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On a differential system arising in the network control theory

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Abstract. We investigate the three-dimensional dynamical system occurring in the network regulatory systems theory for specific choices of regulatory matrix $\{\{0,1,1\}\{1,0,1\}\{1,1,0\}\}$ and sigmoidal regulatory function $f(z) = 1/(1+\mathrm{e}^{-\mu z})$, where $z = \sum W_{ij}x_j - \theta$. The description of attracting sets is provided. The attracting sets consist of respectively one, two or three critical points. This depends on whether the parameters (μ, θ) belong to a set Ω or to the complement of Ω or to the boundary of Ω, where Ω is fully defined set.

Keywords: network control, attracting sets, dynamical system, phase portrait.

1 Introduction

In the articles [5, 6], the following problem is studied. To accommodate traffic on a wavelength-routed optical network, one can construct an optimal virtual network topology (VNT) by establishing a set of lightpaths between nodes. To treat changing in time (fluctuating) traffic on a VNT, adaptive VNT control methods should be invented, which reconfigure VNTs according to traffic conditions on VNTs. To develope such methods, one way is to observe *attractor selection* in biological systems that *adapt to unknown changes in their surrounding environments and recover their conditions*.

Mechanism and concept of Attractor Selection are described in [6]:

"1) Concept of Attractor Selection: The dynamic system that is driven by attractor selection uses noise to adapt to environmental changes. In attractor selection, attractors are a part of the equilibrium points in the solution space in which the system conditions are preferable. The basic mechanism consists of two behaviors, i.e., deterministic and stochastic behaviors. When the current system conditions are suitable for the environment, i.e., the system state is close to one of the attractors, deterministic behavior drives the system to the attractor. Where the current system conditions are poor, stochastic behavior dominates over deterministic behavior. While stochastic behavior is dominant in controlling the system, the system state fluctuates randomly due to noise and the system searches for a new attractor. When the system conditions have recovered, deterministic behavior again controls the system. These two behaviors are controlled by simple feedback of the conditions of the system. In this way, attractor selection adapts to environmental changes by selecting attractors using stochastic behavior, deterministic behavior, and simple feedback."

Therefore, the attractor selection represents mechanism of adaptation to unknown and rapid changes in biological systems. The overview of results in this direction can be found in $[1, 3, 4, 7]$. So the problem of describing the structure of attracting sets is actual and useful for understanding the above mentioned biological (and related telecommunication) networks.

We follow the above model in the following setting. We neglect the stochastic behavior in our analysis. Besides we consider the simplified model consisting of three differential equations. This system can be treated by the 3D phase plane analysis [2]. Our goal is to study the structure of attracting sets in this simple model. The system involves two parameters, therefore, changes in the structure of a phase space should be described under the change of parameters. We provide full description of attracting sets depending on the choice of parameters.

2 System

The dynamics of the expression level of the protein on the ith gene, x_i, is described by the differential system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f\Bigl(\sum W_{ij}x_j - \theta\Bigr)v_g - x_i v_g - \eta.$$

The first and second terms at the right-hand side represent the deterministic behavior of gene i; and the third term η represents stochastic behavior. The deterministic behavior controls the x_i due to the effects of activation and inhibition from the other genes. Those regulations of protein expression levels on gene i by other genes are indicated by regulatory matrix W_{ij}, the elements of which take values 1, 0, or -1, corresponding respectively to activation, no regulatory interaction, and inhibition of the ith gene by the jth gene.

The rate of increase in the expression level is given by the sigmoidal regulation function

$$f(z) = \frac{1}{1+\mathrm{e}^{-\mu z}}, \quad \text{where } z = \sum W_{ij}x_j - \theta.$$

Parameter θ is a regulatory parameter, which can be adjusted, and μ indicates the gain parameter of the sigmoidal function.

2.1 Three-dimensional system

We consider the simplified system ($v_g = 1$, $\eta = 0$)

$$\begin{aligned} x_1' &= \frac{1}{1 + \mathrm{e}^{-\mu(W_{11}x_1 + W_{12}x_2 + W_{13}x_3 - \theta)}} - x_1, \\ x_2' &= \frac{1}{1 + \mathrm{e}^{-\mu(W_{21}x_1 + W_{22}x_2 + W_{23}x_3 - \theta)}} - x_2, \\ x_3' &= \frac{1}{1 + \mathrm{e}^{-\mu(W_{31}x_1 + W_{32}x_2 + W_{33}x_3 - \theta)}} - x_3. \end{aligned} \tag{1}$$

The entries of W_{ij} can take values 1, 0 or -1. We consider the specific case

$$W = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}.$$

System (1) looks as

$$\begin{aligned} x_1' &= \frac{1}{1 + \mathrm{e}^{-\mu(x_2 + x_3 - \theta)}} - x_1, \\ x_2' &= \frac{1}{1 + \mathrm{e}^{-\mu(x_1 + x_3 - \theta)}} - x_2, \\ x_3' &= \frac{1}{1 + \mathrm{e}^{-\mu(x_1 + x_2 - \theta)}} - x_3, \end{aligned} \tag{2}$$

where μ and θ are positive parameters. Our goal is to study phase portraits of this system and describe the attracting sets.

2.2 Critical points

Critical points of system (2) are to be determined from

$$\begin{aligned} x_1 &= \frac{1}{1 + \mathrm{e}^{-\mu(x_2 + x_3 - \theta)}}, \\ x_2 &= \frac{1}{1 + \mathrm{e}^{-\mu(x_1 + x_3 - \theta)}}, \\ x_3 &= \frac{1}{1 + \mathrm{e}^{-\mu(x_1 + x_2 - \theta)}}. \end{aligned} \tag{3}$$

Since the right sides in (3) are positive but less than a unity, all critical points locate in the cube $(0;1) \times (0;1) \times (0;1)$.

Lemma 1. *All critical points are on the bisectrix* $x_1 = x_2 = x_3$.

Proof. System (3) can be written as

$$\begin{aligned} 1 &= x_1 + x_1 \mathrm{e}^{-\mu(x_2 + x_3 - \theta)}, \\ 1 &= x_2 + x_2 \mathrm{e}^{-\mu(x_1 + x_3 - \theta)}, \\ 1 &= x_3 + x_3 \mathrm{e}^{-\mu(x_1 + x_2 - \theta)}. \end{aligned} \tag{4}$$

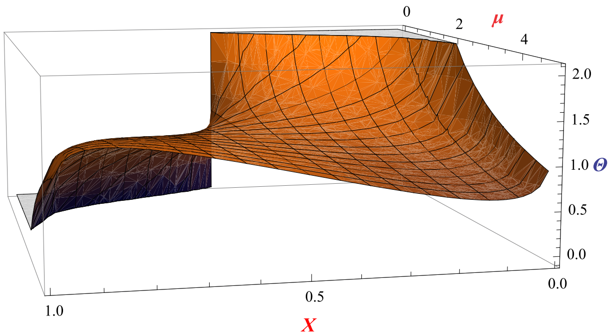

Figure 1. The relation between x, μ and θ.

It follows from (4) that

$$\begin{aligned}\frac{1}{x_1} &= 1 + \mathrm{e}^{-\mu(x_2+x_3-\theta)},\\ \frac{1}{x_2} &= 1 + \mathrm{e}^{-\mu(x_1+x_3-\theta)},\\ \frac{1}{x_3} &= 1 + \mathrm{e}^{-\mu(x_1+x_2-\theta)}.\end{aligned} \tag{5}$$

We will prove that $x_1 = x_2$, in the same way, it follows that $x_1 = x_3$ and then $x_1 = x_2 = x_3$. Divide the first row in (5) by the second one and get

$$\frac{x_2}{x_1} = \frac{1 + \mathrm{e}^{-\mu(x_2+x_3-\theta)}}{1 + \mathrm{e}^{-\mu(x_1+x_3-\theta)}}. \tag{6}$$

Suppose that x_1 and x_2 in equation (6) are distinct. Consider the case $x_2 > x_1$. Then the contradiction

$$1 < \frac{x_2}{x_1} = \frac{1 + \mathrm{e}^{-\mu(x_2+x_3-\theta)}}{1 + \mathrm{e}^{-\mu(x_1+x_3-\theta)}} < 1$$

follows. Similarly, the case $x_2 < x_1$ can be considered. It follows then that $x_1 = x_2$. Similarly, $x_1 = x_3$, and it follows that any critical point is of the form $(x; x; x)$, where

$$x = \frac{1}{1 + \mathrm{e}^{-\mu(2x-\theta)}}. \tag{7}$$

The proof is complete. □

Formula (7) is a key relation between x, θ and μ. This is visualized in Fig. 1.

We claim that there are only three possibilities for the number of critical points of system (2).

Lemma 2. *There are at most three critical points of system* (2).

Proof. Rewrite (7) as

$$\theta = \frac{1}{\mu} \ln\left(\frac{1}{x} - 1\right) + 2x. \tag{8}$$

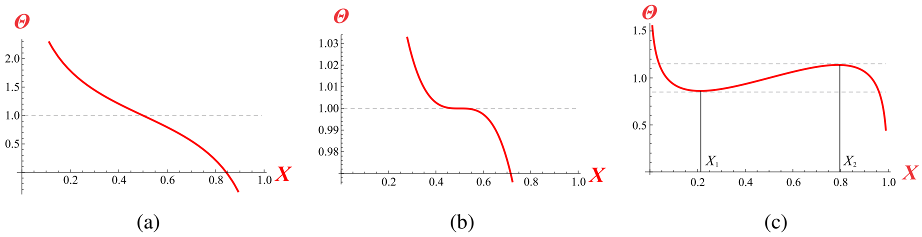

Figure 2. Relation (8) for: (a) $\mu = 1$; (b) $\mu = 2$; (c) $\mu = 3$.

It follows that (for any positive μ) $\theta(x) \to +\infty$ as $x \to 0+$ and $\theta(x) \to -\infty$ as $x \to 1-$. Is $\theta(x)$ monotone? Consider the derivative

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{\mu}\frac{1}{\frac{1}{x} - 1}\left(-\frac{1}{x^2}\right) + 2 = \frac{2\mu(x - x^2) - 1}{\mu(x - x^2)} = 0. \tag{9}$$

The sign of $\theta'(x)$ is determined by the sign of polynomial (10) (recall that the denominator in (9) is positive for $x \in (0, 1)$).

The roots of the equation

$$-x^2 + x - \frac{1}{2\mu} = 0 \tag{10}$$

are

$$x_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{2\mu}}. \tag{11}$$

The function $\theta(x)$ is strictly decreasing for $0 < \mu \leqslant 2$ (since then the discriminant $1/4 - 1/(2\mu)$ is negative) and for $\mu > 2$ if $0 < x < x_1$ or $x > x_2$.

If $\mu \geqslant 2$, equation (10) has real roots (11). The function $\theta(x)$ is monotonically increasing for $\mu > 2$ if $x_1 < x < x_2$.

The behavior of $\theta(x)$ is visualized in Fig. 2. For $\theta = 1$, there is a critical point $(0.5, 0.5, 0.5)$ for any positive μ. For any $\mu \in (0; 2]$, there is exactly one critical point (x, x, x) for every θ, see Fig. 2a. For $\mu \in (2; +\infty)$, if $\theta = \theta(x_1)$ or $\theta = \theta(x_2)$ (horizontal dashed lines in Fig. 2b), then there are two critical points. If $\theta \in (\theta(x_1), \theta(x_2))$ (θ is between dashed lines), then there are exactly three critical points, see Fig. 2c. The case $(\mu, \theta) = (2, 1)$ is the special one. There is exactly one critical point, see Fig. 2b. □

Put values $x_1(\mu)$ and $x_2(\mu)$ into (8) and get

$$\theta_{1,2} = \frac{1}{\mu}\ln\left(\frac{1}{\frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{1}{2\mu}}} - 1\right) + 2\left(\frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{1}{2\mu}}\right).$$

The graphs of these functions θ_1, θ_2 are depicted in Fig. 3. The upper branch is for θ_2, and the lower one is for θ_1.

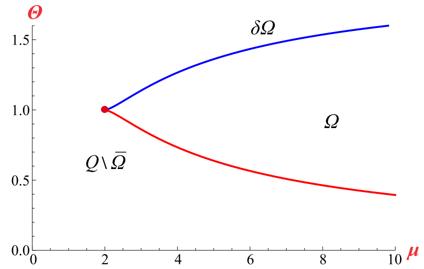

Figure 3. The dependence of θ (for a critical point $(x; x; x)$) of μ.

Now we introduce the following notation. Let Ω denote the open region between θ_1 and θ_2, Q is the positive (open) quadrant of (μ, θ)-plane, $\partial\Omega$ usually denotes the boundary of Ω.

It follows from the proof of Lemma 2 and it is illustrated by Fig. 2 that there is exactly one critical point if $(\mu, \theta) \in Q \setminus \overline{\Omega}$.

There are exactly three critical points if $(\mu, \theta) \in \Omega$.

Besides, there are two critical points (look at Fig. 2c) if $(\mu, \theta) \in \partial\Omega \setminus \{2, 1\}$.

At last, the point $(2, 1)$ corresponds to the critical point $(0.5, 0.5, 0.5)$.

3 Linearized system

In this section, we would like to study the character of possible critical points. For this, consider the linearized system

$$\begin{aligned}
u_1' &= -u_1 + \frac{\mu e^{-\mu(x_2+x_3-\theta)}}{[1+e^{-\mu(x_2+x_3-\theta)}]^2}(u_2+u_3),\\
u_2' &= -u_2 + \frac{\mu e^{-\mu(x_1+x_3-\theta)}}{[1+e^{-\mu(x_1+x_3-\theta)}]^2}(u_1+u_3),\\
u_3' &= -u_3 + \frac{\mu e^{-\mu(x_1+x_2-\theta)}}{[1+e^{-\mu(x_1+x_2-\theta)}]^2}(u_1+u_2).
\end{aligned}$$

By using (5) and $x_1 = x_2 = x_3 = x$ the linearized system can by written as

$$\begin{aligned}
u_1' &= -u_1 + \mu x(1-x)u_2 + \mu x(1-x)u_3,\\
u_2' &= \mu x(1-x)u_1 - u_2 + \mu x(1-x)u_3,\\
u_3' &= \mu x(1-x)u_1 + \mu x(1-x)u_2 - u_3.
\end{aligned} \tag{12}$$

The matrix A of system (12) is

$$A = \begin{vmatrix} -1 & \mu x(1-x) & \mu x(1-x)\\ \mu x(1-x) & -1 & \mu x(1-x)\\ \mu x(1-x) & \mu x(1-x) & -1 \end{vmatrix},$$

$$A - \lambda I = \begin{vmatrix} -1-\lambda & \mu x(1-x) & \mu x(1-x) \\ \mu x(1-x) & -1-\lambda & \mu x(1-x) \\ \mu x(1-x) & \mu x(1-x) & -1-\lambda \end{vmatrix},$$

$$\det|A - \lambda I| = -(1+\lambda)^3 + 2\big(\mu x(1-x)\big)^3 + 3(1+\lambda)\big(\mu x(1-x)\big)^2 = 0. \tag{13}$$

In the new variables $L = (1+\lambda)$ and $M = \mu x(1-x)$, equation (13) takes a simpler form

$$\det|A - \lambda I| = -L^3 + 2M^3 + 3LM^2 = 0. \tag{14}$$

The roots of (14) are $L_1 = -M$, $L_2 = -M$, $L_3 = 2M$. Then λ_i are

$$\begin{aligned} \lambda_1 &= -\mu x(1-x) - 1, \\ \lambda_2 &= -\mu x(1-x) - 1, \\ \lambda_3 &= 2\mu x(1-x) - 1. \end{aligned}$$

It is evident that λ_1 and λ_2 are negative, but λ_3 can be positive, negative and even zero. It follows from the key relation (7) that λ_i can be written also as

$$\begin{aligned} \lambda_1 &= -\mu\left(1 - \frac{1}{1+\mathrm{e}^{-\mu(2x-\theta)}}\right) - 1, \\ \lambda_2 &= -\mu\left(1 - \frac{1}{1+\mathrm{e}^{-\mu(2x-\theta)}}\right) - 1, \\ \lambda_3 &= 2\mu\left(1 - \frac{1}{1+\mathrm{e}^{-\mu(2x-\theta)}}\right) - 1. \end{aligned}$$

The above form is convenient for further analysis of characters of critical points.

It appears that the following three types of critical points are possible for system (2).

The first type of a critical point after reduction to canonical variables (v_1, v_2, v_3) is a stable node in both subspaces (v_1, v_2) and v_3:

$$\begin{vmatrix} \text{negative} & 1 & 0 \\ 0 & \text{negative} & 0 \\ 0 & 0 & \text{negative} \end{vmatrix}. \tag{15}$$

There is a degenerate stable node in the (v_1, v_2)-subspace attracting trajectories in v_3-direction due to negativity of λ_3.

For the purposes of this article, we will denote this type as $(-,-,-)$.

The second type of a critical point is a stable degenerate node in the (v_1, v_2)-subspace with continua of copies in v_3-direction due to zero value of λ_3:

$$\begin{vmatrix} \text{negative} & 1 & 0 \\ 0 & \text{negative} & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{16}$$

In this article, we denote this type as $(-,-,0)$.

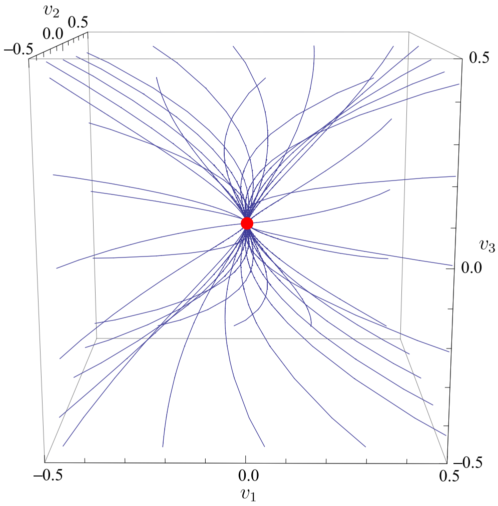

Figure 4. Some trajectories of the linearized system in canonical coordinates (v_1, v_2, v_3) for the critical point $(0.71848; 0.71848; 0.71848)$, $\theta = 0.5$, $\mu = 1$. All trajectories attract to the origin.

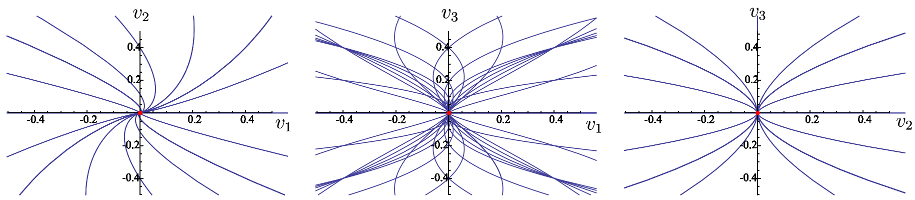

Figure 5. Projections of the phase portrait in Fig. 4 on three coordinate planes.

Figure 6 shows the phase portrait of a linearized system in canonical variables. The respective λ_3 is zero. It demonstrates that in (v_1, v_2) subspace, solutions near the critical point tend to $(0; 0)$ point (this can be seen in the left graph of Fig. 7). This two-dimensional subspace is repeated by parallel transition along the v_3-axis.

The third type of a critical point is a semi-stable node that is stable in the (v_1, v_2)-subspace and unstable in v_3-direction:

$$\begin{vmatrix} \text{negative} & 1 & 0 \\ 0 & \text{negative} & 0 \\ 0 & 0 & \text{positive} \end{vmatrix}. \tag{17}$$

The respective notation will be $(-, -, +)$.

The graph of Fig. 8 is in canonical form. It demonstrates that in $[v_1, v_2]$ subspace, solutions tend to $(0; 0)$ (this can be seen in the left graph of Fig. 9). In the remaining one-dimensional subspace v_3, solutions are repelled by the critical point.

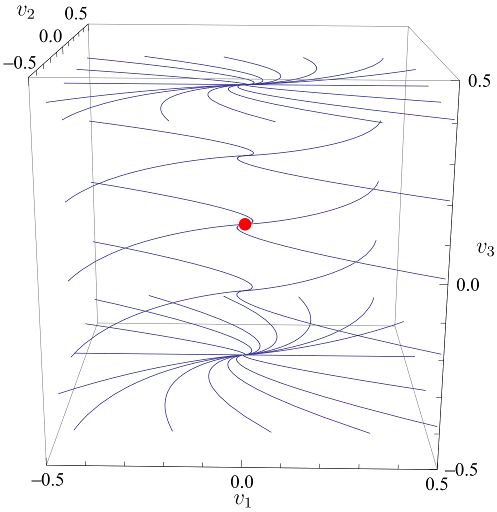

Figure 6. $\theta = 1, \mu = 2$, the critical point is $(0.5; 0.5; 0.5)$.

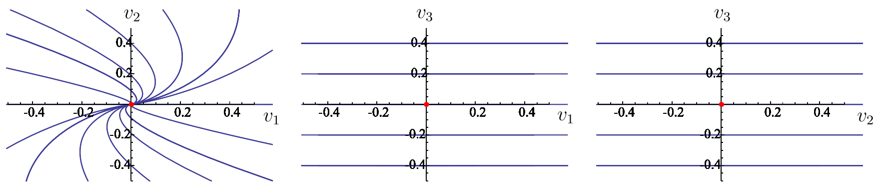

Figure 7. Projections of the phase portrait in Fig. 6 on three coordinate planes.

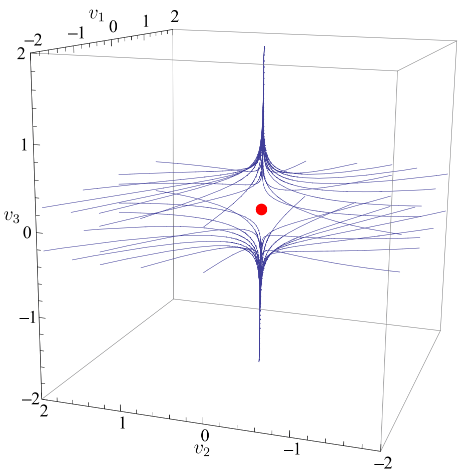

Figure 8. Phase portrait for linearized system in canonical form, $\theta = 1, \mu = 3$, the critical point is $(0.5; 0.5; 0.5)$.

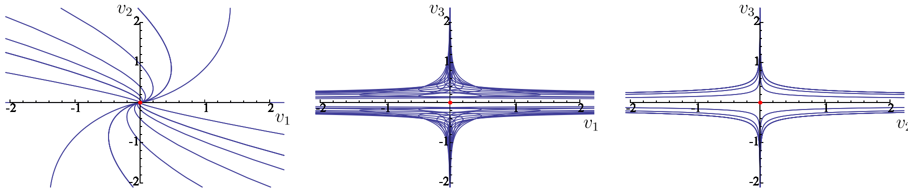

Figure 9. For $\theta = 1$, $\mu = 3$, critical points is $(0.5; 0.5; 0.5)$.

4 Critical points

In this section, we analyze types of critical points. For any critical point $\lambda_1 = \lambda_2 < 0$, it is sufficient to know the sign of λ_3. For brevity and only for purposes of this article, let us denote the type of a critical point by symbols $(-, +, 0)$, where $-$, 0 and $+$ are respectively for λ negative, zero or positive. It is known up to now that any critical point is of the type $(-, -, *)$, where "$*$" may be $-$, 0 or $+$.

Look at the relation

$$\lambda_3 = 2\mu x(1 - x) - 1. \tag{18}$$

The elementary analysis of the quadratic equation

$$x - x^2 - \frac{1}{2\mu} = 0 \tag{19}$$

shows that λ_3 is zero only for

$$x_{1,2} = \frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{1}{2\mu}}$$

if the discriminant

$$D = \frac{1}{4} - \frac{1}{2\mu}$$

is non-negative. The discriminant is zero at $\mu = 2$, then $x = 1/2$ and $\theta = 1$. So the critical point $(1/2, 1/2, 1/2)$ for $(\mu, \theta) = (2, 1)$ (that corresponds to the vertex of region Ω)) is of the type $\lambda_1 = \lambda_2 < 0$, $\lambda_3 = 0$.

If $\mu > 2$, then $\theta(x)$ has local maximum at $x = x_2$ and local minimum at $x = x_1$ as seen in Fig. 2c. It follows that for $\theta = x_2 = 1/2 + \sqrt{1/4 - 1/(2\mu)}$, there are two critical points, namely, one at x_2 (then $\lambda_3 = 0$) and another one corresponding to the intersection of $\theta(x)$ with $\theta(x_2)$ as seen in Fig. 2c. The respective coordinate x is less then x_1, and therefore, the respective λ_3 is negative. Similarly, there are two critical points for $\theta = x_1 = 1/2 - \sqrt{1/4 - 1/(2\mu)}$. The first one is at x_1, and another one is with the coordinate x greater than x_2. The respective λ_3 are zero for the first point and negative for the second one.

Proposition 1. *For* $(\mu, \theta) \in (\partial\Omega \setminus (2, 1))$*, there are exactly two critical points*: *one of the type* $(-, -, 0)$ *and the second one of the type* $(-, -, -)$*. For* $(\mu, \theta) = (2, 1)$*, there exists one critical point* $(1/2, 1/2, 1/2)$ *of the type* $(-, -, 0)$.

The following is also true.

Proposition 2. *For $(\mu,\theta) \in Q \setminus \overline{\Omega}$, there exists only one critical point of the type $(-,-,-)$.*

Proof. The proof follows from the fact that, in the mentioned region, the quadratic polynomial (19) and, therefore, the right-hand side in (18) are negative. □

Proposition 3. *For any $(\mu,\theta) \in \Omega$, there exist exactly three critical points: the middle is of the type $(-,-,+)$, and two side points are of the type $(-,-,-)$.*

Proof. For any $\mu > 2$, consider some θ_* such that

$$\theta_{\min}(\mu) = \theta\big(x_1(\mu)\big) < \theta_* < \theta\big(x_2(\mu)\big) = \theta_{\max}(\mu).$$

The horizontal line $\theta = \theta_*$ in (x,θ)-plane (Fig.2c) has three points of intersection with the curve $\theta(x)$. The middle point x is between x_1 and x_2. Therefore, polynomial (19) is positive, and hence, the respective λ_3 is positive. The two remaining critical points satisfy either $x < x_1$ or $x > x_2$. In both cases, polynomial (19) is negative, and the respective λ_3 are negative also. □

It follows from the above arguments that the following statement is true.

Theorem 1. *The above system has attracting sets of the following structure.*
All critical points are located on the bisectrix $x_1 = x_2 = x_3$.
The open quadrant $Q = \{\mu > 0,\ \theta > 0\}$ contains fully defined region Ω such that:

1) *if $(\mu,\theta) \in Q \setminus \overline{\Omega}$, then the attracting set is only one critical point of the type $(-,-,-)$;*
2) *if $(\mu,\theta) = (2,1)$ (that corresponds to the vertex of region Ω), then only one critical point $(1/2,1/2,1/2)$ is of the type $(-,-,0)$;*
3) *if $(\mu,\theta) \in \partial\Omega \setminus (2,1)$, then the attracting set consists of two critical points: one of them of the type $(-,-,-)$ and another one of the type $(-,-,0)$;*
4) *if $(\mu,\theta) \in \Omega$, then the attracting set consists of three critical points: a point in the middle is of the type $(-,-,+)$, two remaining points are of the type $(-,-,-)$.*

5 Phase portraits

Consider a number of examples illustrating (and confirming) our analysis.

5.1 Particular case I

In the first particular case, the parameter $\mu = 1$ and $\theta = 0.5$. System (2) has one critical point $(0.71848; 0.71848; 0.71848)$. System (2) with parameters $\mu = 1$ and $\theta = 0.5$ has the following visual interpretation.

In Fig. 10, we can see attractor, where all solutions from the first quadrant tend to the critical point $(0.71848, 0.71848, 0.71848)$.

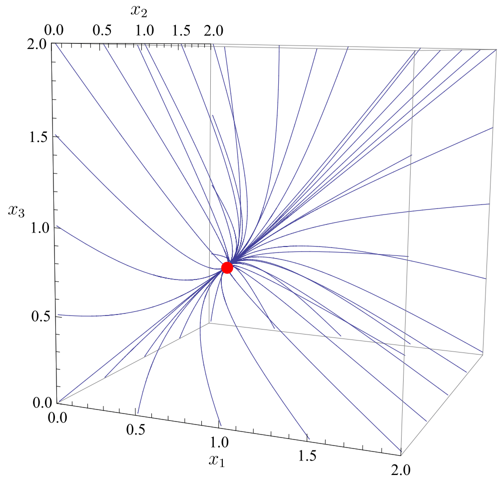

Figure 10. System (2) when $\mu = 1$ and $\theta = 0.5$ with the critical point $(0.71848; 0.71848; 0.71848)$.

The λ matrix for the critical point $(0.71848, 0.71848, 0.71848)$ is

$$\begin{vmatrix} -1.20227 & 1 & 0 \\ 0 & -1.20227 & 0 \\ 0 & 0 & -0.595467 \end{vmatrix}.$$

The type of this critical point is a stable node in both subspaces according to (15).

5.2 Particular case II

In the second particular case, the parameter $\mu = 3$ and $\theta = 1$. The system has three critical points: $(0.5, 0.5, 0.5)$, $(0.07072, 0.07072, 0.07072)$, and the last one is $(0.929279, 0.929279, 0.929279)$.

In Fig. 11, one can see the visual interpretation of the system.

The λ matrix for the critical point $(0.5, 0.5, 0.5)$ is

$$\begin{vmatrix} -1.75 & 1 & 0 \\ 0 & -1.75 & 0 \\ 0 & 0 & 0.5 \end{vmatrix}.$$

The type of this critical point is a semi-stable node that is stable in the one subspace and unstable in the second subspace according to (17).

The λ matrices for the critical points $(0.07072, 0.07072, 0.07072)$ and $(0.929279, 0.929279, 0.929279)$ are equal and the matrix is

$$\begin{vmatrix} -1.19716 & 1 & 0 \\ 0 & -1.19716 & 0 \\ 0 & 0 & -0.605688 \end{vmatrix}.$$

The type of this critical point is a stable node in both subspaces according to (15).

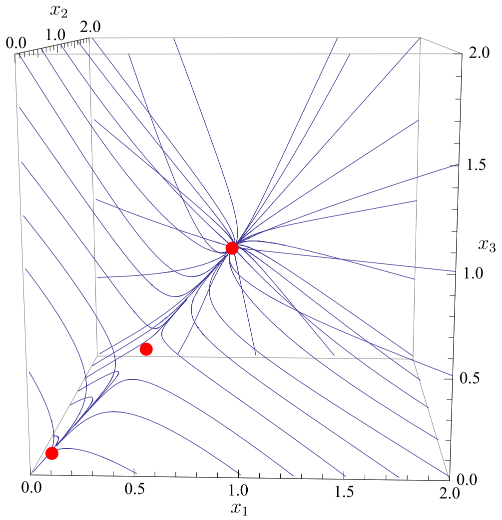

Figure 11. System (2) when $\mu = 3$ and $\theta = 1$ with critical points $(0.5, 0.5, 0.5)$, $(0.07072, 0.07072, 0.07072)$, $(0.929279, 0.929279, 0.929279)$.

5.3 Particular case III

In the third particular case, the parameter $\mu = 3$ and $\theta = (3 + \sqrt{3} + \log[2 - \sqrt{3}])/3 = 1.13836$. The system has two critical points: the first one is $(0.788675, 0.788675, 0.788675)$, and the second is $(0.079487, 0.079487, 0.079487)$.

The visual interpretation of the system is the following:

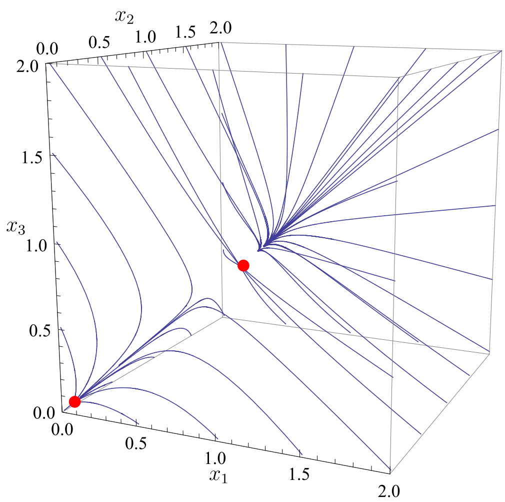

Figure 12. System (2) when $\mu = 3$ and $\theta = 1.13836$ with critical points $(0.788675, 0.788675, 0.788675)$, $(0.079487, 0.079487, 0.079487)$.

The λ matrix for the critical point $(0.788675, 0.788675, 0.788675)$ is

$$\begin{vmatrix} -1.5 & 1 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

The type of the critical point is a stable node in one subspace and a degenerate type in the second subspace according to (16).

The λ matrix for the critical point $(0.079487, 0.079487, 0.079487)$ is

$$\begin{vmatrix} -1.14634 & 1 & 0 \\ 0 & -1.14634 & 0 \\ 0 & 0 & -0.707325 \end{vmatrix}.$$

The type of this critical point is a stable node in both subspaces according to (15).

5.4 Particular case IV

In the fourth particular case, the parameter $\mu = 2$ and $\theta = 1$. System (2) has one critical point. This point is $(0.5, 0.5, 0.5)$. In Fig. 13, one can see the visual interpretation of system (2) with parameters $\mu = 2$ and $\theta = 1$.

The λ matrix for the critical point $(0.5, 0.5, 0.5)$ is

$$\begin{vmatrix} -1.5 & 1 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

The type of this critical point is a stable node in one subspace and a degenerate type in the second subspace according to (16).

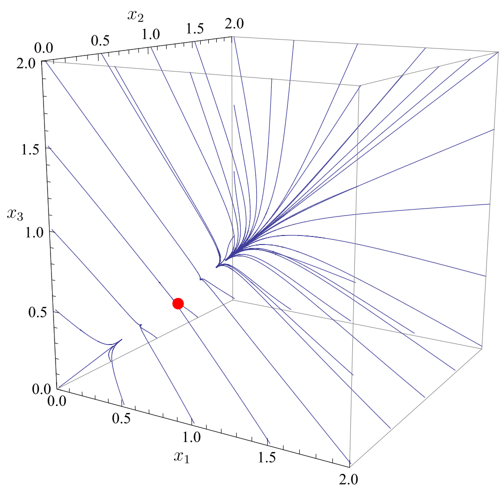

Figure 13. System (2) when $\mu = 2$ and $\theta = 1$ with critical point $(0.5, 0.5, 0.5)$.

6 Conclusions

We have found that structure of the attracting sets is relatively simple, namely, it may consist of one to three critical points, and this depends of the choice of parameters μ and θ. If a point (μ, θ) is located outside the region Ω, then there is exactly one attracting critical point. If (μ, θ) is located inside the region Ω, then there are exactly three critical points, one of them is attracting/repelling. Finally, if (μ, θ) is on the boundary of the region Ω, then some intermediate cases are possible (including degenerate ones). The set Ω is fully described.

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