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On the modification of the universality of the Hurwitz zeta-function

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Abstract. In the paper, the lower limit in the universality inequality for the Hurwitz zeta-function is replaced by an ordinary limit. The cases of continuous and discrete universalities are considered.

Keywords: Hurwitz zeta-function, linear independence, transcendental number, universality.

1 Introduction

The Hurwitz zeta-function $\zeta(s,\alpha)$, $s=\sigma+\mathrm{i}t$, with the parameter α, $0<\alpha\leqslant 1$, is defined for $\sigma>1$ by the Dirichlet series

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s}$$

and can be analytically continued to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1. For $\alpha=1$, the function $\zeta(s,\alpha)$ reduces to the Riemann zeta function $\zeta(s)$.

It is well known that the function $\zeta(s,\alpha)$, for some classes of the parameter α, is universal in the Voronin sense, i.e., its shifts $\zeta(s+\mathrm{i}\tau,\alpha)$, $\tau\in\mathbb{R}$, approximate a wide class of analytic functions. For a precise statement of the universality theorem, we need some notations. Let $D=\{s\in\mathbb{C}\colon 1/2<\sigma<1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements and by $H(K)$, $K\in\mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K. Moreover, let $\operatorname{meas} A$ stand for the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then the universality property of the Hurwitz zeta-function is contained in the following theorem.

Theorem A. *Suppose that the parameter α is transcendental or rational $\neq 1, 1/2$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Bigl\{\tau\in[0;T]\colon\ \sup_{s\in K}\bigl|\zeta(s+\mathrm{i}\tau,\alpha)-f(s)\bigr|<\varepsilon\Bigr\}>0. \tag{1}$$

Theorem A was obtained in a slightly different form by Voronin [17], Gonek [4] and Bagchi [1], see also [9]. We note that the case of algebraic irrational parameter α is an open problem.

The inequality of the theorem shows that the set of shifts $\zeta(s+\mathrm{i}\tau,\alpha)$, $\tau \in \mathbb{R}$, approximating a given analytic function $f(s)$ is infinite and even has a positive lower density. The cases $\alpha = 1$ and $\alpha = 1/2$ are excluded because, as it was noted above, $\zeta(s,\alpha) = \zeta(s)$, and

$$\zeta\left(s, \frac{1}{2}\right) = \left(2^s - 1\right)\zeta(s).$$

In these cases, the function $\zeta(s,\alpha)$ remains universal, however, the approximated function $f(s)$ must be non-vanishing on K.

Let

$$L(\alpha) = \left\{\log(m+\alpha)\colon m \in \mathbb{N}_0\right\}.$$

Then a joint universality theorem of [6] with $r = 1$ implies the following result.

Theorem B. *Suppose that the set $L(\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then inequality* (1) *holds for any $\varepsilon > 0$.*

A theorem of Cassels asserts [3] that if α, $0 < \alpha \leqslant 1$, is algebraic irrational, then at least 51 percent of elements of the set $L(\alpha)$ in the sense of density are linearly independent over $\mathbb{Q}$. Therefore, it is possible that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ with algebraic irrational α, and the function $\zeta(s,\alpha)$ with this α is universal in the sense of Theorem A.

Theorems A and B are the so-called continuous universality theorems, $\tau \in \mathbb{R}$ in the shifts $\zeta(s+\mathrm{i}\tau,\alpha)$ is an arbitrary number. Also, the discrete universality of $\zeta(s,\alpha)$ has been considered. In this case, τ in $\zeta(s+\mathrm{i}\tau,\alpha)$ takes values from the set $\{mh\colon m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$, where $h > 0$ is a fixed number. Here analytic functions are approximated by shifts $\zeta(s+\mathrm{i}kh,\alpha)$, $k \in \mathbb{N}_0$. The following discrete universality theorems for $\zeta(s,\alpha)$ are known.

Theorem C. *Suppose that the parameter α is transcendental or rational $\neq 1, 1/2$, $K \in \mathcal{K}$, and $f(s) \in H(K)$. In the case of rational α, let the number $h > 0$ be arbitrary, while in the case of transcendental α, let $h > 0$ be such that $\exp\{2\pi/h\}$ is a rational number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N\colon \sup_{s\in K} \left|\zeta(s+\mathrm{i}kh,\alpha) - f(s)\right| < \varepsilon\right\} > 0. \tag{2}$$

For rational α, Theorem C was obtained in [1] and, by a different method, in [16]. For transcendental α, the theorem follows from a more general discrete universality theorem proved in [10] for the periodic Hurwitz zeta-function.

In [8], the following version of Theorem C was obtained.

Theorem D. *Suppose that the set*

$$L(\alpha, h, \pi) \stackrel{\mathrm{def}}{=} \left\{\left(\log(m+\alpha)\colon m \in \mathbb{N}_0\right), \frac{\pi}{h}\right\}$$

is linearly independent over $\mathbb{Q}$. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then inequality* (2) *holds for any* $\varepsilon > 0$.

The aim of this paper is to replace "$\liminf$" in Theorems A–D by "$\lim$". In the case of the Riemann zeta-function, this was done in [12] and [14]. As an example, we state the continuous version of a modified universality theorem for $\zeta(s)$ obtained in [12].

Theorem E. *Let* $K \in \mathcal{K}$ *and* $f(s)$ *be a continuous non-vanishing function on* K *which is analytic in the interior of* K. *Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0;T]\colon \sup_{s\in K} \big|\zeta(s+\mathrm{i}\tau) - f(s)\big| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Now we state the versions of Theorem E for the Hurwitz zeta-function.

Theorem 1. *Suppose that the parameter* α *is transcendental or rational* $\neq 1, 1/2$. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0;T]\colon \sup_{s\in K} \big|\zeta(s+\mathrm{i}\tau, \alpha) - f(s)\big| < \varepsilon\Big\} > 0 \tag{3}$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem 2. *Suppose that the set* $L(\alpha)$ *is linearly independent over* $\mathbb{Q}$. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then limit* (3) *exists for all but at most countably many* $\varepsilon > 0$.

Theorem 3. *Suppose that all hypotheses of Theorem* C *are satisfied. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N\colon \sup_{s\in K} \big|\zeta(s+\mathrm{i}kh, \alpha) - f(s)\big| < \varepsilon\Big\} > 0 \tag{4}$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem 4. *Suppose that the set* $L(\alpha, h, \pi)$ *is linearly independent over* $\mathbb{Q}$. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then limit* (4) *exists for all but at most countably many* $\varepsilon > 0$.

There exist two basic methods for proving universality theorems on approximation of analytic functions by shifts of zeta and L-functions. The first of them dates back to Voronin [17], Good [5] and Gonek [4], and is based on various approximation type results, including an approximation by a finite Euler product, the Kronecker approximation theorem, a rearrangement theorem in Hilbert spaces, the Hadamard three circles theorem, the Mongomery method for large values, etc. The second method, proposed by Bagchi [1], is of probabilistic type based on limit theorems for weakly convergent probability measures in functional spaces. It is well known that the weak convergence of probability measures has equivalents in terms of various type sets. For a long time, a more simple equivalent in terms of open sets was exploited, and this gave universality results with "$\liminf$". In [12] and [14], it was observed that an equivalent in terms of continuity sets also can be applied, and this led to the inequality of universality with "lim" in place of "lim inf". However, some problems of probabilistic nature do not allow to obtain the accuracy of approximation with every $\varepsilon > 0$, and the results obtained are valid only "for almost all $\varepsilon > 0$". In general, the latter approach can be applied for all zeta-functions defined by Dirichlet series and satisfying some natural growth hypotheses. The case of Hurwitz zeta-function is exceptional because of its dependence of the parameter α and non-existence of the Euler product. The role of α in both the cases "lim inf" and "lim" is the same, however, limit theorems are not known for all cases of α. The situation becomes more complicated in the case of the discrete universality, when a new parameter $h > 0$ appears. The type of h requires new limit theorems. The problem of h is essentially related to limit theorems on the tori Ω and Ω_1, see Section 2, and appears in the process of the investigation, as $N \to \infty$, of the Fourier transforms

$$\frac{1}{N+1}\sum_{k=0}^{N}\exp\left\{-\mathrm{i}kh\sum_{m=0}^{\infty}k_m\log(m+\alpha)\right\} \tag{5}$$

in the case of non-rational α, and

$$\frac{1}{N+1}\sum_{k=0}^{N}\exp\left\{-\mathrm{i}kh\sum_{m=0}^{\infty}k_p\log p\right\}$$

in the case of rational α, where only a finite number of integers k_m and k_p are distinct from zero.

We note that the hypothesis on h of Theorems C and 3, in the case of transcendental α, implies that of Theorems D and 4. Thus, Theorems D and 4 extend the case of transcendental α. Unfortunately, at the moment, we do not know any example of algebraic irrational α with linearly independent set $L(\alpha, h, \pi)$.

The linear independence of the set $L(\alpha)$ is applied for the proof of a limit theorem of continuous type for the Hurwitz zeta-function. More precisely, the role of $L(\alpha)$ is related to the study of the Fourier transform

$$\frac{1}{T}\int_{0}^{T}\exp\left\{-\mathrm{i}\tau\sum_{m=0}^{\infty}k_m\log(m+\alpha)\right\}$$

as $T \to \infty$. The details can be found in [6]. The linear independence of the set $L(\alpha, h, \pi)$ is used in the proof of a discrete limit theorem for $\zeta(s, \alpha)$, namely, for finding the asymptotics of (5). The details is given in [8].

2 Auxiliary results

For the proof of Theorems 1–4, we will apply a probabilistic approach based on limit theorems for weakly convergent probability measures in the space of analytic functions.

Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ be the unit circle in the complex plane. Define the set

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X. Then the compactness of Ω implies that, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(m)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}_0$. Denote by $H(G)$ the space of analytic functions on the region $G \subset \mathbb{C}$ equipped with the topology of uniform convergence on compacta, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \alpha, \omega)$ by the formula

$$\zeta(s, \alpha, \omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m+\alpha)^s}.$$

Let P_ζ be the distribution of the random element $\zeta(s, \alpha, \omega)$, i.e.,

$$P_\zeta(A) = m_H\bigl(\omega \in \Omega\colon \zeta(s, \alpha, \omega) \in A\bigr), \quad A \in \mathcal{B}\bigl(H(D)\bigr).$$

Lemma 1. *Suppose that the number α is transcendental. Then*

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0; T]\colon \zeta(s + \mathrm{i}\tau, \alpha) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H(D)\bigr),$$

converges weakly to P_ζ as $T \to \infty$. Moreover, the support of the measure P_ζ is the whole of $H(D)$.

The proof of the lemma can be found in [9], Theorem 5.2.3 and Lemma 6.1.7.

Now let $\alpha = a/b$, $(a, b) = 1$, and let $\alpha \neq 1, 1/2$. Then we have that $1 \leqslant a < b$ with $b \geqslant 3$ and, for $\sigma > 1$,

$$\zeta\left(s, \frac{a}{b}\right) = \sum_{m=0}^{\infty} \frac{1}{(m + \frac{a}{b})^s} = b^s \sum_{m=0}^{\infty} \frac{1}{(mb+a)^s} = f_1(s) f_2(s), \tag{6}$$

where $f_1(s) = b^s$ and

$$f_2(s) = \sum_{m=0}^{\infty} \frac{1}{(mb+a)^s}$$

or, in a more convenient form,

$$f_2(s) = \sum_{\substack{m=0 \\ m \equiv a \ (\mathrm{mod}\ b)}}^{\infty} \frac{1}{m^s}.$$

Denote by $\mathbb{P}$ the set of all prime numbers and define

$$\Omega_1 = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. Similarly as in the case of Ω, with the product topology and pointwise multiplication, the torus Ω_1 is a compact topological Abelian group. Therefore, on $(\Omega_1, \mathcal{B}(\Omega_1))$, the probability Haar measure $m_{1\,H}$ can be defined. This gives the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1\,H})$. Let $\omega_1(p)$ be the projection of $\omega_1 \in \Omega_1$ to the coordinate space γ_p, $p \in \mathbb{P}$. Extend the function $\omega_1(p)$ to the set $\mathbb{N}$ by the formula

$$\omega_1(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

On the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1\,H})$, define the $H(D)$-valued random elements $f_1(s, \omega_1) = \overline{\omega_1}(b) b^s$ and

$$f_2(s, \omega_1) = \sum_{\substack{m=0 \\ m \equiv a \ (\mathrm{mod}\, b)}}^{\infty} \frac{\omega_1(m)}{m^s}.$$

Moreover, let

$$\zeta\left(s, \frac{a}{b}, \omega_1\right) = f_1(s, \omega_1) f_2(s, \omega_1), \tag{7}$$

and P_ζ be the distribution of the random element $\zeta(s, a/b, \omega_1)$, i.e.,

$$P_\zeta(A) = m_{1\,H}\left(\omega_1 \in \Omega_1 \colon \zeta\left(s, \frac{a}{b}, \omega_1\right) \in A\right), \quad A \in \mathcal{B}\big(H(D)\big).$$

Lemma 2. *Suppose that the number α is rational $\neq 1, 1/2$. Then P_T converges weakly to P_ζ as $T \to \infty$. Moreover, the support of the measure P_ζ is the whole of $H(D)$.*

Proof. By a standard method, it can be proved similarly as in [9] that

$$\frac{1}{T} \operatorname{meas}\big\{\tau \in [0; T] \colon \big(f_1(s + \mathrm{i}\tau), f_2(s + \mathrm{i}\tau)\big) \in A\big\}, \quad A \in \mathcal{B}\big(H^2(D)\big), \tag{8}$$

converges weakly to the distribution of the $H^2(D)$-valued random element $(f_1(s, \omega_1), f_2(s, \omega_1))$ as $T \to \infty$. Let $u : H^2(D) \to H(D)$ be given by the formula $u(g_1, g_2) = g_1 g_2$. Then the function u is continuous. Therefore, using Theorem 5.1 of [2], the weak convergence of (8) and equalities (6) and (7), we obtain that P_T converges weakly to P_ζ as $T \to \infty$.

It remains to consider the support of P_ζ. Since $(a, b) = 1$, we have that the random variable $\omega_1(b)$ and each random variable $\omega(m)$, $m \equiv a (\mathrm{mod}\, b)$, are independent. From this, it follows that the random elements $f_1(s, \omega_1)$ and $f_2(s, \omega_1)$ are independent.

Define

$$a_m = \begin{cases} 1 & \text{if } m \equiv a \ (\mathrm{mod}\, b), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{a_m\colon\ m \in \mathbb{N}_0\}$ is a periodic sequence, and $(m, b) = 1$ when $m \equiv a \pmod b$ since $(a, b) = 1$. Therefore, for $\sigma > 1$,

$$f_2(s) = \sum_{\substack{m=1\\(m,b)=1}}^{\infty} \frac{a_m}{m^s}.$$

By a standard way, it follows that

$$\frac{1}{T}\,\mathrm{meas}\bigl\{\tau \in [0;T]\colon\ f_2(s + \mathrm{i}\tau, \alpha) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H(D)\bigr), \tag{9}$$

converges weakly to the distribution P_{f_2} of the random element $f_2(s, \omega_1)$ as $T \to \infty$. It remains to find the support of P_{f_2}. Having in mind that non-vanishing of a polynomial in a bounded region can be controlled by its constant term, we replace the strip D by a bounded rectangle. Let $V > 0$ be an arbitrary number, and $D_V = \{s \in \mathbb{C}\colon\ 1/2 < \sigma < 1,\ |t| < V\}$. Since the mapping $u : H(D) \to H(D_V)$ given by the formula $u(g(s)) = g(s)|_{s \in D_V}$ is continuous, we find from the weak convergence of the measure (9) and Theorem 5.1 of [2] that

$$P_{T,V}(A) = \frac{1}{T}\,\mathrm{meas}\bigl\{\tau \in [0;T]\colon\ f_2(s + \mathrm{i}\tau, \alpha) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H(D_V)\bigr),$$

also converges weakly to $P_{f_2,V}$ as $T \to \infty$, where

$$P_{f_2,V}(A) = m_{1H}\bigl(\omega_1 \in \Omega_1\colon\ f_2(s, \omega_1) \in A\bigr), \quad A \in \mathcal{B}\bigl(H(D_V)\bigr).$$

We will prove that the support of $P_{f_2,V}$ is the whole of $H(D_V)$. Denote by $\chi_1, \ldots, \chi_v$ all Dirichlet characters modulo b. Then there exist complex numbers $c_1, \ldots, c_v$ such that, for $1 \leqslant m \leqslant b$, $(m, b) = 1$,

$$a_m = \sum_{l=1}^{v} c_l \chi_l(m). \tag{10}$$

Since a_m and $\chi_v(m)$ are periodic, equality (10) remains true for all $m \in \mathbb{N}$, $(m, b) = 1$. Hence,

$$f_2(s) = \sum_{l=1}^{v} c_l L(s, \chi_l), \tag{11}$$

where $L(s, \chi_l)$ denotes the Dirichlet L-function. For $A \in \mathcal{B}(H^v(D_V))$, let

$$P_L(A) = m_{1H}\bigl(\omega_1 \in \Omega_1\colon\ \bigl(L(s, \omega_1, \chi_1), \ldots, L(s, \omega_1, \chi_v)\bigr) \in A\bigr),$$

where

$$L(s, \omega_1, \chi_l) = \prod_p \left(1 - \frac{\omega_1(p)\chi_l(p)}{p^s}\right)^{-1}, \quad l = 1, \ldots, v.$$

Define

$$S_V = \bigl\{g \in H(D_V)\colon\ g^{-1}(s) \in H(D_V) \text{ or } g(s) \equiv 0\bigr\}.$$

Then, by Lemma 13 of [7], the support of the measure P_L is the set S_V^v. Clearly, in (10), at least two numbers c_l, say c_1 and c_2, are non-zeros. The operator $F: H^v(D_V) \to H(D_V)$ given by the formula

$$F(g_1, \dots, g_v) = \sum_{l=1}^{v} c_l g_l, \quad g_1, \dots, g_v \in H(D_V),$$

is continuous. Moreover, for each polynomial $q = q(s)$, there exists $g_1, \dots, g_v \in S_V$ such that

$$F(g_1, \dots, g_v) = q.$$

For example, we may take that $g_1(s) = (q(s)+C)/C_1$, $g_2(s) = -(C+c_3+\dots+c_v)/c_2$, $g_3(s) = \dots = g_v(s) = 1$, where $|C|$ is rather large. Therefore, in view of (11) and Lemma 16 of [7], we have that the support of $P_{f_2,V}$ is the whole of $H(D_V)$. Here $V > 0$ is an arbitrary number. Letting $V \to \infty$, we obtain that $H(D_V)$ coincides with $H(D)$, and $P_{f_2,V}$ becomes P_{f_2}. Thus, the support of the random element $f_2(s, \omega_1)$ is the whole of $H(D)$. Since $f_1(s, \omega_1)$ is not degenerated at zero and $f_1(s, \omega_1)$ and $f_2(s, \omega_1)$ are independent random elements, this shows that the support of the product $f_1(s, \omega_1) f_2(s, \omega_1)$ is the whole of $H(D)$. Therefore, in view of (7), the support of P_ζ is the whole of $H(D)$. □

Lemma 3. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then P_T converges weakly to the measure P_ζ as $T \to \infty$.*

The lemma is a case of Theorem 4 from [6] with $r = 1$.

Lemma 4. *Suppose that the parameter α is transcendental, and $h > 0$ be such that* $\exp\{2\pi/h\}$ *is a rational number. Then*

$$P_N(A) \stackrel{\text{def}}{=} \frac{1}{N+1} \#\bigl\{0 \leqslant k \leqslant N\colon \zeta(s + \mathrm{i}kh, \alpha) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H(D)\bigr),$$

converges weakly to the measure P_ζ as $N \to \infty$.

Proof. Let $\mathfrak{a} = \{a_m\colon m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers. The periodic Hurwitz zeta-function $\zeta(s, \alpha, \mathfrak{a})$ is defined for $\sigma > 1$ by the series

$$\zeta(s, \alpha, \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}$$

and can be analytically continued to the whole complex plane, except for a possible pole at the point $s = 1$. Define the $H(D)$-valued random element $\zeta(s, \alpha, \omega, \mathfrak{a})$ by the formula

$$\zeta(s, \alpha, \omega, \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}$$

and denote by $P_{\zeta,\mathfrak{a}}$ its distribution. Then in [10], Theorem 6.1, it was proved that, under hypotheses of the lemma,

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N\colon \zeta(s+\mathrm{i}kh,\alpha,\mathfrak{a}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\zeta,\mathfrak{a}}$ as $N \to \infty$. Obviously, if $a_m \equiv 1$, then the function $\zeta(s,\alpha,\mathfrak{a})$ becomes the Hurwitz zeta-function $\zeta(s,\alpha)$, and $\zeta(s,\alpha,\omega,\mathfrak{a})$ becomes $\zeta(s,\alpha,\omega)$. Therefore, the lemma is a partial case of Theorem 6.1 from [10]. □

Lemma 5. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Then P_N converges weakly to the measure P_ζ as $N \to \infty$.*

The proof of the lemma is given in [8], Theorem 2.1.

The number $h > 0$ is called of type 1 if $\exp\{2\pi m/h\}$ is an irrational number for all $m \in \mathbb{Z} \setminus \{0\}$, and of type 2 if there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $\exp\{2\pi m/h\}$ is a rational number. Let Ω_{1h} be the closed subgroup of Ω_1 generated by $(p^{-ih}\colon p \in \mathbb{P})$. It is known [1] that if h is of type 1, then $\Omega_{1h} = \Omega_1$. Now suppose that h is of type 2. Then there exists the least $m_0 \in \mathbb{N}$ such that $\exp\{2\pi m_0/h\}$ is rational. Let

$$\exp\left\{\frac{2\pi m_0}{h}\right\} = \frac{u}{v}, \quad u,v \in \mathbb{N},\ (u,v)=1.$$

It is known [1] that, in this case,

$$\Omega_{1h} = \{\omega_1 \in \Omega_1\colon \omega_1(u) = \omega_1(v)\}.$$

Denote by m_{1H}^h the probability Haar measure on $(\Omega_{1h}, \mathcal{B}(\Omega_{1h}))$, and, on the probability space $(\Omega_{1h}, \mathcal{B}(\Omega_{1h}), m_{1H}^h)$, define the $H(D)$-valued random element

$$\zeta_{1h}\left(s, \frac{a}{b}, \omega_1\right) = f_1(s,\omega_1) f_2(s,\omega_1), \quad \omega_1 \in \Omega_{1h}.$$

Let $P_{\zeta_{1h}}$ stand for the distribution of $\zeta_{1h}(s, a/b, \omega_1)$.

Lemma 6. *Suppose that α is rational $\neq 1, 1/2$, and $h > 0$ is an arbitrary number. Then P_N converges weakly to $P_{\zeta_{1h}}$ as $N \to \infty$. Moreover, the support of $P_{\zeta_{1h}}$ is the whole of $H(D)$.*

Proof. If h is of type 1, the proof, in view of the equality $\Omega_{1h} = \Omega_1$, coincides with that of Lemma 2. Therefore, it remains to consider the case of h of type 2. For this, we will apply the following assertion from [11, Lemma 2]:

$$Q_{N,h}(A) \stackrel{\text{def}}{=} \frac{1}{N+1}\#\{0 \leqslant k \leqslant N\colon (p^{-\mathrm{i}kh}\colon p \in \mathbb{P}) \in A\}, \quad A \in \mathcal{B}(\Omega_{1h}),$$

converges weakly to the Haar measure m_{1H}^h as $N \to \infty$. To prove this, as usual, the method of Fourier transforms is applied. The main difficulty is to describe the characters of the group Ω_{1h}. Define

$$\mathbb{P}_0 = \left\{ p \in \mathbb{P}\colon\ \alpha_p \neq 0 \text{ in } \frac{u}{v} = \prod_{p\in\mathbb{P}} p^{\alpha_p} \right\}.$$

Let Ω_1^* be the dual group of Ω_1, $\chi_{m_0} \in \Omega_1^*$ be given by the formula

$$\chi_{m_0}(\omega_1) = \prod_{p\in\mathbb{P}_0} \omega_1^{\alpha_p}(p) = \frac{\omega_1(u)}{\omega_1(v)},$$

and $\Omega_{1h}^{\perp} = \{\chi \in \Omega_1^*\colon\ \chi(\omega_1) = 1,\ \omega_1 \in \Omega_{1h}\}$. If h is of type 2, then is not difficult to see that

$$\Omega_{1h}^{\perp} = \bigl\{\chi_{m_0}^{l}\colon\ l \in \mathbb{Z}\bigr\}.$$

In view of [15, Thm. 27], we have that the factor group $\Omega_1^*/\Omega_{1h}^{\perp}$ is the dual group of Ω_{1h}. Hence, the characters of the group Ω_{1h} are of the form

$$\chi(\omega_1) = \prod_{p\in\mathbb{P}\setminus\mathbb{P}_0} \omega_1^{k_p}(p) \prod_{p\in\mathbb{P}_0} \omega_1^{k_p+l\alpha_p}(p), \quad l \in \mathbb{Z},$$

where only finite number of integers k_p are distinct from zero. Therefore, the Fourier transform $\varphi_{N,h}(\underline{k})$, $\underline{k} = (k_p\colon\ p \in \mathbb{P})$, of $Q_{N,h}$ is of the form

$$\begin{aligned}\varphi_{N,h}(\underline{k}) &= \int_{\Omega_{1h}} \chi(\omega_1)\,\mathrm{d}Q_{N,h} \\ &= \frac{1}{N+1}\sum_{k=0}^{N} \prod_{p\in\mathbb{P}\setminus\mathbb{P}_0} p^{-ikk_ph} \prod_{p\in\mathbb{P}_0} p^{-\mathrm{i}kh(k_p+l\alpha_p)}, \quad l \in \mathbb{Z}.\end{aligned}$$

Now properties of α_p and h imply that $\varphi_{N,h}(\underline{k})$, as $N \to \infty$, converges to the Fourier transform of the measure m_{1H}, and this prove the claim.

Further, by a standard method, it follows that

$$\frac{1}{N+1}\#\bigl\{0 \leqslant k \leqslant N\colon\ \bigl(f_1(s+\mathrm{i}kh),\ f_2(s+\mathrm{i}kh)\bigr) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H^2(D)\bigr),$$

and, for $\omega_1 \in \Omega_{1h}$,

$$\frac{1}{N+1}\#\bigl\{0 \leqslant k \leqslant N\colon\ \bigl(f_1(s+\mathrm{i}kh,\omega_1),\ f_2(s+\mathrm{i}kh,\omega_1)\bigr) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H^2(D)\bigr),$$

converges weakly to the same probability measure P on $(H^2(D), \mathcal{B}H^2(D))$ as $N \to \infty$. For the identification of the limit measure P, we apply Lemma 11 of [11] on the ergodicity of the transformation φ_h of Ω_{1h} defined by

$$\varphi_h(\omega) = \bigl(p^{-\mathrm{i}kh}\colon\ p \in \mathbb{P}\bigr)\omega_1, \quad \omega_1 \in \Omega_{1h}.$$

This application, together with the classical Birkhoff–Khintchine ergodic theorem, shows that P is the distribution of the random element $(f_1(s,\omega_1), f_2(s,\omega_1))$, $\omega_1 \in \Omega_{1h}$. Hence, it follows that P_N converges weakly to $P_{\zeta_{1h}}$ as $N \to \infty$.

For the proof that the support of $P_{\zeta_{1h}}$ is the whole of $H(D)$, we use arguments of discrete type analogous to those applied in the proof of Lemma 2. □

3 Proofs of universality theorems

Proofs of Theorems 1–4 are based on limit theorems for P_T and P_N and on the equivalent of the weak convergence of probability measures in terms of continuity sets. We state this equivalent as the following lemma. We recall that $A \in \mathcal{B}(X)$ is a continuity set of the probability measure P on $(X, \mathcal{B}(X))$ if $P(\partial A) = 0$, where ∂A is the boundary of A.

Lemma 7. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. Then P_n, as $n \to \infty$, converges weakly to P if and only if, for every continuity set A of P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 of [2].

We also use the famous Mergelyan theorem [13] on the approximation of analytic functions by polynomials which is contained in the next lemma.

Lemma 8. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $g(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $q(s)$ such that*

$$\sup_{s\in K} \bigl|g(s) - q(s)\bigr| < \varepsilon.$$

Proof of Theorem 1. Let

$$G_\varepsilon = \Bigl\{g \in H(D)\colon\ \sup_{s\in K} \bigl|g(s) - f(s)\bigr| < \varepsilon\Bigr\}.$$

Then G_ε is an open set in $H(D)$, moreover,

$$\partial G_\varepsilon = \Bigl\{g \in H(D)\colon\ \sup_{s\in K} \bigl|g(s) - f(s)\bigr| = \varepsilon\Bigr\}.$$

Therefore, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \emptyset$ for $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Hence, $P_\zeta(\partial G_\varepsilon) > 0$ for at most a countably set of $\varepsilon > 0$, i.e., the set G_ε is a continuity set of P_ζ for all but at most countably many $\varepsilon > 0$. Thus, in view of Lemmas 1, 2 and 7,

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0;T]\colon\ \zeta(s+\mathrm{i}\tau,\alpha) \in G_\varepsilon\bigr\} \\
&\quad= \lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\Bigl\{\tau \in [0;T]\colon\ \sup_{s\in K} \bigl|\zeta(s+\mathrm{i}\tau,\alpha) - f(s)\bigr| > \varepsilon\Bigr\} \\
&\quad= P_\zeta(G_\varepsilon) \qquad (12)
\end{aligned}$$

for all but at most countably many $\varepsilon > 0$. By Lemma 8, there exists a polynomial $q(s)$ such that

$$\sup_{s\in K}\bigl|f(s)-q(s)\bigr| < \frac{\varepsilon}{2}. \tag{13}$$

Since, by Lemmas 1 and 2, $q(s)$ is an element of the support of the measure P_ζ, we have that $P_\zeta(\widehat{G}_\varepsilon) > 0$, where

$$\widehat{G}_\varepsilon = \Bigl\{g\in H(D)\colon \sup_{s\in K}\bigl|g(s)-q(s)\bigr| < \frac{\varepsilon}{2}\Bigr\}.$$

Clearly, for $g \in \widehat{G}_\varepsilon$, by (13),

$$\sup_{s\in K}\bigl|g(s)-q(s)\bigr| < \varepsilon.$$

Therefore, $\widehat{G}_\varepsilon \subset G_\varepsilon$. Hence, $P_\zeta(G_\varepsilon) \geqslant P_\zeta(\widehat{G}_\varepsilon) > 0$, and the theorem follows from equality (12). □

Proof of Theorem 2. We use Lemma 3 and follow the proof of Theorem 1. □

Proof of Theorem 3. In virtue of similarity, we consider only the case of transcendental parameter α. Also, we preserve the notation of the proof of Theorem 1. By Lemmas 4 and 7, we have that

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N+1}\#\bigl\{0\leqslant k\leqslant N\colon \zeta(s+\mathrm{i}kh,\alpha)\in G_\varepsilon\bigr\}\\
&\quad=\lim_{N\to\infty}\frac{1}{N+1}\#\Bigl\{0\leqslant k\leqslant N\colon \sup_{s\in K}\bigl|\zeta(s+\mathrm{i}kh,\alpha)-f(s)\bigr|<\varepsilon\Bigr\}\\
&\quad=P_\zeta(G_\varepsilon)
\end{aligned} \tag{14}$$

for all but at most countably many $\varepsilon > 0$. Lemma 4 implies the inequality $P_\zeta(G_\varepsilon) > 0$. Since $P_\zeta(G_\varepsilon) \geqslant P_\zeta(\widehat{G}_\varepsilon)$, (14) proves the theorem. □

Proof of Theorem 4. The theorem follows from Lemma 5 in the same way as Theorem 3. □

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