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Multivalued generalizations of fixed point results in fuzzy metric spaces*

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Abstract. This paper attempts to prove fixed and coincidence point results in fuzzy metric space using multivalued mappings. Altering distance function and multivalued strong $\{b_n\}$-fuzzy contraction are used in order to do that. Presented theorems are generalization of some well known single valued results. Two examples are given to support the theoretical results.

Keywords: fixed point, coincidence point, fuzzy metric space, multivalued mappings, altering distance.

1 Introduction

Banach contraction principle [1] was motivation for many fixed point studies in various spaces [2,3,4,5,6,7,10,12,13,14,15,16,17,18,22,24,25,26,30]. In particular, multivalued generalization of this principle in metric space (X,d) is done by Nadler [27] on the following way: there exist $k \in (0,1)$ so that, for every $x,y \in X$,

$$H(fx,fy) \leqslant kd(x,y), \tag{1}$$

where H is Hausdorff–Pompeiu metric and f is multivalued mapping from X to the family of its non-empty, closed and bounded subsets. Later on, the probabilistic versions

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of condition (1) are given in [12, 13, 15, 16], where notions of weakly demicompact mapping, f-strongly demicompact and weakly commuting mapping are introduced. Further, Hausdorff distance between sets in fuzzy metric spaces is introduced [22] and used in [7] for study of existence of coincidence point using two multivalued and one single valued mappings.

Also, Banach contraction principle in metric spaces is improved by Khan, Swaleh and Sessa [19], where control function, called altering distance function, is introduced. This type of function is used in [30] in fuzzy matric space (X, M, T) with the following condition:

$$\varphi\bigl(M(fx, fy, t)\bigr) \leqslant k(t) \cdot \varphi\bigl(M(x, y, t)\bigr), \quad x, y \in X,\ t > 0,\ 0 < k(t) < 1, \quad (2)$$

where φ is altering distance function. Note that condition (2) is improved in [5]. Moreover, many functions of this type are used in the study of fixed point problems [3, 4, 26].

Another classes of contraction, so called (strong) $\{b_n\}$-probabilistic contraction, are introduced in [6, 24] and used in the study of fixed point problems in multivalued case in probabilistic spaces [25].

Our aim in present paper is to study the multivalued generalization in fuzzy metric spaces of results given in [6, 19, 30]. First, we use altering distance function in the style of condition (2) to obtain coincidence point results. That is realized through two theorems using strong fuzzy metric space with t-norm of H-type in the first and f-strongly demicompact mappings in the second one. On the other side, result given in [6] is transferred to multivalued case by introducing multivalued strong $\{b_n\}$-fuzzy contraction.

2 Preliminaries

In order to make paper more readable, first, we list the definitions of basic notions important to further work. Using the results of Menger and Zadeh [23, 31], Kramosil and Michalek [21] introduced the notion of fuzzy metric space. Later, George and Veermani [8, 9] modified their definition in way to associate each fuzzy metric to a Hausdorff topology.

Definition 1. (See [29].) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (t-norm) if the following conditions are satisfied:

(T1) $T(a, 1) = a,\ a \in [0, 1]$,
(T2) $T(a, b) = T(b, a),\ a, b \in [0, 1]$,
(T3) $a \geqslant b,\ c \geqslant d \Rightarrow T(a, c) \geqslant T(b, d),\ a, b, c, d \in [0, 1]$,
(T4) $T(a, T(b, c)) = T(T(a, b), c),\ a, b, c \in [0, 1]$.

Definition 2. (See [21].) The 3-tuple (X, M, T) is said to be a KM fuzzy metric space in the sense of Kramosil and Michalek if X is an arbitrary set, T is a t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(KM1) $M(x, y, 0) = 0,\ x, y \in X$,
(KM2) $M(x, y, t) = 1,\ t > 0 \Leftrightarrow x = y$,

(KM3) $M(x, y, t) = M(y, x, t),\ x, y \in X,\ t > 0,$
(KM4) $T(M(x, y, t), M(y, z, s)) \leqslant M(x, z, t + s),\ x, y, z \in X,\ t, s > 0,$
(KM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left-continuous for every $x, y \in X$.

Definition 3. (See [8, 9].) The 3-tuple (X, M, T) is said to be a fuzzy metric space in the sense of George and Veeramani if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(GV1) $M(x, y, t) > 0,\ x, y \in X,\ t > 0,$
(GV2) $M(x, y, t) = 1,\ t > 0 \Leftrightarrow x = y,$
(GV3) $M(x, y, t) = M(y, x, t),\ x, y \in X,\ t > 0,$
(GV4) $T(M(x, y, t), M(y, z, s)) \leqslant M(x, z, t + s),\ x, y, z \in X,\ t, s > 0,$
(GV5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous for every $x, y \in X$.

If (GV4) is replaced by condition

(GV4') $T(M(x, y, t), M(y, z, t)) \leqslant M(x, z, t),\ x, y, z \in X,\ t > 0,$

then (X, M, T) is called a strong fuzzy metric space [11].

Moreover, if (X, M, T) is a fuzzy metric space, then M is a continuous function on $X \times X \times (0, \infty)$ [28] and $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$ [10].

If (X, M, T) is a fuzzy metric space, then M generates the Hausdorff topology on X (see [8, 9]) with base of open sets $\{U(x, r, t)\colon\ x \in X,\ r \in (0, 1),\ t > 0\}$, where $U(x, r, t) = \{y\colon\ y \in X,\ M(x, y, t) > 1 - r\}$.

A function $\varphi : [0, 1] \to [0, 1]$ is called an altering distance function [26, 30] if it satisfies the following properties:

(AD1) φ is strictly decreasing and left continuous;
(AD2) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

It is obvious that $\lim_{\lambda \to 1^-} \varphi(\lambda) = \varphi(1) = 0$.

Definition 4. (See [8, 9]) Let (X, M, T) be a fuzzy metric space.

(a) A sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, M, T) if, for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, $n, m \geqslant n_0$, $t > 0$.
(b) A sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x in (X, M, T) if, for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$, $n \geqslant n_0$, $t > 0$. Then we say that $\{x_n\}_{n \in \mathbb{N}}$ is convergent. Every convergent sequence is a Cauchy sequence.
(c) A fuzzy metric space (X, M, T) is complete if every Cauchy sequence in (X, M, T) is convergent.

Definition 5. (See [14].) Let T be a t-norm and $T_n : [0, 1] \to [0, 1]$, $n \in \mathbb{N}$, be defined in the following way:

$$T_1(x) = T(x, x), \quad T_{n+1}(x) = T\big(T_n(x), x\big), \quad n \in \mathbb{N},\ x \in [0, 1].$$

We say that t-norm T is of H-type if the family $\{T_n(x)\}_{n \in \mathbb{N}}$ is equicontinuous at $x = 1$.

Each t-norm T can be extended (see [20]) (by associativity) in a unique way to an n-ary operation taking for $(x_1, \dots, x_n) \in [0,1]^n$ the values

$$\mathbf{T}_{i=1}^{0} x_i = 1, \qquad \mathbf{T}_{i=1}^{n} x_i = T\big(\mathbf{T}_{i=1}^{n-1} x_i, x_n\big).$$

A t-norm T can be extended to a countable infinite operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $[0,1]$ the value

$$\mathbf{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=1}^{n} x_i.$$

The sequence $(\mathbf{T}_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is non-increasing and bounded from below. Hence, the limit $\mathbf{T}_{i=1}^{\infty} x_i$ exists.

In the fixed point theory (see [15, 17]), it is of interest to investigate the classes of t-norms T and sequences (x_n) from the interval $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=1}^{\infty} x_{n+i} = 1. \tag{3}$$

In [15], the following proposition is obtained.

Proposition 1. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and t-norm T is of H-type. Then $\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=1}^{\infty} x_{n+i} = 1$.*

Definition 6. (See [12,15].) Let (X, M, T) be a fuzzy metric space, A a non-empty subset of X and $f: A \to 2^X \backslash \{\emptyset\}$. The mapping f is weakly demicompact if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ from A such that $x_{n+1} \in fx_n$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} M(x_{n+1}, x_n, t) = 1$, $t > 0$, there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$.

Throughout the paper by $C(X)$ is denoted a family of all non-empty and closed subsets of X.

Definition 7. (See [22].) Let (X, M, T) be a fuzzy metric space, A a non-empty subset of X, $f: A \to A$ and $F: A \to C(A)$. The mapping F is a f-strongly demicompact if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ from A such that $\lim_{n\to\infty} M(fx_n, y_n, t) = 1$, $t > 0$, for some sequence $\{y_n\}_{n\in\mathbb{N}}$, $y_n \in Fx_n$, $n \in \mathbb{N}$, there exists a convergent subsequence $\{fx_{n_k}\}_{k\in\mathbb{N}}$.

Definition 8. (See [13, 15].) A mapping $F: X \to C(X)$ is weakly commuting with $f: X \to X$ if, for all $x \in X$, it holds $f(Fx) \subseteq F(fx)$.

3 Main results

3.1 Multivalued mappings using altering distance

Main result of this section is an extension of results given in [30] to the case of multivalued mappings.

Let A and B be two nonempty subsets of X, define the Hausdorff–Pompeiu fuzzy metric as

$$\widetilde{M}(A,B,t) = \min\Bigl\{\inf_{x\in A} E(x,B,t),\ \inf_{y\in B} E(y,A,t)\Bigr\}, \quad t>0,$$

where $E(x,B,t) = \sup_{y\in B} M(x,y,t)$.

Theorem 1. *Let (X,M,T) be a complete strong fuzzy metric space and T is t-norm of H-type. Let $f : X \to X$ be a continuous mapping and $F,G : X \to C(X)$ are weakly commuting with f. If there exist $k : (0,\infty) \to (0,1)$ and altering distance function φ such that the following condition is satisfied*:

$$\varphi\bigl(\widetilde{M}(Fx,Gy,t)\bigr) \leqslant k(t)\cdot\varphi\bigl(M(fx,fy,t)\bigr), \quad x,y\in X,\ x\neq y,\ t>0, \tag{4}$$

then there exists $x\in X$ such that $fx \in Fx\cap Gx$.

Proof. Let $x_0 \in X$. Since Fx_0 is a non-empty subset of X, there exist $x_1 \in X$ such that $fx_1 \in Fx_0$. Let $t_0 > 0$ be arbitrary. Continuity of M and the fact that $k(t) < 1$, $t > 0$, implies that, for $\varepsilon_1 > 0$, the following inequality holds:

$$k(t_0)\cdot\varphi\bigl(M(fx_0,fx_1,t_0)\bigr) < \varphi\bigl(M(fx_0,fx_1,t_0)+\varepsilon_1\bigr). \tag{5}$$

By definition of Hausdorff fuzzy metric, for $\varepsilon_1 > 0$ given in (5), there exist $x_2 \in X$, $fx_2 \in Gx_1$ and $l_1 \in \mathbb{N}\setminus\{0\}$ such that

$$\widetilde{M}(Fx_0,Gx_1,t_0) \leqslant M(fx_1,fx_2,t_0) + \frac{\varepsilon_1}{2^{l_1}}. \tag{6}$$

Now, by (4), (5) and (6), using that φ is strictly decreasing, we conclude that

$$M(fx_0,fx_1,t_0) < M(fx_1,fx_2,t_0). \tag{7}$$

Similarly, we can find $x_3 \in X$, $fx_3 \in Fx_2$, and $l_2 \in \mathbb{N}$, $l_2 > l_1$ such that

$$k(t)\cdot\varphi\bigl(M(fx_1,fx_2,t_0)\bigr) < \varphi\bigl(M(fx_1,fx_2,t_0)+\varepsilon_1\bigr) \tag{8}$$

and

$$\widetilde{M}(Gx_1,Fx_2,t_0) \leqslant M(fx_2,fx_3,t_0) + \frac{\varepsilon_1}{2^{l_2}}. \tag{9}$$

By (8) and (9) we have

$$M(fx_1,fx_2,t_0) < M(fx_2,fx_3,t_0). \tag{10}$$

Repeating the procedure presented above, we define a sequence $\{x_n\}_{n\in\mathbb{N}}$ from X and strictly increasing sequence $\{l_n\}_{n\in\mathbb{N}}$ from $\mathbb{N}$ such that the following conditions are satisfied:

(i) $fx_{2n+1} \in Fx_{2n},\ fx_{2n+2} \in Gx_{2n+1},\ n\in\mathbb{N}$,
(ii) $M(fx_{n-1},fx_n,t) < M(fx_n,fx_{n+1},t),\ t>0,\ n\in\mathbb{N}$,

where

$$\widetilde{M}(Fx_{2n}, Gx_{2n+1}, t) \leqslant M(fx_{2n+1}, fx_{2n+2}, t) + \frac{\varepsilon_1}{2^{l_n}}, \quad t > 0,\ n \in \mathbb{N}. \tag{11}$$

Hence, the sequence $\{M(fx_n, fx_{n+1}, t)\}_{n\in\mathbb{N}}$, $t > 0$, is non-decreasing and bounded, so there exist $a : (0, \infty) \to [0, 1]$ such that

$$\lim_{n\to\infty} M(fx_n, fx_{n+1}, t) = a(t), \quad t > 0. \tag{12}$$

By (4), (11) and (12), for $n \in \mathbb{N}$, $t > 0$, we have

$$\begin{aligned}\varphi&\left(M(fx_{2n+1}, fx_{2n+2}, t) + \frac{\varepsilon_1}{2^{l_n}}\right)\\ &< \varphi\bigl(\widetilde{M}(Fx_{2n}, Gx_{2n+1}, t)\bigr) < k(t) \cdot \varphi\bigl(M(fx_{2n}, fx_{2n+1}, t)\bigr).\end{aligned} \tag{13}$$

Letting $n \to \infty$ in (13), we get

$$\varphi\bigl(a(t)\bigr) \leqslant k(t) \cdot \varphi\bigl(a(t)\bigr), \quad t > 0, \tag{14}$$

and we conclude that $\varphi(a(t)) = 0$ for all $t > 0$ so that $a \equiv 1$.

Further, we will prove that $\{fx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$ and $s \in \mathbb{N}$. Since t-norm T is of H-type, using (12) and Proposition 1, we have that there exist $n_0 \in \mathbb{N}$ such that

$$\mathbf{T}_{i=n}^{\infty} M(fx_i, fx_{i+1}, t) > 1 - \varepsilon, \quad t > 0,\ n \geqslant n_0. \tag{15}$$

Since (X, M, T) is strong fuzzy metric space and $\{\mathbf{T}_{i=1}^{n} M(fx_i, fx_{i+1}, t)\}_{n\in\mathbb{N}}$ is non-increasing sequence, by (15), we have that

$$M(fx_{n+s+1}, fx_n, t) \geqslant \mathbf{T}_{i=n}^{n+s} M(fx_i, fx_{i+1}, t) > 1 - \varepsilon, \quad t > 0,\ n \geqslant n_0. \tag{16}$$

So, $\{fx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and, since the space (X, M, T) is complete, there exist $x \in X$ such that

$$x = \lim_{n\to\infty} fx_n. \tag{17}$$

It remains to prove that $fx \in Fx \cap Gx$. As $Fx \cap Gx = \overline{Fx} \cap \overline{Gx}$, we need to show that, for every $t > 0$ and $\lambda \in (0, 1)$, there exists $r_1 = r_1(t, \lambda) \in Fx$ and $r_2 = r_2(t, \lambda) \in Gx$ such that $r_1, r_2 \in U(fx, t, \lambda)$, i.e. $M(fx, r_1, t) > 1 - \lambda$ and $M(fx, r_2, t) > 1 - \lambda$.

Let $t_0 > 0$ and $\lambda \in (0, 1)$. Since t-norm T is continuous, it follows that there exist $\delta = \delta(\lambda) \in (0, 1)$ such that

$$T\bigl(1 - \delta,\ T(1 - \delta, 1 - \delta)\bigr) > 1 - \lambda. \tag{18}$$

By the continuity of f and (17) there exist $n_1 \in \mathbb{N}$ such that

$$M\left(fx, ffx_{2n}, \frac{t_0}{3}\right) > 1 - \delta, \quad n \geqslant n_1. \tag{19}$$

By (12) there exists $n_2 \in \mathbb{N}$ such that

$$M\left(ffx_{2n}, ffx_{2n+1}, \frac{t_0}{3}\right) > 1 - \delta, \quad n \geqslant n_2.$$

Since f is weakly commuting with F, we have

$$ffx_{2n+1} \in f(Fx_{2n}) \subseteq F(fx_{2n}). \tag{20}$$

Also, there exist $\varepsilon^* \in (0,1)$ such that

$$k\left(\frac{t_0}{3}\right) \cdot \varphi\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right)\right) < \varphi\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right) + \varepsilon^*\right) \tag{21}$$

for arbitrary $n_0 \geqslant \max\{n_1, n_2\}$. By (20) and definition of Hausdorff fuzzy metric there exist $r_2 \in Gx$ such that, for $\varepsilon^* > 0$ (defined in (21)), the following is satisfied:

$$\widetilde{M}\left(Gx, F(fx_{2n_0}), \frac{t_0}{3}\right) \leqslant M\left(r_2, ffx_{2n_0+1}, \frac{t_0}{3}\right) + \varepsilon^*. \tag{22}$$

By (4), (20) and (21) we have:

$$\begin{aligned}
&\varphi\left(M\left(r_2, ffx_{2n_0+1}, \frac{t_0}{3}\right) + \varepsilon^*\right) \\
&\quad \leqslant \varphi\left(\widetilde{M}\left(Gx, F(fx_{2n_0}), \frac{t_0}{3}\right)\right) \leqslant k\frac{t_0}{3} \cdot \varphi\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right)\right) \\
&\quad < \varphi\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right) + \varepsilon^*\right).
\end{aligned}$$

Now, by (19) follows that

$$M\left(r_2, ffx_{2n_0}, \frac{t_0}{3}\right) > M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right) > 1 - \delta.$$

Finally, using (18), we get

$$\begin{aligned}
M(fx, r_2, t_0) \geqslant T\bigg(&M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right), \\
&T\left(M\left(ffx_{2n_0}, ffx_{2n_0+1}, \frac{t_0}{3}\right), M\left(ffx_{2n_0+1}, r_2, \frac{t_0}{3}\right)\right)\bigg) \\
\geqslant{} &T\big(1-\delta, T(1-\delta, 1-\delta)\big) > 1 - \lambda.
\end{aligned}$$

So, $r_2 \in U(fx, t_0, \lambda)$ for arbitrary $t_0 > 0$ and $\lambda \in (0,1)$, i.e. $fx \in Gx$. Similarly, it can be shown that $r_1 \in U(fx, t, \lambda)$, $t > 0$, $\lambda \in (0,1)$, which implies that $fx \in Fx$, too. □

Theorem 2. *Let (X, M, T) be a complete fuzzy metric space and $f : X \to X$ be a continuous mapping. Let $F, G : X \to C(X)$ are weakly commuting with f and F or G is f-strongly demicompact. If, for some $k : (0, \infty) \to (0, 1)$ and altering distance function φ, the following condition is satisfied*:

$$\varphi\bigl(\widetilde{M}(Fx, Gy, t)\bigr) \leqslant k(t) \cdot \varphi\bigl(M(fx, fy, t)\bigr), \quad x, y \in X,\ x \neq y,\ t > 0, \tag{23}$$

then there exists $x \in X$ such that $fx \in Fx \cap Gx$.

Proof. The proof is similar with that of the Theorem 1, except in the part related to Cauchy sequence. Namely, since F or G is f-strongly demicompact, $fx_{2n+1} \in Fx_{2n}$ or $fx_{2n+2} \in Gx_{2n+1}$ and $\lim_{n\to\infty} M(fx_{2n}, fx_{2n+1}, t) = 1$, $t > 0$, we conclude that there exist convergent subsequence $\{fx_{2n_p}\}_{p\in\mathbb{N}}$ or $\{fx_{2n_p+1}\}_{p\in\mathbb{N}}$, respectively, such that

$$\lim_{p\to\infty} fx_{2n_p} = x. \tag{24}$$

The last part of the proof is analogous as in Theorem 1, where instead of sequence $\{fx_n\}_{n\in\mathbb{N}}$, we deal with subsequences $\{fx_{2n_p}\}_{p\in\mathbb{N}}$ and $\{fx_{2n_p+1}\}_{p\in\mathbb{N}}$. □

If in Theorems 1 and 2, we take that $F = G$ and that f is the identity mapping, we get the following corollary.

Corollary 1. *Let (X, M, T) be a complete fuzzy metric space, $F : X \to C(X)$, and one of the following conditions is satisfied*:

(a) *F is weakly demicompact mapping,*

or

(b) *(X, M, T) is strong fuzzy metric space and T is t-norm of H-type.*

If there exist $k : (0, \infty) \to (0, 1)$ and altering distance function φ such that:

$$\varphi\bigl(\widetilde{M}(Fx, Fy, t)\bigr) \leqslant k(t) \cdot \varphi\bigl(M(x, y, t)\bigr), \quad x, y \in X,\ t > 0, \tag{25}$$

then there exists $x \in X$ such that $x \in Fx$.

Moreover, if the mapping F in Corollary 1 is single-valued we got the result in [30].

Example 1.

(a) Let $X = [0, 2], T = T_P, M(x, y, t) = t/(t + d(x, y))$, where d is Euclidian metric. Then (X, M, T) is a fuzzy metric space. Let $F(x) = \{1, 2\}$, $x \in X$. Since F is weakly demicompact and condition (25) is satisfied, by Corollary 1(a) follows that there exists $x \in X$ such that $x \in Fx$.

(b) Let $X = [0, 2], T = T_M, M^*(x, y, t) = t/(t + d^*(x, y))$, where d^* is ultrametric. Ultrametric space is metric space, where instead of triangle inequality condition, the following is satisfied: $d^*(x, z) \leqslant \max\{d^*(x, y), d^*(y, z)\}$. Then (X, M^*, T) is a strong fuzzy metric space [11]. For $F(x) = \{1, 2\}$, $x \in X$, condition (25) is satisfied and by Corollary 1(b) follows that there exists $x \in X$ such that $x \in Fx$.

3.2 Multivalued strong $\{b_n\}$-fuzzy contraction

In this part, we present multivalued extension of results given in [6] using multivalued strong $\{b_n\}$-fuzzy contraction.

Definition 9. Let (X, M, T) be a fuzzy metric space and $\{b_n\}_{n\in\mathbb{N}}$ a sequence from $(0,1)$ such that $\lim_{n\to\infty} b_n = 1$. The mapping $F : X \to C(X)$ is a multivalued strong $\{b_n\}$-fuzzy contraction if there exist $q \in (0,1)$ such that

$$M(x,y,t) > b_n \quad \Longrightarrow \quad \widetilde{M}(Fx,Fy,qt) > b_{n+1}, \quad x,y \in X,\ t > 0,\ n \in \mathbb{N}. \tag{26}$$

Theorem 3. *Let (X, M, T) be a complete KM fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$, $x, y \in X$, $\sup_{a<1} T(a,a) = 1$. Let $\{b_n\} \subset (0,1)$ be a sequence such that $\lim_{n\to\infty} b_n = 1$ and $F : X \to C(X)$ be a multivalued strong $\{b_n\}$-fuzzy contraction. If t-norm T satisfies the following condition*:

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} b_i = 1, \tag{27}$$

then there exists $x \in X$ such that $x \in Fx$.

Proof. Let $x_0, x_1 \in X$, where $x_1 \in Fx_0$. By (27), for arbitrary $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $t_0 > 0$ such that

$$\mathbf{T}_{i=n_0}^{\infty} b_i > 1 - \varepsilon \quad \text{and} \quad M(x_0, x_1, t_0) > b_{n_0}. \tag{28}$$

Then by condition (26), for some $q \in (0,1)$ and $\varepsilon_0 > 0$, we have

$$\widetilde{M}(Fx_0, Fx_1, qt_0) > b_{n_0+1} + \varepsilon_0. \tag{29}$$

Keeping the same ε_0 and using definition of Hausdorff metric, we can find $x_2 \in Fx_1$ such that

$$\widetilde{M}(Fx_0, Fx_1, qt_0) \leqslant M(x_1, x_2, qt_0) + \varepsilon_0. \tag{30}$$

By (26), (29) and (30) we obtain

$$M(x_1, x_2, qt_0) > b_{n_0+1} \quad \Longrightarrow \quad \widetilde{M}(Fx_1, Fx_2, q^2 t_0) > b_{n_0+2}.$$

Repeating the same procedure, we get

$$M(x_k, x_{k+1}, q^k t_0) > b_{n_0+k}, \quad k \in \mathbb{N}. \tag{31}$$

Let $\varepsilon > 0$ and $t > 0$. If we choose $k_0 \in \mathbb{N}$, $k_0 > n_0$, such that $\sum_{k=k_0}^{\infty} q^k < t/t_0$, then, for every $l, r \in \mathbb{N}$, $r > 1$, we have

$$\begin{aligned}
&M(x_{k_0+l}, x_{k_0+l+r}, t) \\
&\quad \geqslant M\left(x_{k_0+l}, x_{k_0+l+r}, t_0 \sum_{k=k_0}^{\infty} q^k\right) \geqslant M\left(x_{k_0+l}, x_{k_0+l+r}, t_0 \sum_{k=k_0+l}^{k_0+l+r-1} q^k\right) \\
&\quad \geqslant \underbrace{T\bigl(T \ldots T}_{(r-1)-\text{times}} \bigl(M\bigl(x_{k_0+l}, x_{k_0+l+1}, t_0 q^{k_0+l}\bigr), \ldots\bigr), \\
&\qquad\qquad M\bigl(x_{k_0+l+r-1}, x_{k_0+l+r}, t_0 q^{k_0+l+r-1}\bigr)\bigr) \\
&\quad \geqslant \mathbf{T}_{i=n_0}^{\infty} b_i > 1 - \varepsilon,
\end{aligned}$$

where is used (28) and (31). So, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and, since (X, M, T) is complete, there exist $x \in X$ so that

$$\lim_{n\to\infty} x_n = x. \tag{32}$$

It is remain to prove that $x \in Fx$. As $Fx = \overline{Fx}$, it is enough to show that, for every $\lambda \in (0,1)$ and $t > 0$, there exists $r = r(t,\lambda) \in Fx$ such that $M(x,r,t) > 1-\lambda$.

Let $t_0 > 0$ and $\lambda \in (0,1)$. Since $\sup_{a<1} T(a,a) = 1$, there exist $\delta = \delta(\lambda) \in (0,1)$ such that

$$T\bigl(T(1-\delta, 1-\delta),\, 1-\delta\bigr) > 1-\lambda. \tag{33}$$

From $\lim_{n\to\infty} b_n = 1$, for δ defined in (33), there exist $p_0 \in \mathbb{N}$ such that

$$b_p > 1-\delta, \quad p \geqslant p_0. \tag{34}$$

By (32), for p_0 given above, it is possible to find $n_0 \in \mathbb{N}$ such that

$$M\biggl(x_n, x, \frac{t_0}{3}\biggr) > b_{p_0} > 1-\delta, \quad n \geqslant n_0, \tag{35}$$

and

$$M\biggl(x_n, x_{n+1}, \frac{t_0}{3}\biggr) > b_{p_0} > 1-\delta, \quad n \geqslant n_0. \tag{36}$$

Now, by (26) there exist $\varepsilon^* > 0$ such that

$$\widetilde{M}\biggl(Fx_n, Fx, q\frac{t_0}{3}\biggr) > b_{p_0+1} + \varepsilon^*, \quad n \geqslant n_0.$$

For the same ε^* there exist $r \in Fx$ such that

$$M\biggl(x_{n+1}, r, q\frac{t_0}{3}\biggr) + \varepsilon^* \geqslant \widetilde{M}\biggl(Fx_n, Fx, q\frac{t_0}{3}\biggr) > b_{p_0+1} + \varepsilon^*,$$

i.e.

$$M\biggl(x_{n+1}, r, \frac{t_0}{3}\biggr) > M\biggl(x_{n+1}, r, q\frac{t_0}{3}\biggr) > b_{p_0+1} > 1-\delta, \quad n \geqslant n_0. \tag{37}$$

Finally, by (33), (35), (36) and (37) we get

$$\begin{aligned} M(x, r, t_0) &\geqslant T\biggl(T\biggl(M\biggl(x, x_n, \frac{t_0}{3}\biggr), M\biggl(x_n, x_{n+1}, \frac{t_0}{3}\biggr)\biggr), M\biggl(x_{n+1}, r, \frac{t_0}{3}\biggr)\biggr) \\ &> 1-\lambda, \end{aligned}$$

which means $x \in Fx$. □

4 Conclusion

In this paper we prove several fixed point and coincidence point results, which presented fuzzy generalization of Nadler fixed point result using altering distance function, as well as a multivalued generalizations of strong fuzzy $\{b_n\}$-contractions.

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