Nonlinear Analysis: Modelling and Control, Vol. 21, No. 5, 651–672 ISSN 1392-5113
http://dx.doi.org/10.15388/NA.2016.5.6

Solvability of boundary value problem for second order impulsive differential equations with one-dimensional p-Laplacian on whole line*

Yuji Liu

Department of Mathematics,
Guangdong University of Finance and Economics,
Guangzhou 510320, China
liuyuji888@sohu.com

Received: January 9, 2015 / **Revised:** July 3, 2015 / **Published online:** June 28, 2016

Abstract. This paper is concerned with a class of boundary value problems of the impulsive differential equations with one-dimensional p-Laplacian on whole line with a non-Carathéodory nonlinearity. Sufficient conditions to guarantee the existence of solutions are established. Some examples are given to illustrate the main results.

Keywords: second order impulsive differential equation on whole line, one-dimensional p-Laplacian, boundary value problem, solution, fixed point theorem.

1 Introduction

The motivation for the present work stems from both practical and theoretical aspects. In fact, boundary value problems on the whole line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations modelling various physical phenomena such as unsteady flow of gas though a whole, porous media, and the theory of drain flows.

The asymptotic theory of ordinary differential equations is an area in which there is great activity among a large number of investigators. In this theory, it is of great interest to investigate, in particular, the existence of solutions with prescribed asymptotic behavior, which are global in the sense that they are solutions on the whole line (half line). The existence of global solutions with prescribed asymptotic behavior is usually formulated as the existence of solutions of boundary value problems on the whole line (half line).

In recent years, the existence of solutions of boundary value problems of the differential equations governed by nonlinear differential operator $[\Phi(u')]' = [|u'|^{p-2}u']'$ has been studied by many authors, see $[5, 7, 8, 9, 10, 11, 12, 13, 15, 17]$.

*This research was supported by the Natural Science Foundation of Guangdong province (grant No. S2011010001900) and Natural science research project for colleges and universities of Guangdong Province (No. 2014KTSCX126).

Impulsive differential equation is one of the main tools to study the dynamics of processes in which sudden changes occur. The theory of impulsive differential equation has recently received considerable attention. However, the study on existence of positive solutions of nonlocal boundary value problems for impulsive differential equations on whole real line has not been sufficiently developed [1, 2, 3, 4, 5, 6, 16].

In all above mentioned papers, the boundary conditions are subjected to the two end points 0 and $+\infty$ (or $-\infty$ and $+\infty$) and the obtained solutions are defined on $[0,+\infty)$ (or $\mathbb{R}$). An interesting question occurs: when one subjects the boundary conditions on two intermediate points ξ,η, how can we get solutions defined on $\mathbb{R}$ of a boundary value problem of differential equations on whole line? On the other hand, in known papers [7, 8, 9, 10, 11, 12, 13, 14], concerning the differential equations $[\Phi(\rho(t)x'(t))]' + p(t)f(t,x(t),\rho(t)x'(t)) = 0$, it is supposed that $(t,u,v)\to p(t)f(t,u,v)$ is a Carathéodory function. To the best of our knowledge, there has been no paper concerning the solvability of $[\Phi(\rho(t)x'(t))]' + p(t)f(t,x(t),\rho(t)x'(t)) = 0$ with $(t,u,v)\to p(t)f(t,u,v)$ being a non-Carathéodory function.

Motivated by mentioned papers, to fill this gap, we consider the following boundary value problem for the impulsive singular differential equation on the whole line:

$$
\begin{aligned}
&\big[\Phi\big(\rho(t)x'(t)\big)\big]' + p(t)f\big(t,x(t),\rho(t)x'(t)\big) = 0, \quad \text{a.e. } t\in\mathbb{R},\\
&x(\xi) = \int_{-\infty}^{+\infty} m(s)\phi\big(s,x(s),\rho(s)x'(s)\big)\,\mathrm{d}s,\\
&x(\eta) = \int_{-\infty}^{+\infty} n(s)\psi\big(s,x(s),\rho(s)x'(s)\big)\,\mathrm{d}s,\\
&\Delta x(t_i) = I\big(t_i,x(t_i),\rho(t_i)x'(t_i)\big), \quad i\in\mathbb{Z},\\
&\Delta\Phi\big(\rho(t_i)x'(t_i)\big) = J\big(t_i,x(t_i),\rho(t_i)x'(t_i)\big), \quad i\in\mathbb{Z},
\end{aligned}
\tag{1}
$$

where $\Phi(x) = |x|^{k-2}x$ with $k>1$, the inverse of Φ is denoted by Φ^{-1} and $\Phi^{-1}(x) = |x|^{l-2}x$ with $1/k+1/l=1$, $\xi<\eta$ are constants, p is nonnegative and satisfies $p\in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\int_{-\infty}^0 p(t)\,\mathrm{d}t = \int_0^{+\infty} p(s)\,\mathrm{d}s = +\infty$, $\rho:\mathbb{R}\to[0,\infty)$ with $\Phi^{-1}(\tau(\cdot))/\rho(\cdot)\in L^1_{\mathrm{loc}}(\mathbb{R})$, $\tau(t) = 1+|\int_\eta^t p(s)\,\mathrm{d}s|$, and the following cases will be discussed:

Case 1. $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u)\,\mathrm{d}u = \int_0^{+\infty}\Phi^{-1}(\tau(u))/\rho(u)\,\mathrm{d}u = +\infty$;
Case 2. $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u)\,\mathrm{d}u < +\infty$, $\int_0^{+\infty}\Phi^{-1}(\tau(u))/\rho(u)\,\mathrm{d}u < +\infty$;
Case 3. $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u)\,\mathrm{d}u = +\infty$, $\int_0^{+\infty}\Phi^{-1}/(\tau(u))\rho(u)\,\mathrm{d}u < +\infty$;
Case 4. $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u)\,\mathrm{d}u < +\infty$, $\int_0^{+\infty}\Phi^{-1}/(\tau(u))\rho(u)\,\mathrm{d}u = +\infty$;

$f,\phi,\psi:\mathbb{R}^3\to\mathbb{R}$ are strong Carathéodory functions, $m,n\in L^1(\mathbb{R})$, $\mathbb{Z}=\{0,\pm1,\pm2,\dots\}$, $\{t_i\colon i\in\mathbb{Z}\}$ is a increasing sequence with $\lim_{i\to-\infty} t_i = -\infty$ and $\lim_{i\to+\infty} t_i = +\infty$, $\Delta x(t_i) = \lim_{t\to t_i^+} x(t) - x(t_i)$, $\Delta\Phi(\rho(t_i)x'(t_i)) = \lim_{t\to t_i^+}\Phi(\rho(t)x'(t)) - \Phi(\rho(t_i)x'(t_i))$, $I,J:\{t_i\colon i\in\mathbb{Z}\}\times\mathbb{R}^2\to\mathbb{R}$ are discrete Carathéodory functions.

The homogeneous boundary conditions $x(\xi) = 0, x(\eta) = 0$ of special case in (1) come from the four-point boundary conditions $a\lim_{t\to-\infty} x(t) - bx(\xi) = c\lim_{t\to+\infty}(t) - dx(\eta) = 0$ (if $a = c = 0$ and $b = d = 1$), which arise in the study of heat flow problems involving a bar of unit length with two controllers at $t = -\infty$ and $t = +\infty$ adding or removing heat according to the temperatures detected by two sensors at $t = \xi$ and $t = \eta$. It is well known that

$$\begin{aligned}
&\big[\Phi\big(\rho(t)x'(t)\big)\big]' + p(t)f\big(t, x(t), \rho(t)x'(t)\big) = 0, \quad \text{a.e. } t \in [\xi, \eta],\\
&x(\xi) = \int_\xi^\eta m(s)\phi\big(s, x(s), \rho(s)x'(s)\big)\,\mathrm{d}s,\\
&x(\eta) = \int_\xi^\eta n(s)\psi\big(s, x(s), \rho(s)x'(s)\big)\,\mathrm{d}s
\end{aligned}$$

is called Dirichlet boundary value problem with integral boundary conditions whose solutions are defined on $[\xi, \eta]$. In this sense, BVP (1) is a generalization of Dirichlet boundary value problem.

Consider the problem $(tx'(t))' = 1$, a.e. $t \in \mathbb{R}$, $x(-1) = x(1) = 1$. It is easy to know from $(tx'(t))' = 1$, a.e. $t \in \mathbb{R}$, that $x(t) = c_1 - c_2 \ln|t| + t$. Thus this problem has no continuous solution. Consider the problem $(\sqrt{|t|}x'(t))' = 1$, a.e. $t \in \mathbb{R}$, $x(-1) = x(1) = 1$. One can get from $(\sqrt{|t|}x'(t))' = 1$, a.e. $t \in \mathbb{R}$, that $x(t) = (2/3)t^{3/2} + 2c_1\sqrt{t} + c_2$ for $t > 0$ and $x(t) = -(2/3)|t|^{3/2} - 2c_1\sqrt{|t|} + c_2$ for $t < 0$. Thus the mentioned problem has infinitely many continuous solutions

$$x(t) = \begin{cases} \frac{2}{3}t^{3/2} + 2c_1\sqrt{t} + c_2, & t \geqslant 0,\\ -\frac{2}{3}|t|^{3/2} - 2c_3\sqrt{|t|} + c_2, & t < 0. \end{cases}$$

Here $c_1, c_2, c_3 \in \mathbb{R}$. So this kind of problem is interesting.

Our purpose is to establish sufficient conditions for the existence of solutions of BVP (1) in Cases 1–4, respectively. The remainder of this paper is organized as follows: the first result are given in Section 2 in Case 1, the existence result of solutions of BVP (1) in Cases 2 is given in Section 3, and similarly, we can establish existence results in Cases 3 and 4, respectively, we omit the details. Finally, in Section 4, two examples are given to illustrate the main results.

2 Solvability of (1) in Case 1

In this section, we present existence result of BVP (1) in Case 1. Denote

$$\tau(t) = 1 + \left|\int_\eta^t p(s)\,\mathrm{d}s\right|, \qquad \sigma(t) = 1 + \left|\int_\xi^t \frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u\right|. \tag{2}$$

It is easy to show that σ, τ are continuous on $\mathbb{R}$ and $\lim_{t\to\pm\infty}\sigma(t)=\lim_{t\to\pm\infty}\tau(t)=+\infty$.

Definition 1. $F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a strong Carathédory function if

(i) $t\to F(t,\sigma(t)u,\Phi^{-1}(\tau(t))v)$ is measurable on $\mathbb{R}$ for any $u,v\in\mathbb{R}$;
(ii) $(u,v)\to F(t,\sigma(t)u,\Phi^{-1}(\tau(t))v)$ is continuous on $\mathbb{R}^2$ for a.e. $t\in\mathbb{R}$;
(iii) for each $r>0$, there exists nonnegative function $M_r\geqslant 0$ such that $|u|,|v|\leqslant r$ implies $|F(t,\sigma(t)u,\Phi^{-1}(\tau(t))v)|\leqslant M_r$, a.e. $t\in\mathbb{R}$.

Definition 2. $H:\{t_i\colon i\in\mathbb{Z}\}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a discrete Carathédory function if

(i) $(x,y)\to H(t_s,\sigma(t_s)x,\Phi^{-1}(\tau(t_s))y)$ is continuous on $\mathbb{R}^2$ for all $s\in\mathbb{Z}$;
(ii) for each $r>0$, there exists nonnegative constants $M_{ir}\geqslant 0$ $(i\in\mathbb{Z})$ such that $|x|,|y|\leqslant r$ implies $|H(t_s,\sigma(t_s)x,\Phi^{-1}(\tau(t_s))y)|\leqslant M_{sr}$, $s\in\mathbb{Z}$, $\sum_{s=-\infty}^{+\infty}M_{sr}<+\infty$.

Definition 3. Let X be a real Banach space. An operator $T:X\to X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Choose

$$X=\biggl\{x:\mathbb{R}\to\mathbb{R}\colon x|_{(t_s,t_{s+1}]},\ \rho x'|_{(t_s,t_{s+1}]}\text{ is continuous},\ s\in\mathbb{Z};\\ \text{the following limits exist and are finite: }\lim_{t\to t_s^+}x(t),\ \lim_{t\to t_s^+}\rho(t)x'(t),\ s\in\mathbb{Z},\\ \lim_{t\to-\infty}\frac{x(t)}{\sigma(t)},\ \lim_{t\to+\infty}\frac{x(t)}{\sigma(t)},\ \lim_{t\to-\infty}\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))},\ \lim_{t\to+\infty}\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\biggr\}.$$

For $x\in X$, define $\|x\|=\max\{\sup_{t\in\mathbb{R}}|x(t)|/\sigma(t),\ \sup_{t\in\mathbb{R}}\rho(t)|x'(t)|/\Phi^{-1}(\tau(t))\}$.

Lemma 1. *X is a Banach space with $\|\cdot\|$ defined.*

Proof. It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X. Then $\|x_u-x_v\|\to 0$, $u,v\to+\infty$. We will prove that there $x_0\in X$ such that $x_u\to x_0$ as $u\to+\infty$. Since $x_u\in X$, we have

$$\sup_{t\in\mathbb{R}}\frac{|x_u(t)-x_v(t)|}{\sigma(t)}\to 0,\quad \sup_{t\in\mathbb{R}}\frac{\rho(t)|x_u'(t)-x_v'(t)|}{\Phi^{-1}(\tau(t))}\to 0,\quad u,v\to+\infty.$$

So

$$\sup_{t\in(t_s,t_{s+1}]}\frac{|x_u(t)-x_v(t)|}{\sigma(t)}\to 0,\quad u,v\to+\infty,$$
$$\sup_{t\in(t_s,t_{s+1}]}\frac{\rho(t)|x_u'(t)-x_v'(t)|}{\Phi^{-1}(\tau(t))}\to 0,\quad u,v\to+\infty,$$
$$\lim_{t\to t_s^+}\frac{x_u(t)}{\sigma(t)},\ \lim_{t\to t_s^+}\frac{\rho(t)x_u'(t)}{\Phi^{-1}(\tau(t))}\ \text{exists},\quad s\in\mathbb{Z}.$$

Then there exists functions $x_{s,0}, y_{s,0} \in C^0[t_s, t_{s+1}]$ such that $\lim_{u\to+\infty} x_u(t)/\sigma(t) = x_{s,0}(t)$ and $\lim_{u\to+\infty} \rho(t)x'_u(t)/\Phi^{-1}(\tau(t)) = y_{s,0}(t)$ uniformly on $[t_s, t_{s+1}]$.

Define $x_0(t) = x_{s,0}(t)$, $y_0(t) = y_{s,0}(t)$ for all $t \in (t_s, t_{s+1}] (s \in \mathbb{Z})$. Then $x_0, y_0 : \mathbb{R} \to \mathbb{R}$ is well defined on $\mathbb{R}$, and

$$\lim_{u\to+\infty} \frac{x_u(t)}{\sigma(t)} = x_0(t), \quad \lim_{u\to+\infty} \frac{\rho(t)x'_u(t)}{\Phi^{-1}(\tau(t))} = y_0(t), \quad t \in \mathbb{R}.$$

It follows that

$$\sup_{t\in(t_s,t_{s+1}]} \left| \frac{x_u(t)}{\sigma(t)} - x_0(t) \right| \to 0, \qquad \sup_{t\in(t_s,t_{s+1}]} \left| \frac{\rho(t)x_u(t)}{\Phi^{-1}(\tau(t))} - y_0(t) \right|, \quad u \to +\infty.$$

Now we do the following three steps:

Step 1. Prove that $\sigma(\cdot)x_0(\cdot), \Phi^{-1}(\tau(\cdot))y_0(\cdot) \in C^0(t_s, t_{s+1}]$ and $\lim_{t\to t_s^+} \sigma(t)x_0(t)$ and $\lim_{t\to t_s^+} \Phi^{-1}(\tau(t))y_0(t)$ exist.

Step 2. Prove that the limits $\lim_{t\to-\infty} x_0(t)$, $\lim_{t\to+\infty} x_0(t)$, $\lim_{t\to-\infty} y_0(t)$, $\lim_{t\to+\infty} y_0(t)$ exist.

Step 3. Prove that $y_0(t) = \rho(t)[\sigma(t)x_0(t)]'/\Phi^{-1}(\tau(t))$.

The details are omitted. It follows that $x_u \to x_0$ as $u \to +\infty$. So X is a Banach space. □

Lemma 2. *Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied*:

(i) *both* $\{t \to x(t)/\sigma(t)\colon x \in M\}$ *and* $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ *are uniformly bounded*;

(ii) *both* $\{t \to x(t)/\sigma(t)\colon x \in M\}$ *and* $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ *are equicontinuous in* $(t_s, t_{s+1}]$ *(*$s \in N$*)*;

(iii) *both* $\{t \to x(t)/\sigma(t)\colon x \in M\}$ *and* $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ *are equi-convergent as* $t \to \pm\infty$.

Proof. (⇐) From Lemma 1 we know X is a Banach space. In order to prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is, for all $\epsilon > 0$, M has a finite ϵ-net.

For any given $\epsilon > 0$, by (i)–(iii) there exist constants $M > 0$, $\delta > 0$, $t_{s_0} > 0$ and $t_{-s_0} < 0$ such that

$$\sup_{t\in\mathbb{R}} \frac{|x(t)|}{\sigma(t)}, \ \sup_{t\in\mathbb{R}} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\tau(t))} \leqslant M, \quad x \in M,$$

$$\left| \frac{x(w_1)}{\sigma(w_1)} - \frac{x(w_2)}{\sigma(w_2)} \right| \leqslant \frac{\epsilon}{3}, \quad \left| \frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))} - \frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(\tau(w_2))} \right| < \frac{\epsilon}{3}, \quad x \in M,$$
$$w_1, w_2 \in (t_s, t_{s+1}], \ |w_1 - w_2| < \delta, \ s = -s_0, -s_0+1, \ldots, s_0-1,$$

$$\left|\frac{x(w_1)}{\sigma(w_1)} - \frac{x(w_2)}{\sigma(w_2)}\right| \leqslant \frac{\epsilon}{3}, \quad \left|\frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))} - \frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(\tau(w_2))}\right| < \frac{\epsilon}{3}, \quad x \in M,$$
$$w_1, w_2 \leqslant t_{-s_0} \text{ or } w_1, w_2 \geqslant t_{s_0}.$$

Define

$$X|_{(-t_{s_0}, t_{s_0}]} = \Big\{x\colon x, \rho(t)x' \in C\big(t_s, t_{s+1}\big],\ s = -s_0, -s_0+1, \dots, s_0-1,$$
$$\lim_{t \to t_s^+} \frac{x(t)}{\sigma(t)} \text{ exist},\ s = -s_0, -s_0+1, \dots, s_0-1,$$
$$\lim_{t \to t_s^+} \frac{\rho(t)x'(t)}{\Phi^1(\tau(t))} \text{ exist},\ s = -s_0, -s_0+1, \dots, s_0-1\Big\}.$$

For $x \in X|_{(-t_{s_0}, t_{s_0}]}$, define

$$\|x\|_{s_0} = \max\left\{\max_{t \in (-t_{s_0}, t_{s_0}]} \frac{|x(t)|}{\sigma(t)}, \max_{t \in (-t_{s_0}, t_{s_0}]} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\tau(t))}\right\}.$$

Similarly to Lemma 1, we can prove that $X_{(-t_{s_0}, t_{s_0}]}$ is a Banach space.

Let $M|_{(-t_{s_0}, t_{s_0}]} = \{t \to x(t),\ t \in (-t_{s_0}, t_{s_0}]\colon x \in M\}$. Then $M|_{(-t_{s_0}, t_{s_0}]}$ is a subset of $X|_{(-t_{s_0}, t_{s_0}]}$. By (i), (ii), and Ascoli–Arzela theorem, we can know that $M|_{(-t_{s_0}, t_{s_0}]}$ is relatively compact. Thus there exist $x_1, x_2, \dots, x_k \in M$ such that, for any $x \in M$, we have that there exists some $i = 1, 2, \dots, k$ such that

$$\|x - x_i\|_{s_0}$$
$$= \max\left\{\sup_{t \in (-t_{s_0}, t_{s_0}]} \frac{|x(t) - x_i(t)|}{\sigma(t)}, \sup_{t \in (-t_{s_0}, t_{s_0}]} \frac{\rho(t)|x'(t) - x_i'(t)|}{\Phi^{-1}(\tau(t))}\right\} < \frac{\epsilon}{3}.$$

Therefore, for $x \in M$, we can get $\|x - x_i\|_X \leqslant \epsilon$. So, for any $\epsilon > 0$, M has a finite ϵ-net $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$, that is, M is totally bounded in X. Hence M is relatively compact in X.

($\Rightarrow$) Assume that M is relatively compact, then for any $\epsilon > 0$, there exists a finite ϵ-net of M. Let the finite ϵ-net be $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$ with $x_i \subset M$. Then for any $x \in M$, there exists U_{x_i} such that $x \in U_{x_i}$ and $\|x\| \leqslant \|x - x_i\| + \|x_i\| \leqslant \epsilon + \max\{\|x_i\|\colon i = 1, 2, \dots, k\}$. It follows that both $\{t \to x(t)/\sigma(t)\colon x \in M\}$ and $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ are uniformly bounded. Then (i) holds.

Furthermore, there exists $t_{-s_0} < 0$ and $t_{s_0} > 0$ such that $|x_i(w_1) - x_i(w_2)| < \epsilon$ for all $w_1, w_2 \geqslant t_{s_0}$ and all $w_1, w_2 \leqslant t_{-s_0}$ and $i = 1, 2, \dots, k$. Then we have for $w_1, w_2 \geqslant t_{s_0}$, all $w_1, w_2 \leqslant t_{-s_0}$ and $x \in M$ that

$$\left|\frac{x(w_1)}{\sigma(w_1)} - \frac{x(w_2)}{\sigma(w_2)}\right| \leqslant \left|\frac{x(w_1)}{\sigma(w_1)} - \frac{x_i(w_1)}{\sigma(w_2)}\right| + \left|\frac{x_i(w_1)}{\sigma(w_1)} - \frac{x_i(w_2)}{\sigma(w_2)}\right|$$
$$+ \left|\frac{x_i(w_2)}{\sigma(w_2)} - \frac{x(w_2)}{\sigma(w_2)}\right| < 3\epsilon.$$

Similarly, for $w_1, w_2 \geqslant t_{s_0}$, all $w_1, w_2 \leqslant t_{-s_0}$ and $x \in M$, we have that

$$\left|\frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))} - \frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(\tau(w_2))}\right| < 3\epsilon.$$

Thus (iii) is valid. Similarly, we can prove that (ii) holds. Consequently, the lemma is proved. □

For ease expression, denote $G_x(t) = G(t, x(t), \rho(t)x'(t))$ for a function $G : \mathbb{R}^3 \to \mathbb{R}$ and $x \in X$.

Lemma 3. *Suppose that $x \in X$. Then there exists a unique $A_x \in \mathbb{R}$ such that*

$$\int_\xi^\eta \frac{\Phi^{-1}(A_x + \sum_{\xi \leqslant t_s < u} J_x(t_s) - \int_\xi^u p(w) f_x(w)\,\mathrm{d}w)}{\rho(u)}\,\mathrm{d}u + \int_{-\infty}^{+\infty} m(w)\phi_x(w)\,\mathrm{d}w - \int_{-\infty}^{+\infty} n(w)\psi_x(w)\,\mathrm{d}w + \sum_{\xi \leqslant t_s < \eta} I_x(t_s) = 0, \tag{3}$$

and A_x satisfies

$$|A_x| \leqslant \sum_{\xi \leqslant t_s < \eta} \left|J_x(t_s)\right| + \int_\xi^\eta p(w)\left|f_x(w)\right|\mathrm{d}w + \Phi\left(\frac{\int_{-\infty}^{+\infty} |m(w)||\phi_x(w)|\,\mathrm{d}w + \int_{-\infty}^{+\infty} |n(w)||\psi_x(w)|\,\mathrm{d}w + \sum_{\xi \leqslant t_s < \eta} |I_x(t_s)|}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\right). \tag{4}$$

Proof. Denote

$$\sigma_1 = \sum_{\xi \leqslant t_s < \eta} \left|J_x(t_s)\right| + \int_\xi^\eta p(w)\left|f_x(w)\right|\mathrm{d}w + \Phi\left(\frac{\int_{-\infty}^{+\infty} n(w)\psi_x(w)\,\mathrm{d}w - \int_{-\infty}^{+\infty} m(w)\phi_x(w)\,\mathrm{d}w - \sum_{\xi \leqslant t_s < \eta} I_x(t_s)}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\right),$$

$$\sigma_2 = -\sum_{\xi \leqslant t_s < \eta} \left|J_x(t_s)\right| - \int_\xi^\eta p(w)\left|f_x(w)\right|\mathrm{d}w + \Phi\left(\frac{\int_{-\infty}^{+\infty} n(w)\psi_x(w)\,\mathrm{d}w - \int_{-\infty}^{+\infty} m(w)\phi_x(w)\,\mathrm{d}w - \sum_{\xi \leqslant t_s < \eta} I_x(t_s)}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\right).$$

Let

$$G(c)=\int_{\xi}^{\eta}\frac{1}{\rho(u)}\Phi^{-1}\left(c+\sum_{\xi\leqslant t_s<u}J_x(t_s)-\int_{\xi}^{u}p(w)f_x(w)\,\mathrm{d}w\right)\mathrm{d}u$$
$$+\int_{-\infty}^{+\infty}m(w)\phi_x(w)\,\mathrm{d}w-\int_{-\infty}^{+\infty}n(w)\psi_x(w)\,\mathrm{d}w+\sum_{\xi\leqslant t_s<\eta}I_x(t_s).$$

Then we find that G is increasing on $\mathbb{R}$, $G(\sigma_1)\geqslant 0$ and $G(\sigma_2)\leqslant 0$. So there exists a unique $A_x\in[\sigma_2,\sigma_1]$ such that $G(A_x)=0$. This is (3). Furthermore, we get from the definitions of σ_1, σ_2 that (4) holds. The proof is completed. ☐

We define $\sum_{a\leqslant t_s<b}k_s:=-\sum_{b\leqslant t_s<a}k_s$ for $a>b$. For $x\in X$, define $(Tx)(t)$ by

$$(Tx)(t)=\int_{-\infty}^{+\infty}m(s)\phi_x(s)\,\mathrm{d}s+\sum_{\xi\leqslant t_s<t}I_x(t_s)$$
$$+\int_{\xi}^{t}\frac{1}{\rho(u)}\Phi^{-1}\left(A_x+\sum_{\eta\leqslant t_s<u}J_x(t_s)-\int_{\eta}^{u}p(w)f_x(w)\,\mathrm{d}w\right)\mathrm{d}u,\quad t\in\mathbb{R},$$

where A_x is defined by (3).

Lemma 4. *Suppose that f is a strong Carathéodory function, I, J are discrete Carathéodory functions, and for each $r>0$, $f(t,\sigma(t)u,\Phi^{-1}(\tau(t))v)$ converges uniformly as $t\to\pm\infty$ on $[-r,r]\times[-r,r]$. Then*

(i) *$T:X\to X$ is well defined*;
(ii) *$x\in X$ is a solution of* (1) *if and only if $x\in X$ is a fixed point of T in X*;
(iii) *T is completely continuous.*

Proof. (i) From Lemma 1, X is a Banach space. For $x\in X$, we have $\|x\|\leqslant r$ for some $r\geqslant 0$. Then there exists constants $M_{r,f}\geqslant 0$, $M_{r,J,s}\geqslant 0$ and $M_{r,I,s}\geqslant 0$ such that

$$\begin{aligned}
&\left|f_x(t)\right|=\left|f\left(t,\sigma(t)\frac{x(t)}{\sigma(t)},\Phi^{-1}(\tau(t))\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\right)\right|\leqslant M_{r,f},\quad \text{a.e. } t\in\mathbb{R},\\
&\left|\phi_x(t)\right|=\left|\phi\big(t,x(t),\rho(t)x'(t)\big)\right|\leqslant M_{r,\phi},\quad \text{a.e. } t\in\mathbb{R},\\
&\left|\psi_x(t)\right|=\left|\psi\big(t,x(t),\rho(t)x'(t)\big)\right|\leqslant M_{r,\psi},\quad \text{a.e. } t\in\mathbb{R},\\
&\left|I_x(t_s)\right|=\left|I\big(t_s,x(t_s),\rho(t_s)x'(t_s)\big)\right|\leqslant M_{r,I,s},\quad s\in\mathbb{Z},\ \sum_{s=-\infty}^{+\infty}M_{r,I,s}<+\infty,\\
&\left|J_x(t_s)\right|=\left|J\big(t_s,x(t_s),\rho(t_s)x'(t_s)\big)\right|\leqslant M_{r,J,s},\quad s\in\mathbb{Z},\ \sum_{s=-\infty}^{+\infty}M_{r,J,s}<+\infty.
\end{aligned}\tag{5}$$

One finds from the definition of Tx that $(Tx)(\cdot)|_{(t_s,t_{s+1}]}$, $\rho(\cdot)(Tx)'(\cdot)|_{(t_s,t_{s+1}]} \in C^0(t_s,t_{s+1}]$, and the limits $\lim_{t\to t_s^+}(Tx)(t)(s\in\mathbb{Z})$ and $\lim_{t\to t_s^+}\rho(t)(Tx)'(t)(s\in\mathbb{Z})$ exist. One can show easily from $\lim_{t\to\pm\infty}\sigma(t)=+\infty$, $\lim_{t\to\pm\infty}\tau(t)=+\infty$ and (5) that

$$\lim_{t\to\pm\infty}\frac{\int_{-\infty}^{+\infty}m(s)\phi_x(s)\,\mathrm{d}s+\sum_{\xi\leqslant t_s<t}I_x(t_s)}{\sigma(t)}=0,$$
$$\lim_{t\to\pm\infty}\frac{A_x+\sum_{\eta\leqslant t_s<t}J_x(t_s)}{\tau(t)}=0.$$

On the other hand, for $t>\xi$, we have by using l'Hôpital's rule that

$$\lim_{t\to+\infty}\frac{\int_\xi^t\frac{1}{\rho(u)}\Phi^{-1}(A_x+\sum_{\eta\leqslant t_s<u}J_x(t_s)-\int_\eta^u p(w)f_x(w)\,\mathrm{d}w)\,\mathrm{d}u}{\sigma(t)}$$
$$=-\lim_{t\to+\infty}\Phi^{-1}\left(f\left(t,\sigma(t)\frac{x(t)}{\sigma(t)},\Phi^{-1}\big(\tau(t)\big)\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\right)\right).$$

Since for each $r>0$, $f(t,\sigma(t)u,\Phi^{-1}(\tau(t))v)$ converges uniformly as $t\to\pm\infty$ on $[-r,r]\times[-r,r]$, then

$$\lim_{t\to+\infty}\frac{\int_\xi^t\frac{1}{\rho(u)}\Phi^{-1}(A_x+\sum_{\eta\leqslant t_s<u}J_x(t_s)-\int_\eta^u p(w)f_x(w)\,\mathrm{d}w)\,\mathrm{d}u}{\sigma(t)}\quad\text{exists.}$$

So $\lim_{t\to+\infty}(Tx)(t)/\sigma(t)$ exists. Similarly we can show that $\lim_{t\to-\infty}(Tx)(t)/\sigma(t)$ and $\lim_{t\to\pm\infty}\rho(t)(Tx)'(t)/\Phi^{-1}(\tau(t))$ exist. It follows that $Tx\in X$. Hence $T:X\to X$ is well defined.

(ii) By direct computation, we can get

$$\begin{aligned}
&\big[\Phi\big(\rho(t)(Tx)'(t)\big)\big]'+p(t)f\big(t,x(t),\rho(t)x'(t)\big)=0,\quad \text{a.e. } t\in\mathbb{R},\\
&(Tx)(\xi)=\int_{-\infty}^{+\infty}m(s)\phi\big(s,x(s),\rho(s)x'(s)\big)\,\mathrm{d}s,\\
&\big(Tx\big)(\eta)=\int_{-\infty}^{+\infty}n(s)\psi\big(s,x(s),\rho(s)x'(s)\big)\,\mathrm{d}s,\\
&\Delta(Tx)(t_i)=I\big(t_i,x(t_i),\rho(t_i)x'(t_i)\big),\quad i\in\mathbb{Z},\\
&\Delta\Phi\big(\rho(t_i)(Tx)'(t_i)\big)=J\big(t_i,x(t_i),\rho(t_i)x'(t_i)\big),\quad i\in\mathbb{Z}.
\end{aligned}$$

Thus it follows that $x\in X$ is a solution of (1) if and only if $x\in X$ is a fixed point of T in X.

(iii) Now we prove that T is completely continuous. The following five steps are needed (Steps 1–2 imply that $T:X\to X$ is continuous, and Steps 3–5 imply that T maps bounded sets into relatively compact sets). We omit the details of the proofs.

Step 1. We prove that the function $A_x : X \to \mathbb{R}$ is continuous in x.
Step 2. We show that T is continuous on X. Since A_x is continuous, f, ϕ, ψ are strong Carathéodory functions, I, J are discrete Carathédory functions, then the result follows.
Step 3. We show that T is maps bounded subsets into bounded sets.
Step 4. We prove that both $\{t \to x(t)/\sigma(t)\colon x \in M\}$ and $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ are equicontinuous in $(t_s, t_{s+1}](s \in \mathbb{Z})$.
Step 5. We show that both $\{t \to x(t)/\sigma(t)\colon x \in M\}$ and $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ are equi-convergent as $t \to \pm\infty$.

From Steps 3–5 and Lemma 2 we see that T maps bounded sets into relatively compact sets.

Therefore, the operator $T : X \to X$ is completely continuous. The proof of (iii) is complete. The proof is complete. □

Now, we address the first result of this paper. We need the following assumption.

Assumption A. There exist nonnegative constants $A_j, a_{ij}, b_{ij} \geqslant 0$ $(i = 1, 2,\ j = 0, 1, 2, \ldots, m)$, $\phi_s \geqslant 0$, $\psi_s \geqslant 0$ $(s \in \mathbb{Z})$ and $k_j, l_j \geqslant 0$ $(j = 1, 2, \ldots, m)$ with $k_j + l_j > 0$ and $\sum_{s=-\infty}^{+\infty} \phi_s < +\infty$, $\sum_{s=-\infty}^{+\infty} \psi_s < +\infty$ and

$$\bigl|f\bigl(t, \sigma(t)u, \Phi^{-1}\bigl(\tau(t)\bigr)v\bigr)\bigr| \leqslant \Phi\Biggl(A_0 + \sum_{j=1}^{m} A_j|u|^{k_j}|v|^{l_j}\Biggr), \quad u, v \in \mathbb{R},\ \text{a.e. } t \in \mathbb{R},$$

$$\bigl|I\bigl(t_s, \sigma(t_s)u, \Phi^{-1}\bigl(\tau(t_s)\bigr)v\bigr)\bigr| \leqslant \phi_s\Biggl[b_{10} + \sum_{j=1}^{m} b_{1j}|u|^{k_j}|v|^{l_j}\Biggr], \quad u, v \in \mathbb{R},\ s \in \mathbb{Z},$$

$$\bigl|J\bigl(t_s, \sigma(t_s)u, \Phi^{-1}\bigl(\tau(t_s)\bigr)v\bigr)\bigr| \leqslant \psi_s\Phi\Biggl(b_{20} + \sum_{j=1}^{m} b_{2j}|u|^{k_j}|v|^{l_j}\Biggr), \quad u, v \in \mathbb{R},\ s \in \mathbb{Z},$$

$$\bigl|\phi\bigl(t, \sigma(t)u, \Phi^{-1}\bigl(\tau(t)\bigr)v\bigr)\bigr| \leqslant a_{10} + \sum_{j=1}^{m} a_{1j}|u|^{k_j}|v|^{l_j}, \quad u, v \in \mathbb{R},\ \text{a.e. } t \in \mathbb{R},$$

$$\bigl|\psi\bigl(t, \sigma(t)u, \Phi^{-1}\bigl(\tau(t)\bigr)v\bigr)\bigr| \leqslant a_{20} + \sum_{j=1}^{m} a_{2j}|u|^{k_j}|v|^{l_j}, \quad u, v \in \mathbb{R},\ \text{a.e. } t \in \mathbb{R}.$$

We denote $\sigma = \max\{k_j + l_j\colon j = 1, 2, \ldots, m\}$ and

$$\begin{aligned}\overline{A}_1 = {}& \Biggl(\frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} + 1\Biggr)\|m\|_1 a_{10} + \sum_{s=-\infty}^{+\infty} \phi_s\Biggl(1 + \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\Biggr)b_{10} + \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\|n\|_1 a_{20} \\ & + 2\Phi^{-1}\Biggl(\sum_{s=-\infty}^{+\infty} \psi_s\Biggr)c_l b_{2,0} + \Biggl(1 + \Phi^{-1}\Biggl(\int_\xi^\eta p(w)\,\mathrm{d}w\Biggr)\Biggr)c_l A_0,\end{aligned}$$

$$\overline{B}_{1,j} = \left(\frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} + 1\right)\|m\|_1 a_{1j} + \sum_{s=-\infty}^{+\infty} \phi_s \left(\frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} + 1\right) b_{1j} + \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\|n\|_1 a_{2j}$$
$$+ 2\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) c_l b_{2,j} + \left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right) + 1\right) c_l A_j,$$

$$\overline{A}_2 = \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\|m\|_1 a_{10} + \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\|n\|_1 a_{20} + \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} \sum_{s=-\infty}^{+\infty} \phi_s b_{10}$$
$$+ 2c_l\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) b_{20} + \left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right) + 1\right) c_l A_0,$$

$$\overline{B}_{2,j} = \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\|m\|_1 a_{1j} + \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\|n\|_1 a_{2j} + \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} \sum_{s=-\infty}^{+\infty} \phi_s b_{1j}$$
$$+ 2c_l\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty} \psi_s\right) b_{2j} + \left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right) + 1\right) c_l A_j,$$

and

$$A = \max\{\overline{A}_1, \overline{A}_2\}, \qquad B = \max\left\{\sum_{j=1}^m \overline{B}_{1,j}, \sum_{j=1}^m \overline{B}_{2,j}\right\}.$$

Theorem 1. *Suppose that Assumption* A *holds, f is a strong Carathéodory function, I, J are discrete Carathéodory functions, and for each $r > 0$, $f(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)$ converges uniformly as $t \to \pm\infty$ on $[-r, r] \times [-r, r]$. Then BVP* (1) *has at least one solution if*

(i) $\sigma \in (0, 1)$ *or*
(ii) $\sigma = 1$ *and* $B < 1$ *or*
(iii) $\sigma > 1$ *and* $B(A + B)^{\sigma-1} \leqslant (\sigma - 1)^{\sigma-1}/\sigma^\sigma$.

Proof. Let X and T be defined above. From Lemma 3, $T : X \to X$ is well defined and is a completely continuous operator. We prove that T has a fixed point in X to get a solution of BVP (1). For $x \in X$, we have $\|x\| \leqslant r < +\infty$. Then Assumption A implies that

$$\begin{aligned}\left|f\big(t, x(t), \rho(t)x'(t)\big)\right| &= \left|f\left(t, \sigma(t)\frac{x(t)}{\sigma(t)}, \Phi^{-1}\big(\tau(t)\big)\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\right)\right| \\ &\leqslant \Phi\left(A_0 + \sum_{j=1}^m A_j \left|\frac{x(t)}{\sigma(t)}\right|^{k_j} \left|\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\right|^{l_j}\right) \\ &\leqslant \Phi\left(A_0 + \sum_{j=1}^m A_j r^{k_j + l_j}\right), \quad \text{a.e. } t \in \mathbb{R},\end{aligned}$$

$$\left|I\big(t_s,x(t_s),\rho(t_s)x'(t_s)\big)\right| \leqslant \phi_s\left(b_{10}+\sum_{j=1}^{m} b_{1j}r^{k_j+l_j}\right), \quad s\in\mathbb{Z},$$

$$\left|J\big(t_s,x(t_s),\rho(t_s)x'(t_s)\big)\right| \leqslant \psi_s\Phi\left(b_{20}+\sum_{j=1}^{m} b_{2j}r^{k_j+l_j}\right), \quad s\in\mathbb{Z},$$

$$\left|\phi\big(t,x(t),\rho(t)x'(t)\big)\right| \leqslant a_{10}+\sum_{j=1}^{m} a_{1j}r^{k_j+l_j}, \quad \text{a.e. } t\in\mathbb{R},$$

$$\left|\psi\big(t,x(t),\rho(t)x'(t)\big)\right| \leqslant a_{20}+\sum_{j=1}^{m} a_{2j}r^{k_j+l_j}, \quad \text{a.e. } t\in\mathbb{R}.$$

By the definition of T, we get by using (4) that

$$\begin{aligned}
|A_x| \leqslant{}& \sum_{\xi\leqslant t_s<\eta} \psi_s\Phi\left(b_{20}+\sum_{j=1}^{m} b_{2j}r^{k_j+l_j}\right)+\int\limits_{\xi}^{\eta} p(w)\,\mathrm{d}w\,\Phi\left(A_0+\sum_{j=1}^{m} A_j r^{k_j+l_j}\right)\\
&+\Phi\bigg(\frac{\|m\|_1(a_{10}+\sum_{j=1}^m a_{1j}r^{k_j+l_j})+\|n\|_1(a_{20}+\sum_{j=1}^m a_{2j}r^{k_j+l_j})}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\\
&+\frac{\sum_{\xi\leqslant t_s<\eta}\phi_s(b_{10}+\sum_{j=1}^m b_{1j}r^{k_j+l_j})}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\bigg).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{|(Tx)(t)|}{\sigma(t)} \leqslant{}& \|m\|_1\left[a_{10}+\sum_{j=1}^{m} a_{1j}r^{k_j+l_j}\right]+\sum_{s=-\infty}^{+\infty}\phi_s\left[b_{1,0}+\sum_{j=1}^{m} b_{1,j}r^{k_j+l_j}\right]\\
&+\Phi^{-1}\Bigg(\sum_{\xi\leqslant t_s<\eta}\psi_s\Phi\left(b_{20}+\sum_{j=1}^{m} b_{2j}r^{k_j+l_j}\right)\\
&+\int\limits_{\xi}^{\eta} p(w)\,\mathrm{d}w\,\Phi\left(A_0+\sum_{j=1}^{m} A_j r^{k_j+l_j}\right)\\
&+\Phi\bigg(\frac{\|m\|_1(a_{10}+\sum_{j=1}^m a_{1j}r^{k_j+l_j})+\|n\|_1(a_{20}+\sum_{j=1}^m a_{2j}r^{k_j+l_j})}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\\
&+\frac{\sum_{\xi\leqslant t_s<\eta}\phi_s(b_{10}+\sum_{j=1}^m b_{1j}r^{k_j+l_j})}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\bigg)\\
&+\sum_{s=-\infty}^{+\infty}\psi_s\Phi\left(b_{2,0}+\sum_{j=1}^{m} b_{2,j}r^{k_j+l_j}\right)+\Phi\left(A_0+\sum_{j=1}^{m} A_j r^{k_j+l_j}\right)\Bigg).
\end{aligned}$$

One knows that

$$\Phi^{-1}(u+v) \leqslant c_l\big[\Phi^{-1}(u)+\Phi^{-1}(v)\big], \quad u,v \geqslant 0 \text{ with } c_l = \begin{cases} 1, & 1<l<2, \\ 2^{l-1}, & l \geqslant 2. \end{cases}$$

It follows that

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \frac{|(Tx)(t)|}{\sigma(t)} \leqslant{}& \left(\frac{c_l\|m\|_1}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} + \|m\|_1\right)a_{10} + \left(\sum_{s=-\infty}^{+\infty}\phi_s + \frac{c_l\sum_{\xi\leqslant t_s<\eta}\phi_s}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\right)b_{10} \\
&+ \frac{c_l\|n\|_1}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}a_{20} + \left(c_l\Phi^{-1}\left(\sum_{\xi\leqslant t_s<\eta}\psi_s\right) + c_l\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty}\psi_s\right)\right)b_{2,0} \\
&+ \left(c_l + c_l\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right)\right)A_0 \\
&+ \sum_{j=1}^m \left[a_{1j}\left(\frac{c_l\|m\|_1}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} + \|m\|_1\right) + b_{1j}\left(\frac{c_l\sum_{\xi\leqslant t_s<\eta}\phi_s}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} + \sum_{s=-\infty}^{+\infty}\phi_s\right)\right. \\
&+ a_{2j}\frac{c_l\|n\|_1}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}} + b_{2,j}\left(c_l\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty}\psi_s\right) + c_l\Phi^{-1}\left(\sum_{\xi\leqslant t_s<\eta}\psi_s\right)\right) \\
&\left.+ A_j\left(c_l\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right) + c_l\right)\right] r^{k_j+l_j}.
\end{aligned}$$

Then

$$\sup_{t\in\mathbb{R}} \frac{|(Tx)(t)|}{\sigma(t)} \leqslant \overline{A}_1 + \sum_{j=1}^m \overline{B}_{1,j} r^{k_j+l_j}. \tag{6}$$

On the other hand, we have

$$\begin{aligned}
\frac{\rho(t)|(Tx)'(t)|}{\Phi^{-1}(\tau(t))} \leqslant{}& \frac{\|m\|_1 c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}a_{10} + \frac{\|n\|_1 c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}a_{20} + \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\sum_{s=-\infty}^{+\infty}\phi_s b_{10} \\
&+ 2c_l\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty}\psi_s\right)b_{20} + \left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right)+1\right)c_l A_0 \\
&+ \sum_{j=1}^m \left[c_l\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right)A_j + \frac{\|m\|_1 c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}a_{1j} + \frac{\|n\|_1 c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}a_{2j}\right. \\
&\left.+ \frac{c_l}{\int_\xi^\eta \frac{\mathrm{d}u}{\rho(u)}}\sum_{s=-\infty}^{+\infty}\phi_s b_{1j} + 2c_l\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty}\psi_s\right)b_{2j} + c_l A_j\right] r^{k_j+l_j}.
\end{aligned}$$

It follows that
$$\sup_{t\in\mathbb{R}}\frac{\rho(t)|(Tx)'(t)|}{\Phi^{-1}(\tau(t))}\leqslant\overline{A}_2+\sum_{j=1}^{m}\overline{B}_{2,j}\|x\|^{k_j+l_j}. \tag{7}$$
It follows from (6) and (7) that $\|Tx\|\leqslant A+B\max\{\|(x,y)\|^{\sigma},1\}\leqslant A+B+B\|x\|^{\sigma}$.

(i) $\sigma\in(0,1)$. Since $\sigma\in(0,1)$, choose $r_0>0$ such that $A+B+Br_0^{\sigma}\leqslant r_0$. Let $\Omega_0=\{x\in X\colon \|x\|\leqslant r_0\}$. Then we get $\|Tx\|\leqslant A+B+Br_0^{\sigma}\leqslant r_0$. So $T\overline{\Omega_0}\subset\overline{\Omega_0}$. Thus Schauder's fixed point theorem implies that the operator T has at least one fixed point in $\overline{\Omega_0}$. So BVP (1) has at least one solution.

(ii) $\sigma=1$ and $B<1$. Let $r_0=(A+B)/(1-B)$ such that $A+B+Br_0=r_0$. Let $\Omega_0=\{x\in X\colon \|x\|\leqslant r_0\}$. Then we get $\|Tx\|\leqslant A+B+Br_0^{\sigma}\leqslant r_0$. So $T\overline{\Omega_0}\subset\overline{\Omega_0}$. Thus Schauder's fixed point theorem implies that the operator T has at least one fixed point in $\overline{\Omega_0}$. So BVP (1) has at least one solution.

(iii) $\sigma>1$ and $B(A+B)^{\sigma-1}\leqslant(\sigma-1)^{\sigma-1}/\sigma^{\sigma}$. Let $r_0=((A+B)/(B(\sigma-1))^{1/\sigma}$. It is easy to show from $(A+B)^{\sigma-1}\sigma^{\sigma}/(\sigma-1)^{\sigma-1}\leqslant 1/B$ that $A+B+Br_0^{\sigma}\leqslant r_0$. Let $\Omega_0=\{x\in X\colon \|x\|\leqslant r_0\}$. Then we get $\|Tx\|\leqslant A+B+Br_0^{\sigma}\leqslant r_0$. So $T\overline{\Omega_0}\subset\overline{\Omega_0}$. Thus Schauder's fixed point theorem implies that the operator T has at least one fixed point in $\overline{\Omega_0}$. So BVP(1) has at least one solution.

The proof of Theorem 1 is completed. □

3 Solvability of (1) in Case 2

In this section, we present existence result of BVP (1) in Case 2. Denote $\tau(t)=1+|\int_{\eta}^{t}p(s)\,\mathrm{d}s|$. It is easy to show that τ are continuous on $\mathbb{R}$ and $\lim_{t\to\pm\infty}\tau(t)=+\infty$.

Definition 4. $F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a strong Carathédory function if

(i) $t\to F(t,u,\Phi^{-1}(\tau(t))v)$ is measurable on $\mathbb{R}$ for any $u,v\in\mathbb{R}$;
(ii) $(u,v)\to F(t,u,\Phi^{-1}(\tau(t))v)$ is continuous on $\mathbb{R}^2$ for a.e. $t\in\mathbb{R}$;
(iii) for each $r>0$, there exists nonnegative function $M_r\geqslant 0$ such that $|u|,|v|\leqslant r$ implies $|F(t,u,\Phi^{-1}(\tau(t))v)|\leqslant M_r$, a.e. $t\in\mathbb{R}$.

Definition 5. $H:\{t_i\colon i\in\mathbb{Z}\}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a discrete Carathédory function if

(i) $(x,y)\to H(t_s,x,\Phi^{-1}(\tau(t_s))y)$ is continuous on $\mathbb{R}^2$ for all $s\in\mathbb{Z}$;
(ii) for each $r>0$, there exists nonnegative constants $M_{ir}\geqslant 0$ $(i\in\mathbb{Z})$ such that $|x|,|y|\leqslant r$ implies $|H(t_s,x,\Phi^{-1}(\tau(t_s))y)|\leqslant M_{sr}$, $s\in\mathbb{Z}$, $\sum_{s=-\infty}^{+\infty}M_{sr}<+\infty$.

Choose
$$X=\Bigg\{x:\mathbb{R}\to\mathbb{R}\colon\ x|_{(t_s,t_{s+1}]},\ \rho x'|_{(t_s,t_{s+1}]}\ \text{is continuous},\ s\in\mathbb{Z};$$
$$\text{the following limits exist and are finite:}$$
$$\lim_{t\to t_s^+}x(t),\ \lim_{t\to t_s^+}\rho(t)x'(t),\ s\in\mathbb{Z},\ \lim_{t\to\pm\infty}x(t),\ \lim_{t\to\pm\infty}\frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\Bigg\}.$$

For $x\in X$, define $\|x\|=\max\{\sup_{t\in\mathbb{R}}|x(t)|,\ \sup_{t\in\mathbb{R}}\rho(t)|x'(t)|/\Phi^{-1}(\tau(t))\}$.

Lemma 5. *X is a Banach space with $\|\cdot\|$ defined.*

Proof. It is similar to the proof of Lemma 1 and is omitted. □

Lemma 6. *Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied*:

(i) *both $\{t \to x(t)\colon x \in M\}$ and $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ are uniformly bounded*;
(ii) *both $\{t \to x(t)\colon x \in M\}$ and $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ are equi-continuous in $(t_s, t_{s+1}]$ ($s \in N$)*;
(iii) *both $\{t \to x(t)\colon x \in M\}$ and $\{t \to \rho(t)x'(t)/\Phi^{-1}(\tau(t))\colon x \in M\}$ are equi-convergent as $t \to \pm\infty$.*

Proof. It is similar to the proof of Lemma 2 and is omitted. □

For $x \in X$, let $(Tx)(t)$ be defined by

$$(Tx)(t) = \int_{-\infty}^{+\infty} m(s)\phi_x(s)\,\mathrm{d}s + \sum_{\xi \leqslant t_s < t} I_x(t_s) + \int_{\xi}^{t} \frac{1}{\rho(u)}\Phi^{-1}\left(A_x + \sum_{\eta \leqslant t_s < u} J_x(t_s) - \int_{\eta}^{u} p(w)f_x(w)\,\mathrm{d}w\right)\mathrm{d}u, \quad t \in \mathbb{R},$$

where A_x is defined by (3).

Lemma 7. *Suppose that f is a strong Carathéodory function and I, J are discrete Carathéodory functions, and for each $r > 0$, $f(t, u, \Phi^{-1}(\tau(t))v)$ converges uniformly as $t \to \pm\infty$ on $[-r, r] \times [-r, r]$. Then $T : X \to X$ is well defined and is completely continuous, $x \in X$ is a solution of* (1) *if and only if $x \in X$ is a fixed point of T in X.*

Proof. It is similar to the proof of Lemma 4 and is omitted. □

To state and prove Theorem 2, we need the following assumption.

Assumption B. There exist nonnegative constants $A_j, a_{ij}, b_{ij} \geqslant 0$ ($i = 1, 2,\ j = 0, 1, 2, \dots, m$), $\phi_s \geqslant 0$, $\psi_s \geqslant 0$ ($s \in \mathbb{Z}$) and $k_j, l_j \geqslant 0$ ($j = 1, 2, \dots, m$) with $k_j + l_j > 0$ and $\sum_{s=-\infty}^{+\infty} \phi_s < +\infty$, $\sum_{s=-\infty}^{+\infty} \psi_s < +\infty$ and

$$\left|f\left(t, u, \Phi^{-1}\left(\tau(t)\right)v\right)\right| \leqslant \Phi\left(A_0 + \sum_{j=1}^{m} A_j|u|^{k_j}|v|^{l_j}\right), \quad u, v \in \mathbb{R},\ \text{a.e. } t \in \mathbb{R},$$

$$\left|I\left(t_s, u, \Phi^{-1}\left(\tau(t_s)\right)v\right)\right| \leqslant \phi_s\left[b_{10} + \sum_{j=1}^{m} b_{1j}|u|^{k_j}|v|^{l_j}\right], \quad u, v \in \mathbb{R},\ s \in \mathbb{Z},$$

$$\left|J\bigl(t_s,u,\Phi^{-1}\bigl(\tau(t_s)\bigr)v\bigr)\right| \leqslant \psi_s \Phi(b_{20}+\sum_{j=1}^{m} b_{2j}|u|^{k_j}|v|^{l_j}), \quad u,v\in\mathbb{R},\ s\in\mathbb{Z},$$

$$\left|\phi\bigl(t,u,\Phi^{-1}\bigl(\tau(t)\bigr)v\bigr)\right| \leqslant a_{10}+\sum_{j=1}^{m} a_{1j}|u|^{k_j}|v|^{l_j}, \quad u,v\in\mathbb{R},\ \text{a.e. } t\in\mathbb{R},$$

$$\left|\psi\bigl(t,u,\Phi^{-1}\bigl(\tau(t)\bigr)v\bigr)\right| \leqslant a_{20}+\sum_{j=1}^{m} a_{2j}|u|^{k_j}|v|^{l_j}, \quad u,v\in\mathbb{R},\ \text{a.e. } t\in\mathbb{R}.$$

We denote $\sigma=\max\{k_j+l_j\colon\ j=1,2,\dots,m\}$ and

$$\begin{aligned}
\overline{A}_1 &= \left(\frac{c_l\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}+1\right)\|m\|_1 a_{10}+\sum_{s=-\infty}^{+\infty}\phi_s\left(1+\frac{c_l\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\mathrm{d}u}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\right)b_{10}\\
&\quad+\frac{c_l\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\mathrm{d}u}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\|n\|_1 a_{20}+2\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty}\psi_s\right)\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u c_l b_{2,0}\\
&\quad+\left(1+\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right)\right)\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u c_l A_0,
\end{aligned}$$

$$\begin{aligned}
\overline{B}_{1,j} &= \left(\frac{c_l\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\mathrm{d}u}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}+1\right)\|m\|_1 a_{1j}+\sum_{s=-\infty}^{+\infty}\phi_s\left(\frac{c_l\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\mathrm{d}u}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}+1\right)b_{1j}\\
&\quad+\frac{c_l\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\|n\|_1 a_{2j}+2\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty}\psi_s\right)\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u c_l b_{2,j}\\
&\quad+\left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right)+1\right)\int_{-\infty}^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u c_l A_j,
\end{aligned}$$

$$\begin{aligned}
\overline{A}_2 &= \frac{c_l}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\|m\|_1 a_{10}+\frac{c_l}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\|n\|_1 a_{20}+\frac{c_l}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\sum_{s=-\infty}^{+\infty}\phi_s b_{10}\\
&\quad+2c_l\Phi^{-1}\left(\sum_{s=-\infty}^{+\infty}\psi_s\right)b_{20}+\left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right)+1\right)c_l A_0,
\end{aligned}$$

$$\begin{aligned}
\overline{B}_{2,j} &= \frac{c_l}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\|m\|_1 a_{1j}+\frac{c_l}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\|n\|_1 a_{2j}+\frac{c_l}{\int_\xi^\eta\frac{\mathrm{d}u}{\rho(u)}}\sum_{s=-\infty}^{+\infty}\phi_s b_{1j}\\
&\quad+2c_l\Phi^{-1}(\sum_{s=-\infty}^{+\infty}\psi_s)b_{2j}+\left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right)+1\right)c_l A_j,
\end{aligned}$$

and

$$A = \max\{\overline{A}_1, \overline{A}_2\}, \qquad B = \max\left\{\sum_{j=1}^{m} \overline{B}_{1,j}, \sum_{j=1}^{m} \overline{B}_{2,j}\right\}.$$

Theorem 2. *Suppose that Assumption* B *holds, f is a strong Carathéodory function, I, J are discrete Carathéodory functions, and for each $r > 0$, $f(t, u, \Phi^{-1}(\tau(t))v)$ converges uniformly as $t \to \pm\infty$ on $[-r, r] \times [-r, r]$. Then BVP* (1) *has at least one solution if*

(i) $\sigma \in (0, 1)$ *or*
(ii) $\sigma = 1$ *and* $B < 1$ *or*
(iii) $\sigma > 1$ *and* $B(A + B)^{\sigma-1} \leqslant (\sigma - 1)^{\sigma-1}/\sigma^{\sigma}$.

Proof. Let X and T be defined above. From Lemma 7, $T : X \to X$ is well defined and is a completely continuous operator. We prove that T has a fixed point in X to get a solution of BVP (1). The proof is similar to that of Theorem 1 and is omitted. □

Remark 1. By constructing suitable Banach spaces similar to X in Sections 2 and 3, one can establish existence results on solvability of BVP (1) for Cases 3 and 4, respectively.

4 Examples

In this section, we present examples to illustrate the main result.

Example 1. We consider the following BVP:

$$\begin{aligned}
&\big[|t-1.5|\big|x'(t)\big|x'(t)\big]' + \frac{1}{2\sqrt{|t-1.5|}}\frac{1}{1+|t|}g\big(t, x(t), x'(t)\big) = 0,\\
&\quad \text{a.e. } t \in \mathbb{R},\\
&x(-1.5) = \sqrt{\pi}, \qquad x(1.5) = 2\sqrt{\pi},\\
&\Delta x(s) = 2^{-|s|}, \quad \Delta\big[|s-1.5|\big|x'(s)\big|x'(s)\big] = 2^{-|s|}, \quad s \in \mathbb{Z},
\end{aligned} \tag{8}$$

where

$$g(t, u, v) = \left(A_0 + \sum_{j=1}^{m} A_j \left|\frac{u}{\sigma_0(t)}\right|^{k_j} \left|\frac{v}{\sqrt{1+|t-1.5|}}\right|^{l_j}\right)^2,$$

$$\sigma_0(t) = \begin{cases} -\frac{5}{3} + \frac{4}{3}(1+\sqrt{t-1.5})^{3/2} + \frac{4}{3}(1+\sqrt{3})^{3/2}, & t \geqslant 1.5,\\ 1 + \frac{4}{3}(1+\sqrt{3})^{3/2} - \frac{4}{3}(1+\sqrt{1.5-t})^{3/2}, & t \in [-1.5, 1.5),\\ 1 + \frac{4}{3}(1+\sqrt{1.5-t})^{3/2} - \frac{4}{3}(1+\sqrt{3})^{3/2}, & t \leqslant -1.5, \end{cases}$$

$A_j \geqslant 0$, $k_j, l_j \geqslant 0$ $(j = 0, 1, 2, \dots, m)$.

Corresponding to BVP (1), we have

$$\Phi(x)=|s|s,\qquad \Phi^{-1}(x)=|s|^{-1/2}s,$$
$$\rho(t)=\sqrt{|t-1.5|},\qquad p(t)=\frac{1}{2\sqrt{|t-1.5|}},\qquad f(t,u,v)=\frac{1}{1+|t|}g(t,u,v),$$
$$\xi=-1.5,\qquad \eta=1.5,\qquad t_s=s,\quad s\in\mathbb{Z},$$
$$\phi(t,u,v)=1,\qquad \psi(t,u,v)=2,\qquad m(t)=n(t)=\mathrm{e}^{-t^2},$$
$$I(s,u,v)=J(s,u,v)=2^{-|s|},\quad s\in\mathbb{Z}.$$

One sees that

(i) $\Phi(x)=|x|^{k-2}x$ with $k=3>1$, the inverse of Φ is denoted by Φ^{-1} and $\Phi^{-1}(x)=|x|^{l-2}x$ with $l=3/2$, then $c_l=1$.

(ii) p is nonnegative and satisfies $p\in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\int_{-\infty}^0 p(t)\,\mathrm{d}t=\int_0^{+\infty}p(s)\,\mathrm{d}s=+\infty$.

(iii) $\rho:\mathbb{R}\to[0,+\infty)$ and

$$\tau(t)=1+\left|\int\limits_{\eta}^{t}p(s)\,\mathrm{d}s\right|=1+\left|\int\limits_{1.5}^{t}p(s)\,\mathrm{d}s\right|=1+\sqrt{|t-1.5|},$$
$$\sigma(t)=\begin{cases}-\frac{5}{3}+\frac{4}{3}(1+\sqrt{t-1.5})^{3/2}+\frac{4}{3}(1+\sqrt{3})^{3/2}, & t\geqslant 1.5,\\ 1+\frac{4}{3}(1+\sqrt{3})^{3/2}-\frac{4}{3}(1+\sqrt{1.5-t})^{3/2}, & t\in[-1.5,1.5),\\ 1+\frac{4}{3}(1+\sqrt{1.5-t})^{3/2}-\frac{4}{3}(1+\sqrt{3})^{3/2}, & t\leqslant -1.5,\end{cases}$$
$$=\sigma_0(t)$$

with $\Phi^{-1}(\tau(\cdot))/\rho(\cdot)\in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\int_{-\infty}^0\Phi^{-1}(\tau(u))/\rho(u)\,\mathrm{d}u=\int_0^{+\infty}\Phi^{-1}(\tau(u))/\rho(u)\,\mathrm{d}u=+\infty$, and f is a strong Carathéodory function and I,J are discrete Carathéodory functions and for each $r>0$,

$$f\bigl(t,\sigma(t)u,\Phi^{-1}\bigl(\tau(t)\bigr)v\bigr)=\frac{1}{1+|t|}\left(A_0+\sum_{j=1}^{m}A_j|u|^{k_j}|v|^{l_j}\right)^2\to 0$$

uniformly as $t\to\pm\infty$ on $[-r,r]\times[-r,r]$.

(iv) $f,\phi,\psi:\mathbb{R}^3\to\mathbb{R}$ are strong Carathéodory functions, $m,n\in L^1(\mathbb{R})$.

(v) $\{t_s\colon s\in\mathbb{Z}\}$ is a increasing sequence with $\lim_{s\to-\infty}t_s=-\infty$ and $\lim_{s\to+\infty}t_s=+\infty$.

(vi) $I,J:\{t_s\colon s\in\mathbb{Z}\}\times\mathbb{R}^2\to\mathbb{R}$ are discrete Carathéodory functions.

Choose A_j $(j=0,1,2,\dots,m)$ and

$$b_{1,0}=b_{2,0}=1,\quad b_{1,j}=b_{2,j}=0=0,\quad j=1,2,\dots,m,\qquad \phi_s=2^{-|s|},\quad s\in\mathbb{Z},$$
$$a_{1,0}=1,\quad a_{2,0}=2,\quad a_{1,j}=a_{2,j}=0,\quad j=1,2,\dots,m,\qquad \psi_s=2^{-|s|},\quad s\in\mathbb{Z}.$$

Then Assumption A holds and $\sum_{s=-\infty}^{+\infty}\phi_s = \sum_{s=-\infty}^{+\infty}\psi_s = 3$. Denote $\sigma = \max\{k_j + l_j\colon\ j = 1,2,\dots,m\}$ and by direct computation, we get

$$\int_\xi^\eta \rho(w)\,\mathrm{d}w = \int_{-1.5}^{1.5}\sqrt{|s-1.5|}\,\mathrm{d}s = 2\sqrt{3},$$

$$\int_\xi^\eta p(w)\,\mathrm{d}w = \int_{-1.5}^{1.5}\frac{1}{2\sqrt{|s-1.5|}}\,\mathrm{d}s = \sqrt{3},$$

$$\overline{A}_1 = \frac{2+\sqrt{3}}{2}\sqrt{\pi} + \frac{6+5\sqrt{3}}{2} + (1+\sqrt[4]{3})A_0, \qquad \overline{B}_{1,j} = (1+\sqrt[4]{3})A_j,$$
$$\overline{A}_2 = \frac{\sqrt{3}}{2}\sqrt{\pi} + \frac{5\sqrt{3}}{2} + (1+\sqrt[4]{3})A_0, \qquad \overline{B}_{2,j} = (1+\sqrt[4]{3})A_j,$$

and

$$A = \max\{\overline{A}_1,\ \overline{A}_2\} = \frac{2+\sqrt{3}}{2}\sqrt{\pi} + \frac{6+5\sqrt{3}}{2} + (1+\sqrt[4]{3})A_0,$$
$$B = \max\left\{\sum_{j=1}^m \overline{B}_{1,j},\ \sum_{j=1}^m \overline{B}_{2,j}\right\} = (1+\sqrt[4]{3})\sum_{j=1}^m A_j.$$

By Theorem 1, BVP (8) has at least one solution if (i) $\sigma \in (0,1)$ or (ii) $\sigma = 1$ and $(\sqrt[4]{3}+1)\sum_{j=1}^m A_j < 1$ or (iii) $\sigma > 1$ and

$$\left(\frac{2+\sqrt{3}}{2}\sqrt{\pi} + \frac{6+5\sqrt{3}}{2} + (1+\sqrt[4]{3})A_0 + (\sqrt[4]{3}+1)\sum_{j=1}^m A_j\right)^{\sigma-1}\sum_{j=1}^m A_j \leqslant \frac{(\sigma-1)^{\sigma-1}}{(\sqrt[4]{3}+1)\sigma^\sigma}.$$

Example 2. We consider the following BVP:

$$\begin{aligned}
&\left[\left(\rho(t)x'(t)\right)^3\right]' + \frac{1}{1+\sqrt{|t|}}g\left(t, x(t), \sqrt{|t|}x'(t)\right) = 0, \quad \text{a.e. } t\in\mathbb{R},\\
&x(-1) = \sqrt{\pi}, \qquad x(1) = 2\sqrt{\pi},\\
&\Delta x(s) = \lim_{t\to s^+} x(t) - x(s) = 2^{-|s|}, \quad s\in\mathbb{Z},\\
&\Delta\left(\sqrt{|s|}x'(s)\right)^3 = \lim_{t\to s^+}\left(\sqrt{|t|}x'(t)\right)^3 - \left(\sqrt{|s|}x'(s)\right)^3 = 2^{-|s|}, \quad s\in\mathbb{Z},
\end{aligned} \tag{9}$$

where

$$\rho(t) = \begin{cases}\sqrt{|t-1|}, & t\in[0,2],\\ (t-1)^4, & |t-1|>1,\end{cases}$$

$$g(t,u,v) = \left(A_0 + \sum_{j=1}^m A_j|u|^{k_j}\left|\frac{v}{\sqrt[3]{1+|t-1|}}\right|^{l_j}\right)^3,$$

and $A_j \geqslant 0$, $k_j, l_j \geqslant 0$ $(j = 0,1,2,\dots,m)$.

Corresponding to BVP (1), we have

$$\Phi(x)=x^3,\qquad \Phi^{-1}(x)=x^{1/3},\qquad \xi=-1,\qquad \eta=1,\qquad t_s=s,\quad s\in\mathbb{Z},$$
$$p(t)=1,\qquad f(t,u,v)=\frac{1}{1+\sqrt{|t|}}g(t,u,v),$$
$$\phi(t,u,v)=1,\qquad \psi(t,u,v)=2,\qquad m(t)=n(t)=\mathrm{e}^{-t^2},$$
$$I(s,u,v)=J(s,u,v)=2^{-|s|},\quad s\in\mathbb{Z}.$$

One sees that

(i) $\Phi(x)=|x|^{k-2}x$ with $k=4>1$, the inverse of Φ is denoted by Φ^{-1} and $\Phi^{-1}(x)=|x|^{l-2}x$ with $l=4/3$, then $c_l=1$.

(ii) $p:\mathbb{R}\to[0,+\infty)$ with $p\in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\int_{-\infty}^0 p(t)\,\mathrm{d}t=\int_0^{+\infty}p(s)\,\mathrm{d}s=+\infty$.

(iii) $\rho:\mathbb{R}\to[0,+\infty)$ with $\Phi^{-1}(\tau(\cdot))/\rho(\cdot)\in L^1_{\mathrm{loc}}(\mathbb{R})$ and

$$\tau(t)=1+\left|\int_\eta^t p(s)\,\mathrm{d}s\right|=1+|t-1|,$$

$$\begin{aligned}\int_0^{+\infty}\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u&=\int_0^2\frac{\sqrt[3]{1+|u-1|}}{\sqrt{|u-1|}}\,\mathrm{d}u+\int_2^{+\infty}\frac{\sqrt[3]{1+|u-1|}}{(u-1)^4}\,\mathrm{d}u\\&<\int_0^1\frac{\sqrt[3]{2}}{\sqrt{1-u}}\,\mathrm{d}u+\int_1^2\frac{\sqrt[3]{2}}{\sqrt{u-1}}\,\mathrm{d}u+\int_1^{+\infty}\frac{\sqrt[3]{2v}}{v^4}\,\mathrm{d}v\\&=\frac{35\sqrt[3]{2}}{8}<+\infty,\\\int_{-\infty}^0\frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u&=\int_{-\infty}^0\frac{\sqrt[3]{1+|u-1|}}{(u-1)^4}\,\mathrm{d}u<\int_{-\infty}^0\frac{1+\sqrt[3]{1-u}}{(u-1)^4}\,\mathrm{d}u\\&=\frac{17}{24}<+\infty.\end{aligned}$$

(iv) $f,\phi,\psi:\mathbb{R}^3\to\mathbb{R}$ are strong Carathéodory functions, $m,n\in L^1(\mathbb{R})$, and for each $r>0$,

$$f\bigl(t,u,\Phi^{-1}\bigl(\tau(t)\bigr)v\bigr)=\frac{1}{1+\sqrt{|t|}}\left(A_0+\sum_{j=1}^m A_j|u|^{k_j}|v|^{l_j}\right)^3\to0$$

uniformly as $t\to\pm\infty$ on $[-r,r]\times[-r,r]$.

(v) $\{t_s\colon s\in\mathbb{Z}\}$ is a increasing sequence with $\lim_{s\to-\infty}t_s=-\infty$ and $\lim_{s\to+\infty}t_s=+\infty$.

(vi) $I,J:\{t_s\colon s\in\mathbb{Z}\}\times\mathbb{R}^2\to\mathbb{R}$ are discrete Carathéodory functions.

Choose A_j ($j = 0, 1, 2, \dots, m$) and

$$b_{1,0} = b_{2,0} = 1, \quad b_{1,j} = b_{2,j} = 0, \quad j = 1, 2, \dots, m, \qquad \phi_s = 2^{-|s|}, \quad s \in \mathbb{Z},$$
$$a_{1,0} = 1, \quad a_{2,0} = 2, \quad a_{1,j} = a_{2,j} = 0, \quad j = 1, 2, \dots, m, \qquad \psi_s = 2^{-|s|}, \quad s \in \mathbb{Z}.$$

Then Assumption B holds and $\sum_{s=-\infty}^{+\infty} \phi_s = \sum_{s=-\infty}^{+\infty} \psi_s = 3$. Denote $\sigma = \max\{k_j + l_j\colon j = 1, 2, \dots, m\}$ and by direct computation and (iii), we get

$$\int_\xi^\eta p(s)\,\mathrm{d}s = 2, \qquad \int_\xi^\eta \rho(s)\,\mathrm{d}s = \frac{103}{15},$$
$$\begin{aligned}
\overline{A}_1 &< \frac{105\sqrt[3]{2} + 12377}{12360}\pi + \frac{315\sqrt[3]{2} + 12401}{4120} + \frac{2\sqrt[3]{3}(105\sqrt[3]{2} + 17)}{24} \\
&\quad + \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24}A_0, \\
\overline{A}_2 &= \frac{45 + 45\pi}{103} + 2\sqrt[3]{3} + (\sqrt[3]{2} + 1)A_0, \\
\overline{B}_{1,j} &= \left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right) + 1\right)\int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)}\,\mathrm{d}u A_j < \frac{(105\sqrt[3]{2}+17)(\sqrt[3]{2}+1)}{24}A_j, \\
\overline{B}_{2,j} &= \left(\Phi^{-1}\left(\int_\xi^\eta p(w)\,\mathrm{d}w\right) + 1\right)A_j = \left(\sqrt[3]{2} + 1\right)A_j,
\end{aligned}$$

and the definition of A, B imply that

$$\begin{aligned}
A &< \frac{105\sqrt[3]{2} + 12377}{12360}\pi + \frac{315\sqrt[3]{2} + 12401}{4120} + \frac{2\sqrt[3]{3}(105\sqrt[3]{2} + 17)}{24} \\
&\quad + \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24}A_0, \\
B &< \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24}\sum_{j=1}^m A_j.
\end{aligned}$$

By Theorem 2, BVP (9) has at least one solution if (i) $\sigma \in (0, 1)$ or (ii) $\sigma = 1$ and $((105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)/24)\sum_{j=1}^m A_j < 1$ or (iii) $\sigma > 1$ and

$$\begin{aligned}
&\left(\frac{105\sqrt[3]{2} + 12377}{12360}\pi + \frac{315\sqrt[3]{2} + 12401}{4120} + \frac{2\sqrt[3]{3}(105\sqrt[3]{2} + 17)}{24}\right. \\
&\qquad \left. + \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24}\sum_{j=0}^m A_j\right)^{\sigma-1}\sum_{j=1}^m A_j \\
&\qquad \leqslant \frac{24}{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}\frac{(\sigma - 1)^{\sigma-1}}{\sigma^\sigma}.
\end{aligned}$$

Acknowledgment. The author would like to thank the editors and referees for their careful reading and some useful comments on improving the presentation of this paper.

References

1. R.P. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer, The Netherlands, 2001.
2. C. Avramescu an C. Vladimirescu, Existence of homoclinic solutions to a nonlinear second order ode, *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.*, **15**:481–491, 2008.
3. C. Avramescu, C. Vladimirescu, Limits of solutions of a perturbed linear differential equation, *Electron. J. Qual. Theory Differ. Equ.*, **2002**:3, 2002.
4. C. Avramescu, C. Vladimirescu, Existence of solutions to second order ordinary differential equations having finite limits at $\pm\infty$, *Electron. J. Differ. Equ.*, **2004**:18, 2004.
5. B. Bianconi, F. Papalini, Non-autonomous boundary value problems on the real line, *Discrete Contin. Dyn. Syst.*, **15**:759–776, 2006.
6. A.A. Boichuk, O.A. Pokutnyi, Dichotomy and boundary value problems on the whole line, *Chaotic Modeling and Simulation (CMSIM)*, **2**:247–255, 2013.
7. A. Cabada, J.A. Cid, Heteroclinic solutions for non-autonomous boundary value problems with singular Φ-Laplacian operators, *Discrete Contin. Dyn. Syst.*, **2009**(Suppl.):118–122, 2009.
8. A. Calamai, Heteroclinic solutions of boundary value problems on the real line involving singular Φ-Laplacian operators, *J. Math. Anal. Appl.*, **378**:667–679, 2011.
9. G. Cupini, C. Marcelli, F. Papalini, Heteroclinic solutions of boundary value problems on the real line involving general nonlinear differential operators, *Differ. Integral Equ.*, **24**(7-8):619–644, 2011.
10. G. Cupini, C. Marcelli, F. Papalini, On the solvability of a boundary value problem on the real line, *Bound. Value Probl.*, **2011**:26, 2011.
11. F.Y. Deren, N.A. Hamal, Second-order boundary-value problems with integral boundary conditions on the real line, *Electron. J. Differ. Equ.*, **2014**:18, 2014.
12. Y. Liu, Multiple positive solutions to mixed boundary value problems for singular ordinary differential equations on the whole line, *Nonlinear Anal. Model. Control*, **17**(4):460–480, 2012.
13. Y. Liu, Solvability of boundary value problems for singular quasi-Laplacian differential equations on the whole, *Math. Model. Anal.*, **17**(3):423–446, 2012.
14. Y. Liu, Existence of solutions of boundary value problems for coupled singular differential equations on whole lines with impulses, *Mediterr. J. Math.*, **12**(3):697–716, 2015.
15. C. Marcelli, The role of boundary data on the solvability of some equations involving non-autonomous nonlinear differential operators, *Bound. Value Probl.*, **2013**:252, 2013.
16. C.G. Philos, I.K. Purnaras, A boundary value problem on the whole line to second order nonlinear differential equations, *Georgian Math. J.*, **17**:241–252, 2010.
17. A. Štikonas, A survey on stationary problems, green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19**(3):301–334, 2014.