + |u_{hx}|^2} - \frac{u_x}{1 + |u_x|^2}, q_x \right) + (hB(v - w), q) \\ &= \mu \left(\frac{1 - u_x u_{hx}}{(1 + |u_{hx}|^2)(1 + |u_x|^2)} q_x, q_x \right) + (hB(v - w), q) \\ &\leq \left\| \frac{1 - u_x u_{hx}}{(1 + |u_{hx}|^2)(1 + |u_x|^2)} \right\|_{L^\infty} \|q_x\|^2 + h \|B(v - w)\| \|q\| \\ &\leq c_0 \|q_x\|^2 + h \|B(v - w)\| \|q\| \\ &\leq \frac{a}{2} \|q_{xx}\|^2 + \left(\frac{c_0^2}{2a} + \frac{1}{4} \right) \|q\|^2 + h^2 \|B(v - w)\|^2. \end{aligned}$$

Hence,

$$\frac{d}{dt} \|q\|^2 + a \|q_{xx}\|^2 \leq \left(\frac{c_0^2}{a} + \frac{1}{2} \right) \|q\|^2 + 2h^2 \|B(v - w)\|^2.$$

Using Gronwall's inequality, it is easy to see that $\|q\|^2 \rightarrow 0$ as $h \rightarrow 0$. Then $u_h \rightarrow u$ strongly in $C([0, T]; H)$ as $h \rightarrow 0$.

Step 2. We prove that $(u_h - u)/h \rightarrow z$ strongly in $W(0, T; V)$. Rewrite (24) in the following form:

$$\begin{aligned} & \left(\frac{d}{dt} \frac{u_h - u}{h}, \eta \right)_{V^*, V} + a \left(\left[\frac{u_h - u}{h} \right]_{xx}, \eta_{xx} \right) \\ & \quad - \left(\frac{\mu}{h} \left[\frac{u_{hx}}{1 + |u_{hx}|^2} - \frac{u_x}{1 + |u_x|^2} \right], \eta_x \right) \\ &= (B(v - w), \eta) \quad \forall \eta \in V, 0 < t \leq T, \\ & \frac{u_h - u}{h}(0) = 0, \quad x \in (0, 1). \end{aligned}$$

We can easily verify that the above problem possesses a unique weak solution in $W(0, T; V)$. On the other hand, it is easy to check that the linear problem (23) possesses a unique weak solution $z \in W(0, T; V)$. Let $r = (u_h - u)/h - z$, thus r satisfies

$$\begin{aligned} & \left(\frac{d}{dt} r, \eta \right)_{V^*, V} + a(r_{xx}, \eta_{xx}) - \mu \left(\frac{1}{h} \left[\frac{u_{hx}}{1 + |u_{hx}|^2} - \frac{u_x}{1 + |u_x|^2} \right] \right. \\ & \quad \left. - \left[\frac{(1 - |u_x|^2)z_x}{(1 + |u_x|^2)^2} \right], \eta_x \right) = 0 \quad \forall \eta \in V, 0 < t \leq T, \\ & r(0) = 0, \quad x \in (0, 1). \end{aligned} \tag{25}$$

Setting $\eta = r$ in (25), $\kappa = u_x + \theta(u_{hx} - u_x)$, $\theta \in (0, 1)$, using differential mean value theorem, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|r\|^2 + a \|r_{xx}\|^2 \\ &= \mu \left(\frac{1}{h} \left(\frac{u_{hx}}{1 + |u_{hx}|^2} - \frac{u_x}{1 + |u_x|^2} \right) - \frac{(1 - |u_x|^2)z_x}{(1 + |u_x|^2)^2}, r_x \right) \\ &= \mu \left(\frac{1 - \kappa^2}{(1 + \kappa^2)^2} \frac{u_{hx} - u_x}{h} - \frac{(1 - |u_x|^2)z_x}{(1 + |u_x|^2)^2}, r_x \right) \\ &= \mu \left(\frac{1 - \kappa^2}{(1 + \kappa^2)^2} \left(\frac{u_{hx} - u_x}{h} - z_x \right), r_x \right) + \mu \left(z_x \left(\frac{1 - \kappa^2}{(1 + \kappa^2)^2} - \frac{(1 - |u_x|^2)}{(1 + |u_x|^2)^2} \right), r_x \right) \\ &=: I_1 + I_2, \end{aligned}$$

Note that $\kappa = u_x + \theta(u_{hx} - u_x)$ satisfies $\|\kappa\|_{H^1} \leq c$ and $\|\kappa\|_\infty \leq c$. Hence, for I_1 , we have

$$\begin{aligned} I_1 &= \mu \left(\frac{1 - \kappa^2}{(1 + \kappa^2)^2} \left(\frac{u_{hx} - u_x}{h} - z_x \right), r_x \right) \\ &= \mu \left(\frac{u_h - u}{h} - z, \frac{1 - \kappa^2}{(1 + \kappa^2)^2} r_{xx} - \frac{2\kappa\kappa_x(3 - \kappa^2)}{(1 + \kappa^2)^3} r_x \right) \\ &\leq c(\|r\| \|r_{xx}\| \|1 - \kappa^2\|_\infty + \|r\| \|r_x\|_\infty \|\kappa_x\| \|\kappa(3 - \kappa^2)\|_\infty) \\ &\leq c_0 \|r\| \|r_{xx}\| \leq \frac{a}{2} \|r_{xx}\|^2 + \frac{c_0^2}{2a} \|r\|^2. \end{aligned}$$

On the other hand, for I_2 , we have

$$\begin{aligned} I_2 &= \mu \left(z_x \left(\frac{1 - \kappa^2}{(1 + \kappa^2)^2} - \frac{(1 - |u_x|^2)}{(1 + |u_x|^2)^2} \right), r_x \right) \\ &\leq c \|r_x\|_\infty \|z_x\| \left\| \frac{1 - \kappa^2}{(1 + \kappa^2)^2} - \frac{(1 - |u_x|^2)}{(1 + |u_x|^2)^2} \right\| \\ &\leq c \|r_{xx}\| \|z_x\| \left\| \frac{1 - \kappa^2}{(1 + \kappa^2)^2} - \frac{(1 - |u_x|^2)}{(1 + |u_x|^2)^2} \right\| \\ &\leq \frac{a}{2} \|r_{xx}\|^2 + c \|z_x\|^2 \left\| \frac{1 - \kappa^2}{(1 + \kappa^2)^2} - \frac{(1 - |u_x|^2)}{(1 + |u_x|^2)^2} \right\|^2 \\ &\rightarrow \frac{a}{2} \|r_{xx}\|^2 \quad \text{as } h \rightarrow 0. \end{aligned}$$

Summing up, we get

$$\begin{aligned} & \frac{d}{dt} \|r\|^2 + \frac{a}{2} \|r_{xx}\|^2 \\ &\leq \frac{c_0^2}{a} \|r\|^2 + c \|z_x\|^2 \left\| \frac{1 - \kappa^2}{(1 + \kappa^2)^2} - \frac{(1 - |u_x|^2)}{(1 + |u_x|^2)^2} \right\|^2 \quad \text{as } h \rightarrow 0. \end{aligned}$$

Using Gronwall’s inequality, it is easy to check that $(u_h - u)/h$ is strongly convergent to z in $C(0, T; H)$. Integrating above inequality over $(0, T)$, we can obtain $(u_h - u)/h$ strongly convergent to z in $L^2(0, T; V)$.

On the other hand, for all $\eta \in V$, we have

$$\begin{aligned} a(r_{xx}, \eta_{xx}) &\leq a\|r_{xx}\|\|\eta\|_{xx} \leq a\|r_{xx}\|\|\eta\|_V, \\ \mu\left(\frac{1}{h}\left[\frac{u_{hx}}{1+|u_{hx}|^2} - \frac{u_x}{1+|u_x|^2}\right] - \left[\frac{(1-|u_x|^2)z_x}{(1+|u_x|^2)^2}\right], \eta_x\right) \\ &= \mu\left(\frac{1-\kappa^2}{(1+\kappa^2)^2} \frac{u_{hx}-u_x}{h} - \frac{(1-|u_x|^2)}{(1+|u_x|^2)^2} z_x, \eta_x\right) \\ &= \mu\left(\frac{1-\kappa^2}{(1+\kappa^2)^2} \left(\frac{u_{hx}-u_x}{h} - z_x\right), \eta_x\right) \\ &\quad + \mu\left(z_x \left(\frac{1-\kappa^2}{(1+\kappa^2)^2} - \frac{(1-|u_x|^2)z_x}{(1+|u_x|^2)^2}\right), \eta_x\right) \\ &\leq c\|r\|\|\eta_{xx}\| \leq C\|r\|\|\eta\|_V. \end{aligned}$$

Hence, using (25), we can obtain

$$\int_0^T \|r\|_{V^*}^2 dt \leq a \int_0^T \|r\|_{xx}^2 dt + c \int_0^T \|r\|^2 dt \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Then $(u_h - u)/h \rightarrow z$ strongly in $W(0, T; V)$, Lemma 1 is proved. □

As in [8], we denote Λ = canonical isomorphism of S onto S^* , where S^* is the dual spaces of S . By calculating the Gateaux derivative of (20) via Lemma 1, we see that the cost $J(v)$ is weakly Gateaux differentiable at w in the direction $v - w$. Then, for all $L^2(Q_0)$, (22) can be rewritten as

$$(C^* \Lambda(Cu(w) - z_d), z)_{W(0,T;V)^*, W(0,T;V)} + \frac{\delta}{2}(w, v - w)_{L^2(Q_0)} \geq 0, \quad (26)$$

where z is the solution of (23), $(\cdot, \cdot)_{W(0,T;V)^*, W(0,T;V)}$ denotes the scalar product between $W(0, T; V)^*$ and $W(0, T; V)$.

Now, we study the necessary conditions of optimality. To avoid the complexity of observation states, we consider the two types of distributive and terminal value observations.

Case 1: $C \in \mathcal{L}(L^2(0, T; V); S)$. In this case, $C^* \in \mathcal{L}(S^*; L^2(0, T; V^*))$ and (26) may be written as

$$\int_0^T (C^* \Lambda(Cu(w) - z_d), z)_{V^*, V} dt + \frac{\delta}{2}(w, v - w)_{L^2(Q_0)} \geq 0 \quad (27)$$

for all $v \in L^2(Q_0)$. Here we give the definition of a solution to the adjoint equation.

Definition 2. Given an optimal control $Bw \in L^2(0, T; H)$ and $u_0 \in V$, there exists a solution $p(v) \in W(0, T; V)$ to the adjoint problem

$$\begin{aligned} & \left(-\frac{d}{dt}p(v), \eta \right)_{V^*, V} + a(p_{xx}(v), \eta_{xx}) - \mu \left(\frac{(1 - |u_x|^2)p_x}{(1 + |u_x|^2)^2}, \eta_x \right) \\ & = (C^* \Lambda(Cu(v) - z_d), \eta)_{V^*, V} \quad \text{in } (0, T) \quad \forall \eta \in V, \\ & p(x, T; v) = 0. \end{aligned} \tag{28}$$

According to Theorem 1, the above problem admits a unique solution (after changing t into $T - t$).

Setting $\eta = z$ in (28) (with $v = w$) and using Lemma 1, we get

$$\int_0^T \left(-\frac{d}{dt}p(w), z \right)_{V^*, V} dt = \int_0^T \left\langle p(w), \frac{d}{dt}z \right\rangle_{V, V^*} dt,$$

and

$$\int_0^T \left(\frac{(1 - |u_x|^2)p_x(w)}{(1 + |u_x|^2)^2}, z_x \right) dt = \int_0^T \left(p_x(w), \frac{(1 - |u_x|^2)z_x}{(1 + |u_x|^2)^2} \right) dt.$$

Then we obtain

$$\begin{aligned} & \int_0^T (C^* \Lambda(Cu(w) - z_d), z)_{V^*, V} dt \\ & = \int_0^T (p(w), z_t)_{V, V^*} dt + a \int_0^T (p_{xx}(w), z_{xx}) dt - \mu \int_0^T \left(p_x(w), \frac{(1 - |u_x|^2)z_x}{(1 + |u_x|^2)^2} \right) dt \\ & = \int_0^T (p(w), Bv - Bw) dt = (B^*p(w), v - w). \end{aligned}$$

Hence, (27) may be written as

$$\int_0^T \int_0^1 B^*p(w)(v - w) dx dt + \frac{\delta}{2}(w, v - w)_{L^2(Q_0)} \geq 0 \quad \forall v \in L^2(Q_0). \tag{29}$$

Therefore, we have proved the following theorem.

Theorem 4. We assume that all conditions of Theorem 3 hold. Let us suppose that $C \in \mathcal{L}(L^2(0, T; V); S)$. The optimal control w is characterized by the system of two PDEs and an inequality: (17), (28) and (29).

Case 2: $C \in \mathcal{L}(H; S)$. In this case, we observe $Cu(v) = Du(T; v)$, $D \in \mathcal{L}(H; H)$. The associated cost function is expressed as

$$J(u, v) = \|Du(T; v) - z\|_S^2 + \frac{\delta}{2} \|v\|_{L^2(Q_0)}^2 \quad \forall v \in L^2(Q_0). \tag{30}$$

Then, for all $v \in L^2(Q_0)$, the optimal control w for (30) is characterized by

$$(Du(T; w) - z, Du(T; v) - Du(T; w))_{V^*, V} + \frac{\delta}{2} (w, v - w)_{L^2(Q_0)} \geq 0. \tag{31}$$

Definition 3. Given an optimal control $Bw \in L^2(0, T; H)$ and $u_0 \in V$, there exists a solution $p(v) \in W(0, T; V)$ to the adjoint problem

$$\begin{aligned} \left(-\frac{d}{dt}p(v), \eta\right)_{V^*, V} + a(p_{xx}(v), \eta_{xx}) - \mu \left(\frac{(1 - |u_x|^2)p_x}{(1 + |u_x|^2)^2}, \eta_x\right) &= 0 \\ \text{in } (0, T) \quad \forall \eta \in V, & \\ p(T; v) = D^*(Du(T; v) - z). & \end{aligned} \tag{32}$$

According to Theorem 1, the above problem admits a unique solution (after changing t into $T - t$).

Let us set $v = w$ in the above equations and set $\eta = u(v) - u(w)$ in (32) and integrate from 0 to T . This gives us,

$$(Du(T; w) - z, Du(T; v) - Du(T; w))_{V^*, V} = \int_0^T (p(w), B(v - w)) dt,$$

since

$$\begin{aligned} \int_0^T \left(-\frac{d}{dt}p(w), z\right)_{V^*, V} dt &= \int_0^T \left(p(w), \frac{d}{dt}z\right)_{V, V^*} dt \\ &\quad - \langle D^*(Du(T; v) - z), u(T; v) - u(T; w) \rangle_{V^*, V}. \end{aligned}$$

Hence, (31) is equivalent to

$$\int_0^T \int_0^1 B^*p(w)(v - w) dx dt + \frac{\delta}{2} (w, v - w)_{L^2(Q_0)} \geq 0 \quad \forall v \in L^2(Q_0). \tag{33}$$

Then we have the following theorem.

Theorem 5. We assume that all conditions of Theorem 3 hold. Let us suppose that $D \in \mathcal{L}(H; H)$. The optimal control w is characterized by the system of two PDEs and an inequality: (17), (32) and (33).

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References

1. R. Becker, B. Vexler, Optimal control of the convection-diffusion equation using stabilized finite element methods, *Numer. Math.*, **106**:349–367, 2007.
2. W.K. Burton, N. Cabrera, F.C. Frank, The growth of crystals and the equilibrium structure of their surfaces, *Philos. Trans. A, R. Soc. Lond.*, **243**:299–358, 1951.
3. N. Duan, W.J. Gao, Optimal control of a modified Swift–Hohenberg equation, *Electron. J. Differ. Equ.*, **2012**(155):1–12, 2012.
4. H. Fujimura, A. Yagi, Asymptotic behavior of solutions for dynamical system for BCF model describing crystal surface growth, *Int. Math. Forum*, **3**:1803–1812, 2008.
5. H. Fujimura, A. Yagi, Homogeneous stationary solution for BCF model describing crystal surface growth, *Sci. Math. Jpn.*, **69**:295–302, 2009.
6. M. Grasselli, G. Mola, A. Yagi, On the longtime behavior of solutions to a model for epitaxial growth, *Osaka J. Math.*, **48**:987–1004, 2011.
7. M.D. Johnson, C. Orme, A.W. Hunt, D. Graff, J. Sudijion, L.M. Sauder, B.G. Orr, Stable and unstable growth in molecular beam epitaxy, *Phys. Rev. Lett.*, **72**:116–119, 1994.
8. J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, Berlin, 1971.
9. O. Pierre-Louis, C. Misbah, Y. Saito, J. Krug, P. Politi, New nonlinear evolution equation for steps during molecular beam epitaxy on vicinal surfaces, *Phys. Rev. Lett.*, **80**:4221–4224, 1998.
10. M. Rost, J. Krug, Coarsening of surface structures in unstable epitaxial growth, *Phys. Rev. E*, **55**:3952–3957, 1997.
11. C. Shen, L. Tian, A. Gao, Optimal control of the viscous Dullin–Gottwald–Holm equation, *Nonlinear Anal., Real World Appl.*, **11**:480–491, 2010.
12. J. Simon, Nonhomogeneous viscous incompressible fluids: Existence of velocity, density, and pressure, *SIAM J. Math. Anal.*, **21**(5):1093–1117, 1990.
13. J. Yong, S. Zheng, Feedback stabilization and optimal control for the Cahn–Hilliard equation, *Nonlinear Anal., Theory Methods Appl.*, **17**(5):431–444, 1991.
14. X. Zhao, C. Liu, Optimal control problem for viscous Cahn–Hilliard equation, *Nonlinear Anal., Theory Methods Appl.*, **74**(17):6348–6357, 2011.
15. X. Zhao, C. Liu, Global attractor for a nonlinear model with periodic boundary value condition, *Port. Math.*, **69**:221–231, 2012.
16. X. Zhao, C. Liu, Optimal control for the convective Cahn–Hilliard equation in 2D case, *Appl. Math. Optim.*, **70**:61–82, 2014.