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Existence of positive solutions for a singular fractional boundary value problem

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Abstract. We study the existence of positive solutions for a nonlinear Riemann–Liouville fractional differential equation with a sign-changing nonlinearity, subject to multi-point fractional boundary conditions.

Keywords: Riemann–Liouville fractional differential equation, fractional boundary conditions, positive solutions, sign-changing nonlinearity.

1 Introduction

We consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha}u(t)+\lambda f\bigl(t,u(t)\bigr)=0,\quad t\in(0,1), \tag{E}$$

with the multi-point fractional boundary conditions

$$\begin{aligned}&u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\&D_{0+}^{p}u(t)\big|_{t=1}=\sum_{i=1}^{m}a_iD_{0+}^{q}u(t)\big|_{t=\xi_i},\end{aligned} \tag{BC}$$

where λ is a positive parameter, $\alpha\in\mathbb{R}$, $\alpha\in(n-1,n]$, $n\in\mathbb{N}$, $n\geqslant 3$, $\xi_i\in\mathbb{R}$ for all $i=1,\dots,m$ ($m\in\mathbb{N}$), $0<\xi_1<\cdots<\xi_m<1$, $p,q\in\mathbb{R}$, $p\in[1,n-2]$, $q\in[0,p]$, D_{0+}^{α} denotes the Riemann–Liouville derivative of order α, and the nonlinearity f may change sign and may be singular at $t=0$ or $t=1$.

We present intervals for parameter λ such that problem (E)–(BC) has at least one positive solution. By a positive solution of (E)–(BC) we mean a function $u\in C([0,1])$ satisfying (E) and (BC) with $u(t)>0$ for all $t\in(0,1]$. This problem is a generalization of the problem studied in [7], where $p \in \mathbb{N}$ and $q = 1$. Other particular cases of problem (E)–(BC) and of problem from [7] were investigated in [13] (where $n = 3$, $p = 0$ and $a_i = 0$ for all $i = 1, \dots, m$) and in [14] (where $p = 0$ and $a_i = 0$ for all $i = 1, \dots, m$ and n is an arbitrary natural number, $n \geqslant 3$). In [11], the author studied the existence of pseudo solutions of problem (E)–(BC) in a reflexive Banach space with $\lambda = 1$, $f(t,u) = g(t)\widetilde{f}(t,u)$, $p = q = 0$ and instead of derivative D^{α}_{0+}, he considered the pseudo fractional derivative D^{α}. For some recent results on the existence and multiplicity of positive solutions for systems of fractional differential equations with various boundary conditions, see the monograph [5].

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [1–3, 8–10, 12, 15]).

The paper is organized as follows. Section 2 contains some auxiliary results, which investigate a nonlocal boundary value problem for fractional differential equations. In Section 3, we give the existence theorems for the positive solutions with respect to a cone for our problem (E)–(BC). Finally, in Section 4, two examples are given to illustrate our main results.

2 Auxiliary results

We present here the definitions of the Riemann–Liouville fractional integral and the Riemann–Liouville fractional derivative and some auxiliary results that will be used to prove our main results.

Definition 1. The (left-sided) fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$\bigl(I^{\alpha}_{0+}f\bigr)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t > 0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t$, $\alpha > 0$.

Definition 2. The Riemann–Liouville fractional derivative of order $\alpha \geqslant 0$ for a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$\begin{aligned}\bigl(D^{\alpha}_{0+}f\bigr)(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \bigl(I^{n-\alpha}_{0+}f\bigr)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s, \quad t > 0,\end{aligned}$$

where $n = \lfloor \alpha \rfloor + 1$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

The notation $\lfloor\alpha\rfloor$ stands for the largest integer not greater than α. If $\alpha = m \in \mathbb{N}$, then $D_{0+}^m f(t) = f^{(m)}(t)$ for $t > 0$, and if $\alpha = 0$, then $D_{0+}^0 f(t) = f(t)$ for $t > 0$.

We consider now the fractional differential equation

$$D_{0+}^{\alpha} u(t) + \widetilde{x}(t) = 0, \quad 0 < t < 1, \tag{1}$$

with the boundary conditions

$$\begin{gathered} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^{p} u(t)\big|_{t=1} = \sum_{i=1}^{m} a_i D_{0+}^{q} u(t)\big|_{t=\xi_i}, \end{gathered} \tag{2}$$

where $\alpha \in (n-1, n]$, $n \geqslant 3$, $0 < \xi_1 < \cdots < \xi_m < 1$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$, and $\widetilde{x} \in C(0,1) \cap L^1(0,1)$. We denote by $\Delta = \Gamma(\alpha)/\Gamma(\alpha-p) - (\Gamma(\alpha)/\Gamma(\alpha-q)) \sum_{i=1}^{m} a_i \xi_i^{\alpha-q-1}$.

Lemma 1. *If $\Delta \neq 0$, then the function $u \in C([0,1])$ given by*

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{x}(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1} \widetilde{x}(s)\,\mathrm{d}s \\ &- \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)} \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-q-1} \widetilde{x}(s)\,\mathrm{d}s, \quad t \in [0,1], \end{aligned} \tag{3}$$

is solution of problem (1)–(2).

Proof. We denote by

$$\begin{aligned} c_1 = &\frac{1}{\Delta\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1} \widetilde{x}(s)\,\mathrm{d}s \\ &- \frac{1}{\Delta\Gamma(\alpha-q)} \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-q-1} \widetilde{x}(s)\,\mathrm{d}s. \end{aligned}$$

Then the continuous function u from (3) can be written as

$$\begin{aligned} u(t) &= c_1 t^{\alpha-1} - I_{0+}^{\alpha} \widetilde{x}(t) \\ &= c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{x}(s)\,\mathrm{d}s, \quad t \in [0,1]. \end{aligned} \tag{4}$$

Because $D_{0+}^{\alpha} u(t) = c_1 D_{0+}^{\alpha}(t^{\alpha-1}) - D_{0+}^{\alpha} I_{0+}^{\alpha} \widetilde{x}(t) = -\widetilde{x}(t)$ for all $t \in (0,1)$, we deduce that u satisfies equation (1). In addition, we have $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$.

Because

$$D_{0+}^{p}u(t) = c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-p)}t^{\alpha-p-1} - I_{0+}^{\alpha-p}\widetilde{x}(t),$$
$$D_{0+}^{q}u(t) = c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t^{\alpha-q-1} - I_{0+}^{\alpha-q}\widetilde{x}(t),$$

by a simple computation we conclude that $D_{0+}^{p}u(t)|_{t=1} = \sum_{i=1}^{m} a_i D_{0+}^{q}u(t)|_{t=\xi_i}$, that is

$$c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-p)} - \frac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s$$
$$= \sum_{i=1}^{m} a_i\left(c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\xi_i^{\alpha-q-1} - \frac{1}{\Gamma(\alpha-q)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-q-1}\widetilde{x}(s)\,\mathrm{d}s\right).$$

Therefore, we deduce that u is solution of problem (1)–(2). □

Lemma 2. *If* $\Delta \neq 0$*, then the solution* u *of problem* (1)–(2) *given by* (3) *can be written as*

$$u(t) = \int_0^1 G(t,s)\widetilde{x}(s)\,\mathrm{d}s, \quad t\in[0,1], \tag{5}$$

where

$$G(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^{m} a_i g_2(\xi_i,s), \tag{6}$$

and

$$\begin{aligned} g_1(t,s) &= \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-1}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{\alpha-1}(1-s)^{\alpha-p-1}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}\\ g_2(t,s) &= \frac{1}{\Gamma(\alpha-q)}\begin{cases} t^{\alpha-q-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-q-1}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{\alpha-q-1}(1-s)^{\alpha-p-1}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}\end{aligned} \tag{7}$$

for all $(t,s)\in[0,1]\times[0,1]$.

Proof. By Lemma 1 and relation (3), we deduce

$$u(t) = \frac{1}{\Gamma(\alpha)}\left\{\int_0^t \left[t^{\alpha-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-1}\right]\widetilde{x}(s)\,\mathrm{d}s\right.$$
$$\left. + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s\right\} - \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s$$

$$+\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s-\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-q-1}\widetilde{x}(s)\,\mathrm{d}s$$

$$=\frac{1}{\Gamma(\alpha)}\Biggl\{\int_0^t\bigl[t^{\alpha-1}(1-s)^{\alpha-p-1}-(t-s)^{\alpha-1}\bigr]\widetilde{x}(s)\,\mathrm{d}s$$

$$+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s\Biggr\}-\frac{1}{\Delta\Gamma(\alpha-p)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s$$

$$+\frac{1}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i\xi_i^{\alpha-q-1}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s$$

$$+\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s-\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-q-1}\widetilde{x}(s)\,\mathrm{d}s$$

$$=\frac{1}{\Gamma(\alpha)}\Biggl\{\int_0^t\bigl[t^{\alpha-1}(1-s)^{\alpha-p-1}-(t-s)^{\alpha-1}\bigr]\widetilde{x}(s)\,\mathrm{d}s$$

$$+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s\Biggr\}$$

$$+\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i\Biggl[\int_0^1\xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s-\int_0^{\xi_i}(\xi_i-s)^{\alpha-q-1}\widetilde{x}(s)\,\mathrm{d}s\Biggr]$$

$$=\frac{1}{\Gamma(\alpha)}\Biggl\{\int_0^t\bigl[t^{\alpha-1}(1-s)^{\alpha-p-1}-(t-s)^{\alpha-1}\bigr]\widetilde{x}(s)\,\mathrm{d}s$$

$$+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s\Biggr\}$$

$$+\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i\Biggl\{\int_0^{\xi_i}\bigl[\xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1}-(\xi_i-s)^{\alpha-q-1}\bigr]\widetilde{x}(s)\,\mathrm{d}s$$

$$+\int_{\xi_i}^1\xi_i^{\alpha-q-1}(1-s)^{\alpha-p-1}\widetilde{x}(s)\,\mathrm{d}s\Biggr\}$$

$$= \int_0^1 g_1(t,s)\widetilde{x}(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^m a_i \int_0^1 g_2(\xi_i,s)\widetilde{x}(s)\,\mathrm{d}s = \int_0^1 G(t,s)\widetilde{x}(s)\,\mathrm{d}s,$$

where G and g_1, g_2 are given in (6) and (7).

Therefore, we obtain expression (5) for the solution u of problem (1)–(2) given by (3). □

Lemma 3. *The functions g_1 and g_2 given by* (7) *have the properties*:

(a) $g_1(t,s) \leqslant h_1(s)$ *for all* $ts \in [0,1]$*, where* $h_1(s) = (1-s)^{\alpha-p-1}(1-(1-s)^p)/\Gamma(\alpha)$*,* $s \in [0,1]$;
(b) $g_1(t,s) \geqslant t^{\alpha-1}h_1(s)$ *for all* $t,s \in [0,1]$;
(c) $g_1(t,s) \leqslant t^{\alpha-1}/\Gamma(\alpha)$ *for all* $t,s \in [0,1]$;
(d) $g_2(t,s) \geqslant t^{\alpha-q-1}h_2(s)$ *for all* $t,s \in [0,1]$*, where* $h_2(s) = (1-s)^{\alpha-p-1} \times (1-(1-s)^{p-q})/\Gamma(\alpha-q)$*,* $s \in [0,1]$;
(e) $g_2(t,s) \leqslant (1/\Gamma(\alpha-q))t^{\alpha-q-1}$ *for all* $t,s \in [0,1]$;
(f) *The functions g_1 and g_2 are continuous on* $[0,1]\times[0,1]$; $g_1(t,s) \geqslant 0$*,* $g_2(t,s) \geqslant 0$ *for all* $t,s \in [0,1]$; $g_1(t,s) > 0$*,* $g_2(t,s) > 0$ *for all* $t,s \in (0,1)$.

Proof. For the proof of properties (a)–(c), see [6].

(d) From (7), if $s \leqslant t$, we obtain

$$\begin{aligned}
g_2(t,s) &= \frac{1}{\Gamma(\alpha-q)}\bigl[t^{\alpha-q-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-q-1}\bigr] \\
&\geqslant \frac{1}{\Gamma(\alpha-q)}\bigl[t^{\alpha-q-1}(1-s)^{\alpha-p-1} - (t-ts)^{\alpha-q-1}\bigr] \\
&= \frac{1}{\Gamma(\alpha-q)}t^{\alpha-q-1}\bigl[(1-s)^{\alpha-p-1} - (1-s)^{\alpha-q-1}\bigr] \\
&= \frac{1}{\Gamma(\alpha-q)}t^{\alpha-q-1}(1-s)^{\alpha-p-1}\bigl(1-(1-s)^{p-q}\bigr) \\
&= t^{\alpha-q-1}h_2(s),
\end{aligned}$$

where $h_2(s) = (1/\Gamma(\alpha-q))(1-s)^{\alpha-p-1}(1-(1-s)^{p-q})$, $s \in [0,1]$.

If $t \leqslant s$, we deduce

$$g_2(t,s) = \frac{1}{\Gamma(\alpha-q)}t^{\alpha-q-1}(1-s)^{\alpha-p-1} \geqslant t^{\alpha-q-1}h_2(s).$$

Hence, we conclude $g_2(t,s) \geqslant t^{\alpha-q-1}h_2(s)$ for all $t,s \in [0,1]$.

(e) We have

$$g_2(t,s) \leqslant \frac{1}{\Gamma(\alpha-q)}t^{\alpha-q-1}(1-s)^{\alpha-p-1} \leqslant \frac{t^{\alpha-q-1}}{\Gamma(\alpha-q)} \quad \forall\, t,s \in [0,1].$$

(f) This property follows from the definitions of g_1 and g_2 and from properties (b) and (d) above. □

Lemma 4. *Assume that* $a_i \geqslant 0$ *for all* $i = 1, \dots, m$ *and* $\Delta > 0$. *Then the function* G *given by* (6) *is a continuous function on* $[0,1] \times [0,1]$ *and satisfies the inequalities*:

(a) $G(t,s) \leqslant J(s)$ *for all* $t, s \in [0,1]$, *where* $J(s) = h_1(s) + (1/\Delta)\sum_{i=1}^m a_i g_2(\xi_i, s)$, $s \in [0,1]$;
(b) $G(t,s) \geqslant t^{\alpha-1}J(s)$ *for all* $t, s \in [0,1]$;
(c) $G(t,s) \leqslant \sigma t^{\alpha-1}$ *for all* $t, s \in [0,1]$, *where* $\sigma = 1/\Gamma(\alpha) + \sum_{i=1}^m a_i \xi_i^{\alpha-q-1}/(\Delta\Gamma(\alpha-q))$.

Proof. By definition of the function G we deduce that G is a continuous function. In addition, by using Lemma 3, we obtain for all $t, s \in [0,1]$

$$\text{(a)}\quad G(t,s) \leqslant h_1(s) + \frac{1}{\Delta}\sum_{i=1}^m a_i g_2(\xi_i, s) = J(s);$$

$$\text{(b)}\quad G(t,s) \geqslant t^{\alpha-1}h_1(s) + \frac{t^{\alpha-1}}{\Delta}\sum_{i=1}^m a_i g_2(\xi_i, s) = t^{\alpha-1}J(s);$$

$$\text{(c)}\quad G(t,s) \leqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-q)}\sum_{i=1}^m a_i \xi_i^{\alpha-q-1} = \sigma t^{\alpha-1}. \qquad \square$$

Lemma 5. *Assume that* $a_i \geqslant 0$ *for all* $i = 1, \dots, m$, $\Delta > 0$, $\widetilde{x} \in C(0,1) \cap L^1(0,1)$ *and* $\widetilde{x}(t) > 0$ *for all* $t \in (0,1)$. *Then the solution* u *of problem* (1)–(2) *given by* (3) *satisfies the inequality* $u(t) \geqslant t^{\alpha-1}u(t')$ *for all* $t, t' \in [0,1]$.

Proof. By using Lemma 4, we obtain for all $t, t' \in [0,1]$

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\widetilde{x}(s)\,\mathrm{d}s \geqslant \int_0^1 t^{\alpha-1}J(s)\widetilde{x}(s)\,\mathrm{d}s \\ &\geqslant t^{\alpha-1}\int_0^1 G(t',s)\widetilde{x}(s)\,\mathrm{d}s = t^{\alpha-1}u(t'). \qquad \square \end{aligned}$$

In the proof of our main results, we shall use the Guo–Krasnosel'skii fixed point theorem presented below (see [4]).

Theorem 1. *Let* X *be a Banach space and let* $C \subset X$ *be a cone in* X. *Assume* Ω_1 *and* Ω_2 *are bounded open subsets of* X *with* $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ *and let* $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ *be a completely continuous operator such that, either*

(i) $\|\mathcal{A}u\| \leqslant \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geqslant \|u\|$, $u \in C \cap \partial\Omega_2$, *or*
(ii) $\|\mathcal{A}u\| \geqslant \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leqslant \|u\|$, $u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ *has a fixed point in* $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Main results

In this section, we investigate the existence of positive solutions for our problem (E)–(BC). First, we present the assumptions that we shall use in the sequel.

(H1) $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, $n \geqslant 3$, $\xi_i \in \mathbb{R}$ for all $i = 1, \ldots, m$ ($m \in \mathbb{N}$), $0 < \xi_1 < \cdots < \xi_m < 1$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$, $a_i \geqslant 0$ for all $i = 1, \ldots, m$, $\lambda > 0$, $\Delta = \Gamma(\alpha)/\Gamma(\alpha-p) - (\Gamma(\alpha)/\Gamma(\alpha-q)) \sum_{i=1}^m a_i \xi_i^{\alpha-q-1} > 0$.

(H2) The function $f \in C((0,1) \times [0, \infty), \mathbb{R})$ may be singular at $t = 0$ and/or $t = 1$, and there exist the functions $r, z \in C((0,1), [0, \infty))$, $g \in C([0,1] \times [0, \infty), [0, \infty))$ such that $-r(t) \leqslant f(t,x) \leqslant z(t) g(t,x)$ for all $t \in (0,1)$ and $x \in [0, \infty)$ with $0 < \int_0^1 r(t)\,\mathrm{d}t < \infty$, $0 < \int_0^1 z(t)\,\mathrm{d}t < \infty$.

(H3) There exists $c \in (0, 1/2)$ such that $f_\infty = \lim_{u\to\infty} \min_{t\in[c,1-c]} f(t,u)/u = \infty$.

(H4) There exists $c \in (0, 1/2)$ such that $\liminf_{u\to\infty} \min_{t\in[c,1-c]} f(t,u) > L_0$, with $L_0 = (2\sigma \int_0^1 r(s)\,\mathrm{d}s)/(c^{\alpha-1} \int_c^{1-c} J(s)\,\mathrm{d}s)$, and $g_\infty = \lim_{u\to\infty} \max_{t\in[0,1]} g(t,u)/u = 0$, where J and σ are given in Section 2 (Lemma 4).

We consider the fractional differential equation

$$D_{0+}^\alpha x(t) + \lambda\bigl(f\bigl(t, \bigl[x(t) - \lambda w(t)\bigr]^*\bigr) + r(t)\bigr) = 0, \quad 0 < t < 1, \tag{8}$$

with the multi-point fractional boundary conditions

$$\begin{aligned} &x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ &D_{0+}^p x(t)\big|_{t=1} = \sum_{i=1}^m a_i D_{0+}^q x(t)\big|_{t=\xi_i}, \end{aligned} \tag{9}$$

where $\lambda > 0$ and $\zeta(t)^* = \zeta(t)$ if $\zeta(t) \geqslant 0$, and $\zeta(t)^* = 0$ if $\zeta(t) < 0$. Here $w(t) = \int_0^1 G(t,s) r(s)\,\mathrm{d}s$, $t \in [0,1]$, is solution of problem

$$\begin{aligned} &D_{0+}^\alpha w(t) + r(t) = 0, \quad 0 < t < 1, \\ &w(0) = w'(0) = \cdots = w^{(n-2)}(0) = 0, \\ &D_{0+}^p w(t)\big|_{t=1} = \sum_{i=1}^m a_i D_{0+}^q w(t)\big|_{t=\xi_i}. \end{aligned}$$

Under assumptions (H1)–(H2), we have $w(t) \geqslant 0$ for all $t \in [0,1]$. We shall prove that there exists a solution x of problem (8)–(9) with $x(t) \geqslant \lambda w(t)$ on $[0,1]$ and $x(t) > \lambda w(t)$ on $(0,1)$. In this case, $u = x - \lambda w$ represents a positive solution of problem (E)–(BC). Therefore, in what follows, we shall investigate problem (8)–(9).

By using Lemma 2, a solution of equation

$$x(t) = \lambda \int_0^1 G(t,s)\bigl(f\bigl(s, \bigl[x(s) - \lambda w(s)\bigr]^*\bigr) + r(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$

is a solution for problem (8)–(9).

We consider the Banach space $X = C([0,1])$ with the supremum norm $\|\cdot\|$, and we define the cone

$$P = \{x \in X\colon\ x(t) \geqslant t^{\alpha-1}\|x\|\ \ \forall t \in [0,1]\}.$$

For $\lambda > 0$, we introduce the operator $T : X \to X$ defined by

$$Tx(t) = \lambda \int_0^1 G(t,s)\bigl(f\bigl(s, \bigl[x(s) - \lambda w(s)\bigr]^*\bigr) + r(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1],\ x \in X.$$

It is clear that if x is a fixed point of operator T, then x is a solution of problem (8)–(9).

Lemma 6. *If* (H1) *and* (H2) *hold, then operator $T : P \to P$ is a completely continuous operator.*

Proof. Let $x \in P$ be fixed. By using (H1) and (H2), we deduce that $Tx(t) < \infty$ for all $t \in [0,1]$. Besides, by Lemma 4, we obtain for all $t, t' \in [0,1]$

$$Tx(t) \leqslant \lambda \int_0^1 J(s)\bigl(f\bigl(s, \bigl[x(s) - \lambda w(s)\bigr]^*\bigr) + r(s)\bigr)\,\mathrm{d}s,$$

$$\begin{aligned} Tx(t) &\geqslant \lambda \int_0^1 t^{\alpha-1} J(s)\bigl(f\bigl(s, \bigl[x(s) - \lambda w(s)\bigr]^*\bigr) + r(s)\bigr)\,\mathrm{d}s \\ &\geqslant t^{\alpha-1} Tx(t'). \end{aligned}$$

Therefore, $Tx(t) \geqslant t^{\alpha-1}\|Tx\|$ for all $t \in [0,1]$. We deduce that $Tx \in P$, and hence, $T(P) \subset P$.

By using standard arguments, we conclude that operator $T : P \to P$ is a completely continuous operator. □

Theorem 2. *Assume that* (H1), (H2) *and* (H3) *hold. Then there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*]$, the boundary value problem* (E)–(BC) *has at least one positive solution.*

Proof. We choose a positive number $R_1 > \sigma \int_0^1 r(s)\,\mathrm{d}s > 0$, and we define the set $\Omega_1 = \{x \in P\colon\ \|x\| < R_1\}$.

We introduce

$$\lambda^* = \min\left\{1, R_1\left(M_1 \int_0^1 J(s)\bigl(z(s) + r(s)\bigr)\,\mathrm{d}s\right)^{-1}\right\}$$

with $M_1 = \max\{\max_{t\in[0,1],\,u\in[0,R_1]} g(t,u), 1\}$.

Let $\lambda \in (0, \lambda^*]$. Because $w(t) \leqslant \sigma t^{\alpha-1} \int_0^1 r(s)\,\mathrm{d}s$ for all $t \in [0,1]$, we deduce for any $x \in P \cap \partial\Omega_1$ and $t \in [0,1]$

$$\bigl[x(t) - \lambda w(t)\bigr]^* \leqslant x(t) \leqslant \|x\| \leqslant R_1$$

and

$$x(t) - \lambda w(t) \geqslant t^{\alpha-1}\|x\| - \lambda\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s = t^{\alpha-1}\left(R_1 - \lambda\sigma\int_0^1 r(s)\,\mathrm{d}s\right)$$
$$\geqslant t^{\alpha-1}\left(R_1 - \lambda^*\sigma\int_0^1 r(s)\,\mathrm{d}s\right) \geqslant t^{\alpha-1}\left(R_1 - \sigma\int_0^1 r(s)\,\mathrm{d}s\right) \geqslant 0.$$

Then for any $x \in P \cap \partial\Omega_1$ and $t \in [0,1]$, we obtain

$$Tx(t) \leqslant \lambda\int_0^1 J(s)\big(z(s)g\big(s, \big[x(s) - \lambda w(s)\big]^*\big) + r(s)\big)\,\mathrm{d}s$$
$$\leqslant \lambda^* M_1\int_0^1 J(s)\big(z(s) + r(s)\big)\,\mathrm{d}s \leqslant R_1 = \|x\|.$$

Therefore, we conclude

$$\|Tx\| \leqslant \|x\| \quad \forall\, x \in P \cap \partial\Omega_1. \tag{10}$$

On the other hand, for c from (H3), we choose a positive constant $L > 0$ such that

$$L \geqslant 2\left(\lambda c^{2(\alpha-1)}\int_c^{1-c} J(s)\,\mathrm{d}s\right)^{-1}.$$

By (H3), we deduce that there exists a constant $M_0 > 0$ such that

$$f(t,u) \geqslant Lu \quad \forall t \in [c, 1-c],\ u \geqslant M_0. \tag{11}$$

Now we define $R_2 = \max\{2R_1, 2M_0/c^{\alpha-1}\}$ and let $\Omega_2 = \{x \in P\colon\ \|x\| < R_2\}$. Then for any $x \in P \cap \partial\Omega_2$, we obtain

$$x(t) - \lambda w(t) \geqslant t^{\alpha-1}\|x\| - \lambda\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s \geqslant t^{\alpha-1}\left(R_2 - \sigma\int_0^1 r(s)\,\mathrm{d}s\right)$$
$$\geqslant t^{\alpha-1}\left(R_1 - \sigma\int_0^1 r(s)\,\mathrm{d}s\right) \geqslant 0 \quad \forall t \in [0,1].$$

Therefore, we conclude

$$\big[x(t) - \lambda w(t)\big]^* = x(t) - \lambda w(t) \geqslant t^{\alpha-1}\left(R_2 - \sigma\int_0^1 r(s)\,\mathrm{d}s\right)$$
$$\geqslant \frac{c^{\alpha-1}R_2}{2} \geqslant M_0 \quad \forall t \in [c, 1-c]. \tag{12}$$

Then for any $x \in P \cap \partial\Omega_2$ and $t \in [c, 1-c]$, by (11) and (12) we deduce

$$\begin{aligned}
Tx(t) &\geqslant \lambda \int_c^{1-c} G(t,s)\bigl(f\bigl(s, \bigl[x(s) - \lambda w(s)\bigr]^*\bigr) + r(s)\bigr)\,\mathrm{d}s \\
&\geqslant \lambda \int_c^{1-c} G(t,s) L\bigl[x(s) - \lambda w(s)\bigr]\,\mathrm{d}s \geqslant \lambda L \int_c^{1-c} t^{\alpha-1} J(s) \frac{c^{\alpha-1} R_2}{2}\,\mathrm{d}s \\
&\geqslant \frac{\lambda L c^{2(\alpha-1)} R_2}{2} \int_c^{1-c} J(s)\,\mathrm{d}s \geqslant R_2 = \|x\|.
\end{aligned}$$

Then

$$\|Tx\| \geqslant \|x\| \quad \forall\, x \in P \cap \partial\Omega_2. \tag{13}$$

By (10), (13) and Theorem 1(i) we conclude that T has a fixed point $x_1 \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$, that is $R_1 \leqslant \|x_1\| \leqslant R_2$. Since $\|x_1\| \geqslant R_1$, we deduce

$$\begin{aligned}
x_1(t) - \lambda w(t) &\geqslant t^{\alpha-1}\left(\|x_1\| - \sigma\lambda \int_0^1 r(s)\,\mathrm{d}s\right) \geqslant t^{\alpha-1}\left(R_1 - \sigma \int_0^1 r(s)\,\mathrm{d}s\right) \\
&= \Lambda_1 t^{\alpha-1},
\end{aligned}$$

and so $x_1(t) \geqslant \lambda w(t) + \Lambda_1 t^{\alpha-1}$ for all $t \in [0,1]$, where $\Lambda_1 = R_1 - \sigma \int_0^1 r(s)\,\mathrm{d}s > 0$.

Let $u_1(t) = x_1(t) - \lambda w(t)$ for all $t \in [0,1]$. Then u_1 is a positive solution of problem (E)–(BC) with $u_1(t) \geqslant \Lambda_1 t^{\alpha-1}$ for all $t \in [0,1]$. This completes the proof of Theorem 2. □

Theorem 3. *Assume that* (H1), (H2) *and* (H4) *hold. Then there exists* $\lambda_* > 0$ *such that, for any* $\lambda \geqslant \lambda_*$*, the boundary value problem* (E)–(BC) *has at least one positive solution.*

Proof. By (H4) there exists $M_2 > 0$ such that

$$f(t,u) \geqslant 2\sigma \int_0^1 r(s)\,\mathrm{d}s \left(c^{\alpha-1} \int_c^{1-c} J(s)\,\mathrm{d}s\right)^{-1} \quad \forall\, t \in [c, 1-c],\ u \geqslant M_2.$$

We define

$$\lambda_* = M_2 \left(c^{\alpha-1}\sigma \int_0^1 r(s)\,\mathrm{d}s\right)^{-1}.$$

We assume now $\lambda \geqslant \lambda_*$. Let $R_3 = 2\lambda\sigma\int_0^1 r(s)\,\mathrm{d}s$ and $\Omega_3 = \{x \in P\colon \|x\| < R_3\}$. Then for any $x \in P \cap \partial\Omega_3$, we deduce

$$\begin{aligned} x(t) - \lambda w(t) &\geqslant t^{\alpha-1}\|x\| - \lambda\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s = t^{\alpha-1}\left(R_3 - \lambda\sigma\int_0^1 r(s)\,\mathrm{d}s\right) \\ &= t^{\alpha-1}\lambda\sigma\int_0^1 r(s)\,\mathrm{d}s \geqslant t^{\alpha-1}\lambda_*\sigma\int_0^1 r(s)\,\mathrm{d}s \\ &= \frac{M_2 t^{\alpha-1}}{c^{\alpha-1}} \geqslant 0 \quad \forall\, t \in [0,1]. \end{aligned}$$

Therefore, for any $x \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we have

$$\left[x(t) - \lambda w(t)\right]^* = x(t) - \lambda w(t) \geqslant \frac{M_2 t^{\alpha-1}}{c^{\alpha-1}} \geqslant M_2.$$

Hence, for any $x \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we conclude

$$\begin{aligned} Tx(t) &\geqslant \lambda\int_c^{1-c} G(t,s) f\left(s, \left[x(s) - \lambda w(s)\right]^*\right)\mathrm{d}s \\ &\geqslant \lambda\int_c^{1-c} t^{\alpha-1} J(s) f\left(s, u(s) - \lambda w(s)\right)\mathrm{d}s \\ &\geqslant \lambda\int_c^{1-c} t^{\alpha-1} J(s)\left(2\sigma\int_0^1 r(s)\,\mathrm{d}s\right)\left(c^{\alpha-1}\int_c^{1-c} J(s)\,\mathrm{d}s\right)^{-1}\mathrm{d}s \\ &= \frac{R_3 t^{\alpha-1}}{c^{\alpha-1}} \geqslant R_3. \end{aligned}$$

Therefore, we obtain

$$\|Tx\| \geqslant \|x\| \quad \forall\, x \in P \cap \partial\Omega_3. \tag{14}$$

On the other hand, we consider the positive number $\varepsilon = (2\lambda\int_0^1 J(s)z(s)\,\mathrm{d}s)^{-1}$. Then by (H4) we deduce that there exists $M_3 > 0$ such that $g(t,u) \leqslant \varepsilon u$ for all $t \in [0,1]$, $u \geqslant M_3$. Therefore, we obtain $g(t,u) \leqslant M_4 + \varepsilon u$ for all $t \in [0,1]$, $u \geqslant 0$, where $M_4 = \max_{t\in[0,1],\, u\in[0,M_3]} g(t,u)$.

We define now

$$R_4 > \max\left\{R_3,\ 2\lambda\max\{M_4, 1\}\int_0^1 J(s)\left(z(s) + r(s)\right)\mathrm{d}s\right\}$$

and $\Omega_4 = \{x \in P\colon \|x\| < R_4\}$.

Then for any $x \in P \cap \partial\Omega_4$, we have

$$\begin{aligned} x(t) - \lambda w(t) &\geqslant t^{\alpha-1}\|x\| - \lambda\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s = t^{\alpha-1}\Biggl(R_4 - \lambda\sigma\int_0^1 r(s)\,\mathrm{d}s\Biggr) \\ &\geqslant t^{\alpha-1}\Biggl(R_3 - \lambda\sigma\int_0^1 r(s)\,\mathrm{d}s\Biggr) = t^{\alpha-1}\lambda\sigma\int_0^1 r(s)\,\mathrm{d}s \\ &\geqslant t^{\alpha-1}\lambda_*\sigma\int_0^1 r(s)\,\mathrm{d}s = \frac{M_2 t^{\alpha-1}}{c^{\alpha-1}} \geqslant 0 \quad \forall t \in [0,1]. \end{aligned}$$

Then for any $x \in P \cap \partial\Omega_4$, we obtain

$$\begin{aligned} Tx(t) &\leqslant \lambda\int_0^1 J(s)\bigl[z(s)g\bigl(s,\bigl[x(s) - \lambda w(s)\bigr]^*\bigr) + r(s)\bigr]\,\mathrm{d}s \\ &\leqslant \lambda\int_0^1 J(s)\bigl[z(s)\bigl(M_4 + \varepsilon\bigl(x(s) - \lambda w(s)\bigr)\bigr) + r(s)\bigr]\,\mathrm{d}s \\ &\leqslant \lambda\max\{M_4, 1\}\int_0^1 J(s)\bigl(z(s) + r(s)\bigr)\,\mathrm{d}s + \lambda\varepsilon R_4\int_0^1 J(s)z(s)\,\mathrm{d}s \\ &\leqslant \frac{R_4}{2} + \frac{R_4}{2} = R_4 = \|x\| \quad \forall\, t \in [0,1]. \end{aligned}$$

Therefore,

$$\|Tx\| \leqslant \|x\| \quad \forall\, x \in P \cap \partial\Omega_4. \tag{15}$$

By (14), (15) and Theorem 1(ii) we conclude that T has a fixed point $x_1 \in P \cap (\bar{\Omega}_4 \setminus \Omega_3)$, so $R_3 \leqslant \|x_1\| \leqslant R_4$.

In addition, we deduce that for all $t \in [0,1]$,

$$\begin{aligned} x_1(t) - \lambda w(t) &\geqslant x_1(t) - \lambda\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s \geqslant t^{\alpha-1}\|x_1\| - \lambda\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s \\ &\geqslant t^{\alpha-1}R_3 - \lambda\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s = \lambda\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s \\ &\geqslant \lambda_*\sigma t^{\alpha-1}\int_0^1 r(s)\,\mathrm{d}s = \frac{M_2 t^{\alpha-1}}{c^{\alpha-1}}. \end{aligned}$$

Let $u_1(t) = x_1(t) - \lambda w(t)$ for all $t \in [0,1]$. Then $u_1(t) \geqslant \widetilde{\Lambda}_1 t^{\alpha-1}$ for all $t \in [0,1]$, where $\widetilde{\Lambda}_1 = M_2/c^{\alpha-1}$. Hence, we conclude that u_1 is a positive solution of problem (E)–(BC), which completes the proof of Theorem 3. □

In a similar manner to that used in the proof of Theorem 3, we deduce the following result.

Theorem 4. *Assume that* (H1), (H2) *and*

($\widetilde{\text{H4}}$) *There exists* $c \in (0, 1/2)$ *such that* $\hat{f}_\infty = \lim_{u\to\infty} \min_{t\in[c,1-c]} f(t,u) = \infty$ *and* $g_\infty = \lim_{u\to\infty} \max_{t\in[0,1]} g(t,u)/u = 0$

hold. Then there exists $\widetilde{\lambda}_* > 0$ *such that, for any* $\lambda \geqslant \widetilde{\lambda}_*$, *the boundary value problem* (E)–(BC) *has at least one positive solution.*

4 Examples

Let $\alpha = 10/3$ ($n = 4$), $p = 3/2$, $q = 4/3$, $m = 3$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $\xi_3 = 3/4$, $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/3$.

We consider the fractional differential equation

$$D_{0+}^{10/3} u(t) + \lambda f\bigl(t, u(t)\bigr) = 0, \quad t \in (0,1), \tag{E*}$$

with the boundary conditions

$$\begin{aligned} &u(0) = u'(0) = u''(0) = 0, \\ &D_{0+}^{3/2} u(t)\big|_{t=1} = D_{0+}^{4/3} u(t)\big|_{t=1/4} + \frac{1}{2} D_{0+}^{4/3} u(t)\big|_{t=1/2} + \frac{1}{3} D_{0+}^{4/3} u(t)\big|_{t=3/4}. \end{aligned} \tag{BC*}$$

Then we obtain $\Delta = \Gamma(10/3)(1/\Gamma(11/6) - 3/4) \approx 0.86980822$. So assumption (H1) is satisfied. Besides, we deduce

$$g_1(t,s) = \frac{1}{\Gamma(10/3)} \begin{cases} t^{7/3}(1-s)^{5/6} - (t-s)^{7/3}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{7/3}(1-s)^{5/6}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$g_2(t,s) = \begin{cases} t(1-s)^{5/6} - t + s, & 0 \leqslant s \leqslant t \leqslant 1, \\ t(1-s)^{5/6}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$G(t,s) = g_1(t,s) + \frac{t^{7/3}}{\Delta}\left(g_2\left(\frac{1}{4}, s\right) + \frac{1}{2} g_2\left(\frac{1}{2}, s\right) + \frac{1}{3} g_2\left(\frac{3}{4}, s\right)\right)$$

for all $t, s \in [0,1]$.

We also obtain $h_1(s) = (1/\Gamma(10/3))(1-s)^{5/6}(1-(1-s)^{3/2})$ for all $s \in [0,1]$, $\sigma = 1/\Gamma(10/3) + 3/(4\Delta) \approx 1.22220971$ and

$$J(s) = \begin{cases} h_1(s) + \frac{1}{\Delta}[\frac{3}{4}(1-s)^{5/6} + \frac{22s-9}{12}], & 0 \leqslant s < \frac{1}{4}, \\ h_1(s) + \frac{1}{\Delta}[\frac{3}{4}(1-s)^{5/6} + \frac{5s-3}{6}], & \frac{1}{4} \leqslant s < \frac{1}{2}, \\ h_1(s) + \frac{1}{\Delta}[\frac{3}{4}(1-s)^{5/6} + \frac{4s-3}{12}], & \frac{1}{2} \leqslant s < \frac{3}{4}, \\ h_1(s) + \frac{3}{4\Delta}(1-s)^{5/6}, & \frac{3}{4} \leqslant s \leqslant 1. \end{cases}$$

Example 1. We consider the function

$$f(t,u) = \frac{u^2+u+1}{\sqrt[3]{t(1-t)^2}} + \ln t, \quad t \in (0,1),\ u \geqslant 0.$$

We have $r(t) = -\ln t$ and $z(t) = 1/\sqrt[3]{t(1-t)^2}$ for all $t \in (0,1)$, $g(t,u) = u^2+u+1$ for all $t \in [0,1]$ and $u \geqslant 0$, $\int_0^1 r(t)\,\mathrm{d}t = 1$, $\int_0^1 z(t)\,\mathrm{d}t = \Gamma(2/3)\Gamma(1/3) \approx 3.63$. Therefore, assumption (H2) is satisfied. In addition, for $c \in (0,1/2)$ fixed, assumption (H3) is also satisfied ($f_\infty = \infty$).

After some computations, we deduce that $\int_0^1 J(s)(z(s)+r(s))\,\mathrm{d}s \approx 1.12036124$. We choose $R_1 = 2$ ($R_1 > \sigma \int_0^1 r(s)\,\mathrm{d}s$), and then we obtain $M_1 = 7$ and $\lambda^* \approx 0.255$. By Theorem 2 we conclude that problem (E*)–(BC*) has at least one positive solution for any $\lambda \in (0, \lambda^*]$.

Example 2. We consider the function

$$f(t,u) = \frac{\sqrt{u+1}}{\sqrt[4]{t^3(1-t)}} - \frac{1}{\sqrt{t}}, \quad t \in (0,1),\ u \geqslant 0.$$

Here we have $r(t) = 1/\sqrt{t}$ and $z(t) = 1/\sqrt[4]{t^3(1-t)}$ for all $t \in (0,1)$, $g(t,u) = \sqrt{u+1}$ for all $t \in [0,1]$ and $u \geqslant 0$. For $c \in (0,1/2)$ fixed, assumptions (H2) and (H4) are satisfied ($\int_0^1 r(t)\,\mathrm{d}t = 2$, $\int_0^1 z(t)\,\mathrm{d}t \approx 4.44$, $\lim_{u\to\infty} \min_{t\in[c,1-c]} f(t,u) = \infty$ and $g_\infty = 0$).

For $c = 1/4$, we obtain $\int_{1/4}^{3/4} J(s)\,\mathrm{d}s \approx 0.23472325$ and $L_0 \approx 529.001$. From the proof of Theorem 3 we deduce that $M_2 \approx 91277.5$ and $\lambda_* \approx 948407$. Then by Theorem 3 we conclude that, for any $\lambda \geqslant \lambda_*$, our problem (E*)–(BC*) has at least one positive solution.

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