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Bifurcation analysis for a singular differential system with two parameters via to topological degree theory*

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Abstract. Based on the relation between Leray–Schauder degree and a pair of strict lower and upper solutions, we focus on the bifurcation analysis for a singular differential system with two parameters, explicit bifurcation points for relative parameters are obtained by using the property of solution for the akin systems and topological degree theory.

Keywords: Leray–Schauder degree, bifurcation analysis, singular differential system, two parameters, strict lower and upper solutions.

1 Introduction

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. Bifurcations can occur in either continuous systems described by ODEs, DDEs, PDEs or discrete systems described by maps. Generally, a bifurcation often occurs when a small smooth change made to the bifurcation parameter values of a system causes a sudden 'qualitative' or topological change in its behaviour [2].

There is considerable interest in understanding bifurcation phenomena because it answers important questions like how many solutions (or steady-states) exist in different

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operating regions, and how the system behaves with changes in various parameters. In fact, a bifurcation study helps provide insights about the physics by classifying the parameter space into different regions and aids further numerical explorations. For example, by using a two-mode lumped model, Alam et al. [1] presented theory and comprehensive bifurcation analysis of thermally coupled homogeneous-heterogeneous combustion of propane and methane in short monolith. Mann et al. [19] stated the bifurcations and limit cycle behavior in milling process can be predicted from a nonlinear time finite element analysis.

In mathematics framework, several interesting theoretical results on bifurcation phenomena of system have been obtained. Among them, João et al. [9] studied a local superlinearity for elliptic systems involving parameters

$$\begin{aligned} &-\Delta u = h\bigl(|x|, u, v\bigr) \quad \text{in } A(r_1, r_2),\\ &-\Delta v = k\bigl(|x|, u, v\bigr) \quad \text{in } A(r_1, r_2),\\ &(u, v) = (0, 0) \quad \text{on } |x| = r_1,\\ &(u, v) = (a, b) \quad \text{on } |x| = r_2, \end{aligned} \tag{1}$$

where a, b are non-negative parameters, $A(r_1, r_2) = \{x \in \mathbb{R}^N \mid r_1 < |x| < r_2\}$ with $N \geqslant 3$ is an annulus. Perform the change of variable

$$t = Ar^{2-N} + B, \qquad A = \frac{(r_1 r_2)^{N-2}}{r_2^{N-2} - r_1^{N-2}}, \qquad B = \frac{r_2^{N-2}}{r_2^{N-2} - r_1^{N-2}},$$

system (1) reduce to the following second-order ordinary differential system:

$$\begin{aligned} &-u''(t) = f(t, u, v, a, b), \quad t \in (0, 1),\\ &-v''(t) = g(t, u, v, a, b), \quad t \in (0, 1),\\ &u(0) = v(0) = u(1) = v(1) = 0, \end{aligned} \tag{2}$$

where the nonlinearities f and g are given by

$$\begin{aligned} f(t, u, v, a, b) &= (1-N)^2 \frac{A^{2/(N-2)}}{(B-t)^{2(N-1)/(N-2)}} h\biggl(\biggl(\frac{A}{B-t}\biggr)^{1/(N-2)}, u + ta, v + tb\biggr),\\ g(t, u, v, a, b) &= (1-N)^2 \frac{A^{2/(N-2)}}{(B-t)^{2(N-1)/(N-2)}} k\biggl(\biggl(\frac{A}{B-t}\biggr)^{1/(N-2)}, u + ta, v + tb\biggr). \end{aligned}$$

Under the conditions where the nonlinearities f and g are superlinear at the origin as well as at infinity, by using the fixed-point theorem together with the upper-lower solutions method and degree arguments, bifurcation arcs of parameters, which guarantee the existence, non-existence, and multiplicity of positive solutions, were obtained, that is, there exists a simple arc Γ, which splits the positive quadrant of the (a, b)-plane into two disjoint sets $\mathcal{S}$ and $\mathcal{R}$ such that (2) has at least two positive solutions when $(a, b) \in \mathcal{S}$, has at least one positive solution when (a, b) is on the boundary of $\mathcal{S}$, and has no positive solutions when $(a, b) \in \mathcal{R}$. Then Liu et al. [17] studied the following singular nonlinear systems on the half-line:

$$\begin{aligned}
&-\big(p_1(t)u'(t)\big)' = \lambda\phi_1(t)f_1\big(t, u(t), v(t), a, b\big), \quad 0 < t < +\infty,\\
&-\big(p_2(t)v'(t)\big)' = \lambda\phi_2(t)f_2\big(t, u(t), v(t), a, b\big), \quad 0 < t < +\infty,\\
&\alpha_{11}u(0) - \beta_{11}\lim_{t\to 0^+} p_1(t)u'(t) = 0,\\
&\alpha_{12}\lim_{t\to+\infty} u(t) + \beta_{12}\lim_{t\to+\infty} p_1(t)u'(t) = 0,\\
&\alpha_{21}v(0) - \beta_{21}\lim_{t\to 0^+} p_2(t)v'(t) = 0,\\
&\alpha_{22}\lim_{t\to+\infty} v(t) + \beta_{22}\lim_{t\to+\infty} p_2(t)v'(t) = 0,
\end{aligned} \tag{3}$$

where $\lambda > 0$ is a parameter, $a, b \geqslant 0$ are constants and $f_1, f_2 \in C((\mathbb{R}^+)^5, \mathbb{R}^+)$ are superlinear at infinity. By using the upper-lower solutions method and the fixed-point theorem on cone in a special space, some results for the existence, nonexistence and multiplicity of positive solutions for the problem are obtained. For other works on the existence of positive solutions, we refer the reader to [3–8, 11–16, 18, 21–24].

Coupled boundary value problems (1)–(3) arise naturally in the research of Sturm–Liouville problems, reaction-diffusion equations, mathematical biology, and so on. Of course, the investigation of the model of abstract dynamics systems for coupled boundary value problems plays important role for getting an in-depth understanding of the systems oneself. Moreover, these theoretical results will then shed light on the development of the related sciences and technologies, further improve the efficiency and reliability of these systems. In view to this aim, in this paper, we are firstly concerned with the multiplicity of positive solutions for the following singular differential system with coupled boundary conditions:

$$\begin{aligned}
&u''(t) + a_1(t)u'(t) + b_1(t)u(t) + c_1(t)f_1\big(t, u(t), v(t)\big) = 0, \quad t \in (0, 1),\\
&v''(t) + a_2(t)v'(t) + b_2(t)v(t) + c_2(t)f_2\big(t, v(t), u(t)\big) = 0, \quad t \in (0, 1),\\
&u(0) = \int_0^1 g_1(s)u(s)\,\mathrm{d}s, \qquad u(1) = \int_0^1 h_1(s)u(s)\,\mathrm{d}s,\\
&v(0) = \int_0^1 g_2(s)v(s)\,\mathrm{d}s, \qquad v(1) = \int_0^1 h_2(s)v(s)\,\mathrm{d}s,
\end{aligned} \tag{4}$$

where $a_i \in C[0, 1]$, $b_i \in C([0, 1], (-\infty, 0))$, $f_i \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}_0^+)$, $c_i \in C((0, 1), \mathbb{R}^+)$, $c_i(t) \not\equiv 0$, $g_i, h_i \in L^1(0, 1)$ are nonnegative, $i = 1, 2$. Then, by using the property of solution for the akin systems (4) as well as topology degree theory, we establish the bifurcation analysis for the corresponding differential system with two parameters

$$
\begin{aligned}
&u''(t) + a_1(t)u'(t) + b_1(t)u(t) + \lambda c_1(t)f_1\big(t, u(t), v(t)\big) = 0, \quad t \in (0,1),\\
&v''(t) + a_2(t)v'(t) + b_2(t)v(t) + \mu c_2(t)f_2\big(t, v(t), u(t)\big) = 0, \quad t \in (0,1),\\
&u(0) = \int_0^1 g_1(s)u(s)\,\mathrm{d}s, \qquad u(1) = \int_0^1 h_1(s)u(s)\,\mathrm{d}s,\\
&v(0) = \int_0^1 g_2(s)v(s)\,\mathrm{d}s, \qquad v(1) = \int_0^1 h_2(s)v(s)\,\mathrm{d}s,
\end{aligned}
\tag{5}
$$

where $\lambda, \mu > 0$ are parameters, $a_i \in C[0,1]$, $b_i \in C([0,1], (-\infty, 0))$, $f_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}_0^+)$, $c_i \in C((0,1), \mathbb{R}^+)$, $c_i(t) \not\equiv 0$, $g_i, h_i \in L^1(0,1)$ $(i = 1,2)$. In the above two systems (4) and (5), $c_i(t)$ $(i = 1,2)$ is allowed to be singular at $t = 0, 1$.

The paper is organized as follows. In Section 2, we give some preliminaries and establish several lemmas. In Section 3, we establish the relation between Leray–Schauder degree and a pair of strict lower and upper solutions for a second-order differential system with integral boundary conditions. The main results shall be established in Sections 4 and 5.

2 Preliminaries and lemmas

Let $E = C[0,1] = \{u \mid u : [0,1] \to \mathbb{R}$ is continuous$\}$ be a Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Throughout the paper, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_0^+ = (0, +\infty)$ and the symbol of the form $C(M, N)$ is denoted as the set of all continuous mappings $u : M \to N$, where M and N are two arbitrary set. In the following, we shall work in Banach space $E \times E$, which is endowed with the norm $\|(u, v)\| = \max\{\|u\|, \|v\|\}$ for any $(u, v) \in E \times E$. We also define a cone on E as follows:

$$
P = \big\{u \in C[0,1] \bigm| u(t) \geqslant 0 \text{ and } u(t) \text{ is concave on } [0,1]\big\}.
$$

In order to establish the bifurcation structure of the coupled boundary value problem (4), we firstly introduce the following lemmas.

Lemma 1. (*See* [18].) *Suppose that* $a_i \in C[0,1]$, $b_i \in C([0,1], (-\infty, 0))$. *Let* $\varphi_{1,i}$ *and* $\varphi_{2,i}$ *be the unique solution of the following problems*:

$$
\begin{aligned}
&\varphi''_{1,i}(t) + a_i(t)\varphi'_{1,i}(t) + b_i(t)\varphi_{1,i}(t) = 0,\\
&\varphi_{1,i}(0) = 0, \qquad \varphi_{1,i}(1) = 1
\end{aligned}
$$

and

$$
\begin{aligned}
&\varphi''_{2,i}(t) + a_i(t)\varphi'_{2,i}(t) + b_i(t)\varphi_{2,i}(t) = 0,\\
&\varphi_{2,i}(0) = 1, \qquad \varphi_{2,i}(1) = 0,
\end{aligned}
$$

respectively. Then $\varphi_{1,i}$ *are strictly increasing on* $[0,1]$, *while* $\varphi_{2,i}$ *are strictly decreasing on* $[0,1]$, *where* $i = 1, 2$.

For convenience, we now list some assumptions to be used in the rest of the paper:

(H1) $a_i \in C[0,1], b_i \in C([0,1],(-\infty,0)), i=1,2$.
(H2) $g_i, h_i \in L^1(0,1)$ are nonnegative with $k_{1,i}>0$, $k_{4,i}>0$, $k_i>0$, where

$$k_{1,i}=1-\int_0^1 \varphi_{2,i}(s)g_i(s)\,\mathrm{d}s, \quad k_{2,i}=\int_0^1 \varphi_{1,i}(s)g_i(s)\,\mathrm{d}s,$$
$$k_{3,i}=\int_0^1 \varphi_{2,i}(s)h_i(s)\,\mathrm{d}s, \quad k_{4,i}=1-\int_0^1 \varphi_{1,i}(s)h_i(s)\,\mathrm{d}s,$$
$$k_i=k_{1,i}k_{4,i}-k_{2,i}k_{3,i}, \quad i=1,2.$$

(H3) $f_i \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $c_i \in C((0,1),\mathbb{R}^+)$, $c_i(t)\not\equiv 0$ and

$$\int_0^1 \mathscr{H}_i(s)c_i(s)\,\mathrm{d}s<+\infty, \quad i=1,2,$$

where $\mathscr{H}_i(s)$ is defined in Lemma 3.

Lemma 2. (*See* [16].) *Assume that* (H1) *and* (H2) *hold. Then, for any* $y_i \in C(0,1)\cap L^1(0,1)$*, the BVP*

$$\begin{gathered} u_i''(t)+a_i(t)u'(t)+b_i(t)u(t)+y_i(t)=0, \quad t\in(0,1),\\ u_i(0)=\int_0^1 g_i(s)u_i(s)\,\mathrm{d}s, \qquad u_i(1)=\int_0^1 h_i(s)u_i(s)\,\mathrm{d}s \end{gathered} \tag{6}$$

has a unique solution

$$u_i(t)=\int_0^1 H_i(t,s)y_i(s)\,\mathrm{d}s, \quad t\in[0,1], \tag{7}$$

where

$$\begin{aligned} H_i(t,s)&=G_i(t,s)p_i(s)+\frac{\varphi_{1,i}(t)k_{3,i}+\varphi_{2,i}(t)k_{4,i}}{k_i}\int_0^1 G_i(\tau,s)p_i(s)g_i(\tau)\,\mathrm{d}\tau\\ &\quad+\frac{\varphi_{1,i}(t)k_{1,i}+\varphi_{2,i}(t)k_{2,i}}{k_i}\int_0^1 G_i(\tau,s)p_i(s)h_i(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{8}$$

$$p_i(t)=\exp\left(\int_0^t a_i(s)\,\mathrm{d}s\right),$$

$$G_i(t,s)=\frac{1}{\rho_i}\begin{cases}\varphi_{1,i}(t)\varphi_{2,i}(s), & 0\leqslant t\leqslant s\leqslant 1,\\ \varphi_{1,i}(s)\varphi_{2,i}(t), & 0\leqslant s\leqslant t\leqslant 1,\end{cases} \qquad \rho_i=\varphi_{1,i}'(0),\ i=1,2.$$

Clearly, $u_i(t) \geqslant 0$ on $[0,1]$ if $y_i(t) \geqslant 0$ on $(0,1)$ $(i=1,2)$.

Lemma 3. (*See* [16].) *Suppose that* (H1) *and* (H2) *hold, then, for any* $t, s \in [0,1]$, *we have*

$$0 \leqslant G_i(t,s) \leqslant G_i(s,s), \qquad 0 \leqslant H_i(t,s) \leqslant \mathscr{H}_i(s), \tag{9}$$

where

$$\mathscr{H}_i(s) = G_i(s,s)p_i(s) + \frac{k_{3,i}+k_{4,i}}{k_i}\int_0^1 G_i(\tau,s)p_i(s)g_i(\tau)\,\mathrm{d}\tau + \frac{k_{1,i}+k_{2,i}}{k_i}\int_0^1 G_i(\tau,s)p_i(s)h_i(\tau)\,\mathrm{d}\tau$$

and

$$H_i(t,s) \geqslant \gamma_i(t)\mathscr{H}_i(s), \tag{10}$$

where $\gamma_i(t) = \min\{\phi_{1,i}(t), \phi_{2,i}(t)\}$, $t \in [0,1]$ $(i=1,2)$.

Since $c_i \in C((0,1),\mathbb{R}^+)$ and $c_i(t) \not\equiv 0$, there exists $t_{0,i} \in (0,1)$ such that $c_i(t_{0,i}) > 0$ $(i=1,2)$. Choose $\delta \in (0,1/2)$ such that $t_{0,i} \in (\delta, 1-\delta)$, then we have

$$H_i(t,s) \geqslant \gamma_\delta \mathscr{H}_i(s), \quad t \in [\delta, 1-\delta],\ s \in [0,1],$$

where

$$0 < \gamma_\delta = \min_{i\in\{1,2\}} \min_{t\in[\delta,1-\delta]} \bigl\{\phi_{1,i}(t), \phi_{2,i}(t)\bigr\} = \min_{i\in\{1,2\}} \min\bigl\{\phi_{1,i}(\delta), \phi_{2,i}(1-\delta)\bigr\} < 1.$$

Let

$$K = \bigl\{u \in P \bigm| u(t) \geqslant \gamma(t)\|u\|,\ t \in [0,1]\bigr\},$$

where $\gamma(t) = \min\{\gamma_1(t), \gamma_2(t)\}$. Then K is a subcone of P. It is easy to verify that, for any $u \in K$, we have $\min_{t\in[\delta,1-\delta]} u(t) \geqslant \gamma_\delta\|u\|$. For any $r > 0$, let $K_r = \{u \in K \mid \|u\| < r\}$, $\partial K_r = \{u \in K \mid \|u\| = r\}$ and $\bar{K}_r = \{u \in K \mid \|u\| \leqslant r\}$.

Next, we define several operators $A_v : K \to P$, $B_u : K \to P$, $T : K \times K \to P \times P$ and $T_i : E \to E$ as follows:

$$A_v(u)(t) = \int_0^1 H_1(t,s)c_1(s)f_1\bigl(s,u(s),v(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1], \tag{11}$$

$$B_u(v)(t) = \int_0^1 H_2(t,s)c_2(s)f_2\bigl(s,v(s),u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1], \tag{12}$$

$$T(u,v)(t) = \bigl(A_v(u)(t), B_u(v)(t)\bigr), \quad t \in [0,1], \tag{13}$$

$$(T_i u)(t) = \int_0^1 H_i(t,s)c_i(s)u(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{14}$$

It is well known that if $(u,v) \in K \times K$ solves the operator equation $(u,v) = T(u,v)$, then (u,v) is a positive solution of system (4).

For any τ: $0 < \tau < \delta$, we define $T_{\tau,i} : E \to E$ by

$$(T_{\tau,i}u)(t) = \int\limits_{\tau}^{1-\tau} H_i(t,s)c_i(s)u(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{15}$$

Lemma 4. (*See* [16].) *Suppose that* (H1)–(H3) *are satisfied, then, for the operators* T_i *defined by* (14) *and* $T_{\tau,i}$ *defined by* (15), *we have*

(i) $T_i : K \to K$ *are completely continuous linear operators.*
(ii) *The spectral radius* $r(T_i) \neq 0$, T_i *has positive eigenfunction corresponding to its first eigenvalue* $\lambda_{1,i} = (r(T_i))^{-1}$, *and* $T_{\tau,i}$ *has positive eigenfunction corresponding to its first eigenvalue* $\lambda_{\tau,i} = (r(T_{\tau,i}))^{-1}$.
(iii) *There exists an eigenvalue* $\widetilde{\lambda}_{1,i}$ *of* T_i *such that* $\lambda_{\tau,i} \to \widetilde{\lambda}_{1,i}$ *as* $\tau \to 0^+$.

The first eigenvalue $\lambda_{1,i}$ and the eigenvalue $\widetilde{\lambda}_{1,i}$ $(i = 1,2)$ will be used in the assumptions of nonlinearity of f_i $(i = 1,2)$ in Section 4.

Lemma 5. (*See* [16].) *If* (H1)–(H3) *hold, then*

(i) *For any* $R > r > 0$ *and* $v \in K$, $A_v : \bar{K}_R \setminus K_r \to K$ *is completely continuous.*
(ii) *For any* $R > r > 0$ *and* $u \in K$, $B_u : \bar{K}_R \setminus K_r \to K$ *is completely continuous.*
(iii) $T : K \times K \to P \times P$ *is completely continuous.*

Lemma 6. (*See* [7].) *Let* X *be a Banach space, and let* $P_i \subset X$ *be a closed convex cone and* W_i *a bounded open subset of* X *with boundary* ∂W_i $(i = 1,2)$. *Suppose that* $A_i : P_i \cap \overline{W_i} \to P_i$ *is a completely continuous mapping and that* $A_i u_i \neq u_i$ *for all* $u_i \in P_i \cap \partial W_i$, *then*

$$i\big(A,\, P_1 \times P_2 \cap (W_1 \times W_2),\, P_1 \times P_2\big) = i(A_1,\, P_1 \cap W_1,\, P_1) \cdot i(A_2,\, P_2 \cap W_2,\, P_2),$$

where $A(u,v) := (A_1u, A_2v)$ *for all* $(u,v) \in (P_1 \times P_2) \cap \overline{(W_1 \times W_2)}$.

Lemma 7. (*See* [7].) *Let* X *be a Banach space, and let* $P_i \subset X$ *be a closed convex cone and* W_i *a bounded open subset of* E *with boundary* ∂W_i $(i = 1,2)$ *and* $P = P_1 \times P_2$, $W = W_1 \times W_2$. *Assume that* $T : P \cap \overline{W} \to P$ *is completely continuous and that there exists compactly continuous mappings* $A_i : P_i \cap \overline{W_i} \to P_i$ *and* $H : (P \cap \overline{W}) \times [0,1] \to P$ *such that*

(i) $H(\cdot,1) = T$, $H(\cdot,0) = A$, *where* $A(u,v) := (A_1u, A_2v)$ *for all* $(u,v) \in P \cap \overline{W}$.
(ii) $A_i u_i \neq u_i$ *for all* $u_i \in P_i \cap \partial W_i$.
(iii) $H(w,\theta) \neq w$ *for all* $(w,\theta) \in (P \cap \partial W) \times [0,1]$.

Then

$$i(T,\, P \cap W,\, P) = i(A_1,\, P_1 \cap W_1,\, P_1) \cdot i(A_2,\, P_2 \cap W_2,\, P_2).$$

Lemma 8. (*See* [10].) *Let P be a cone in Banach space X. For $r > 0$, denote $P_r = \{x \in P \mid \|x\| < r\}$, $\overline{P_r} = \{x \in P \mid \|x\| \leqslant r\}$ and $\partial P_r = \{x \in P \mid \|x\| = r\}$. Suppose that $A : \overline{P}_r \to P$ is a completely continuous operator.*

(i) *If there exists $u_0 \in P \setminus \theta$ such that*

$$u - Au \neq \mu u_0, \quad u \in \partial P_r, \ \mu \geqslant 0,$$

then the fixed point index $i(A, P_r, P) = 0$.

(ii) *If*

$$Au \neq \mu u, \quad u \in \partial P_r, \ \mu \geqslant 1,$$

then the fixed point index $i(A, P_r, P) = 1$.

Lemma 9. (*See* [10].) *Let P be a cone in Banach space X. For $r > 0$, denote $P_r = \{x \in P \mid \|x\| < r\}$, $\overline{P}_r = \{x \in P \mid \|x\| \leqslant r\}$ and $\partial P_r = \{x \in P \mid \|x\| = r\}$. Suppose that $A : \overline{P}_r \to P$ is a completely continuous operator.*

(i) *If $\|Au\| \leqslant \|u\|$ for $u \in \partial P_r$, then the fixed point index $i(A, P_r, P) = 1$.*
(ii) *If $\|Au\| \geqslant \|u\|$ for $u \in \partial P_r$, then the fixed point index $i(A, P_r, P) = 0$.*

3 Upper-lower solution and Leray–Schauder degree

In the following, we will define the (strict) lower solutions and (strict) upper solutions of the integral boundary value problem (4). Theorems 1 and 2 are keys to the existence of at least two positive solutions of systems (4) and (5), see the proof of Theorems 3 and 4.

Definition 1. For $\alpha_u, \alpha_v \in C^2([0,1], \mathbb{R})$, (α_u, α_v) is said to be a lower (strict lower) solution of (4) if

$$\begin{aligned}
&\alpha_u''(t) + a_1(t)\alpha_u'(t) + b_1(t)\alpha_u(t) + c_1(t)f_1\big(t, \alpha_u(t), \alpha_v(t)\big) \geqslant (>)\ 0, \quad t \in (0,1),\\
&\alpha_v''(t) + a_2(t)\alpha_v'(t) + b_2(t)\alpha_v(t) + c_2(t)f_2\big(t, \alpha_u(t), \alpha_v(t)\big) \geqslant (>)\ 0, \quad t \in (0,1),\\
&\alpha_u(0) \leqslant (<) \int_0^1 g_1(s)\alpha_u(s)\,\mathrm{d}s, \qquad \alpha_u(1) \leqslant (<) \int_0^1 h_1(s)\alpha_u(s)\,\mathrm{d}s,\\
&\alpha_v(0) \leqslant (<) \int_0^1 g_2(s)\alpha_v(s)\,\mathrm{d}s, \qquad \alpha_v(1) \leqslant (<) \int_0^1 h_2(s)\alpha_v(s)\,\mathrm{d}s.
\end{aligned}$$

An upper (strict upper) solution $(\beta_u, \beta_v) \in C^2([0,1], \mathbb{R}) \times C^2([0,1], \mathbb{R})$ can also be defined if it satisfies the reverse of the above inequalities.

Definition 2. For a function $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $F(t,u,v)$ is said to be quasi-monotone nondecreasing with respect to v (or u) if, for fixed t,

$$F(t,u,v_1) \leqslant F(t,u,v_2) \quad \text{as } v_1 \leqslant v_2 \qquad \big(\text{or} \quad F(t,u_1,v) \leqslant F(t,u_2,v) \quad \text{as } u_1 \leqslant u_2\big).$$

Let us consider the fixed point of operator associated with (4), i.e., a compact operator $\mathscr{T}: C[0,1]\times C[0,1]\to C[0,1]\times C[0,1]$ defined by $\mathscr{T}(\varphi,\psi):=(u,v)$ if

$$\begin{aligned}
&u''(t)+a_1(t)u'(t)+b_1(t)u(t)+c_1(t)f_1\big(t,\varphi(t),\psi(t)\big)=0, \quad t\in(0,1),\\
&v''(t)+a_2(t)v'(t)+b_2(t)v(t)+c_2(t)f_2\big(t,\varphi(t),\psi(t)\big)=0, \quad t\in(0,1),\\
&u(0)=\int_0^1 g_1(s)\varphi(s)\,\mathrm{d}s, \qquad u(1)=\int_0^1 h_1(s)\varphi(s)\,\mathrm{d}s,\\
&v(0)=\int_0^1 g_2(s)\psi(s)\,\mathrm{d}s, \qquad v(1)=\int_0^1 h_2(s)\psi(s)\,\mathrm{d}s.
\end{aligned} \tag{16}$$

The following theorem provides the relation between Leray–Schauder degree of compactly continuous field $id-\mathscr{T}$ and a pair of strict lower and upper solutions for (4). Firstly, let us define a set

$$\begin{aligned}\Omega=\big\{&(u,v)\in C[0,1]\times C[0,1]\to C[0,1]\times C[0,1]\ \big|\\ &(\alpha_u,\alpha_v)<(u,v)<(\beta_u,\beta_v)\text{ on }[0,1]\big\}.\end{aligned}$$

Theorem 1. *Let (α_u,α_v) and (β_u,β_v) be a strict lower solution and a strict upper solution of* (4)*, respectively, and*

(i) $(\alpha_u(t),\alpha_v(t))<(\beta_u(t),\beta_v(t))$ *for all* $t\in[0,1]$;
(ii) $\mathscr{E}_\alpha^\beta:=\{(t,u,v)\in[0,1]\times\mathbb{R}^2 \mid \alpha_u(t)<u<\beta_u(t),\ \alpha_v(t)<v<\beta_v(t)\}\subset[0,1]\times\mathbb{R}^2$;
(iii) $f_1(t,u,v)$ *is quasi-monotone nondecreasing with respect to v, and $f_2(t,u,v)$ is quasi-monotone nondecreasing with respect to u.*

Then

$$\deg\big(id-\mathscr{T},\ \Omega,\ (\theta,\theta)\big)=1,$$

furthermore, problem (4) *has at least one solution (u,v) such that*

$$\big(\alpha_u(t),\alpha_v(t)\big)<\big(u(t),v(t)\big)<\big(\beta_u(t),\beta_v(t)\big)\quad \forall t\in[0,1].$$

Proof. The proof of this theorem is similar to the proof of Theorem 2.4 in [3]. □

In the similar way, we can prove the following theorem.

Theorem 2. *Let (α_u,α_v) and (β_u,β_v) be a lower solution and an upper solution of* (4)*, respectively, and*

(i) $(\alpha_u(t),\alpha_v(t))\leqslant(\beta_u(t),\beta_v(t))$ *for all* $t\in[0,1]$;
(ii) $\mathscr{D}_\alpha^\beta:=\{(t,u,v)\in[0,1]\times\mathbb{R}^2 \mid \alpha_u(t)\leqslant u\leqslant\beta_u(t),\ \alpha_v(t)\leqslant v\leqslant\beta_v(t)\}\subset[0,1]\times\mathbb{R}^2$;
(iii) $f_1(t,u,v)$ *is quasi-monotone nondecreasing with respect to v, and $f_2(t,u,v)$ is quasi-monotone nondecreasing with respect to u.*

Then problem (4) *has at least one solution* (u, v) *such that*

$$\big(\alpha_u(t), \alpha_v(t)\big) \leqslant \big(u(t), v(t)\big) \leqslant \big(\beta_u(t), \beta_v(t)\big) \quad \forall t \in [0,1]. \tag{17}$$

4 Positive solutions of system (4)

For any $y > 0$, we denote

$$f_{1,0}(y) = \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f_1(t,x,y)}{x}, \qquad f_{1,\infty}(y) = \liminf_{x\to +\infty} \min_{t\in[0,1]} \frac{f_1(t,x,y)}{x},$$
$$f_1^0(y) = \limsup_{x\to 0^+} \max_{t\in[0,1]} \frac{f_1(t,x,y)}{x}, \qquad f_1^\infty(y) = \limsup_{x\to +\infty} \max_{t\in[0,1]} \frac{f_1(t,x,y)}{x}.$$

For any $x > 0$, we denote

$$f_{2,0}(x) = \liminf_{y\to 0^+} \min_{t\in[0,1]} \frac{f_1(t,x,y)}{y}, \qquad f_{2,\infty}(x) = \liminf_{y\to +\infty} \min_{t\in[0,1]} \frac{f_1(t,x,y)}{y},$$
$$f_2^0(x) = \limsup_{y\to 0^+} \max_{t\in[0,1]} \frac{f_1(t,x,y)}{y}, \qquad f_2^\infty(x) = \limsup_{y\to +\infty} \max_{t\in[0,1]} \frac{f_1(t,x,y)}{y}.$$

Lemma 10. (*See* [20].) *Suppose conditions* (H1)–(H3) *are satisfied.*

(i) *If*

$$\inf_{y\in\mathbb{R}^+} f_{1,0}(y) > \lambda_{1,1}, \tag{18}$$

then, for any $v \in K$*, there exists* $r_1 > 0$ *such that*

$$i(A_v, K_r, K) = 0, \quad 0 < r < r_1.$$

(ii) *If*

$$\inf_{x\in\mathbb{R}^+} f_{2,0}(x) > \lambda_{1,2}, \tag{19}$$

then, for any $u \in K$*, there exists* $r_2 > 0$ *such that*

$$i(B_u, K_r, K) = 0, \quad 0 < r < r_2.$$

Lemma 11. (*See* [20].) *Suppose conditions* (H1)–(H3) *are satisfied.*

(i) *If*

$$\inf_{y\in\mathbb{R}^+} f_{1,\infty}(y) > \tilde{\lambda}_{1,1}, \tag{20}$$

then, for any $u \in K \setminus \{0\}$*, there exists* $R_1 > 0$ *such that*

$$i(A_v, K_R, K) = 0 \quad \forall R > R_1.$$

(ii) *If*
$$\inf_{x\in\mathbb{R}^+} f_{2,\infty}(x) > \tilde{\lambda}_{1,2}, \tag{21}$$
then, for any $v \in K \setminus \{0\}$*, there exists* $R_2 > 0$ *such that*
$$i(B_u, K_R, K) = 0 \quad \forall R > R_2.$$

Now let us finish our presentation to announce our main results that can be stated as follows:

Theorem 3. *Suppose that* (H1)–(H3) *and the following conditions hold*:

(H4) $f_1(t,x,y)$ (*resp.* $f_2(t,x,y)$) *is quasi-monotone nondecreasing w.r.t.* x (*resp.* y);
(H5) $\inf_{y\in\mathbb{R}^+} f_{1,0}(y) > \lambda_{1,1},\ \inf_{x\in\mathbb{R}^+} f_{2,0}(x) > \lambda_{1,2}$;
(H6) $\inf_{y\in\mathbb{R}^+} f_{1,\infty}(y) > \tilde{\lambda}_{1,1},\ \inf_{x\in\mathbb{R}^+} f_{2,\infty}(x) > \tilde{\lambda}_{1,2}$;
(H7) $f_i \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}_0^+)$*, where* $\mathbb{R}_0^+ = (0,+\infty)$*,* $i = 1,2$.

If system (4) *has a strict upper solution* (β_u, β_v)*, then system* (4) *has at least two positive solutions.*

Proof. From Lemma 10 and (H5), we can find $r_1, r_2 > 0$ such that
$$i(A_v, K_r, K) = 0 \quad \forall v \in K,\ r \in (0, r_1];$$
$$i(B_u, K_r, K) = 0 \quad \forall u \in K,\ r \in (0, r_2].$$

Denote $r_0 = \min\{r_1, r_2\}$. Since $A_v : K \to K$, $B_u : K \to K$, $T : K\times K \to K\times K$ are completely continuous, from Theorem 6, we get
$$(T,\ K_r\times K_r,\ K\times K) = i(A_v, K_r, K)\times i(B_u, K_r, K) = 0 \quad \forall r \in (0, r_0].$$

Similarly, from Lemma 11 and (H6), we can find $R_0 > 0$ such that
$$(T,\ K_R\times K_R,\ K\times K) = i(A_v, K_R, K)\times i(B_u, K_R, K) = 0 \quad \forall R \in [R_0, +\infty).$$

Since $f_i \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}_0^+)$ $(i = 1,2)$, we can expand f_i as
$$f_i(t,x,y) = \begin{cases} f_i(t,x,y), & x \geqslant 0 \text{ and } y \geqslant 0,\ t \in [0,1], \\ f_i(t,|x|,|y|), & x < 0 \text{ or } y < 0,\ t \in [0,1], \end{cases} \tag{22}$$

then f_i $(i = 1,2)$ is even with respect to u and v. Let $\varepsilon > 0$ small enough and $(\alpha_u, \alpha_v) = (-\varepsilon\beta_u, -\varepsilon\beta_v)$, then we can prove that (α_u, α_v) is a strict lower solution to system (4). Denote
$$\Omega = \bigl\{(u,v) \in C[0,1]\times C[0,1] \bigm| (\alpha_u, \alpha_v) < (u,v) < (\beta_u, \beta_v)\bigr\},$$

then from Theorem 1 we have
$$i\bigl(T,\ (K\times K)\cap\Omega,\ K\times K\bigr) = \deg\bigl(id - T, \Omega, (\theta,\theta)\bigr) = 1. \tag{23}$$

Now select $0 < r < \min\{r_0, \min_{t\in[0,1]}\{\beta_u(t), \beta_v(t)\}\}$ and $R > \max\{R_0, \|(\beta_u, \beta_v)\|\}$, in view of Lemmas 10 and 11, we get

$$i(T,\, K_r \times K_r,\, K \times K) = i(T,\, K_R \times K_R,\, K \times K) = 0. \tag{24}$$

Combining with (23)–(24) and by the additivity of fixed point index, we obtain

$$\begin{aligned} &i\bigl(T, \bigl[(K \times K) \cap \Omega\bigr] \setminus \overline{K_r \times K_r},\, K \times K\bigr) = 1, \\ &i\bigl(T, (K_R \times K_R) \setminus \overline{(K_r \times K_r) \cap \Omega},\, K \times K\bigr) = -1, \end{aligned} \tag{25}$$

which means that system (4) has at least two positive solutions. □

5 Bifurcation analysis for singular differential system with two parameters

In this section, we shall apply the property of solution for the akin systems (4) and topology degree theory to study explicit bifurcation points for relative parameters to system (5).

Theorem 4. *If* (H1)–(H4), (H7) *and the condition*

(H6′) $$\liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f_1(t,x,0)}{x} = +\infty, \qquad \liminf_{y\to\infty} \min_{t\in[0,1]} \frac{f_1(t,0,y)}{y} = +\infty$$

hold, then the following conclusions are valid:

(i) *There exist $\lambda_*, \mu_* \in \mathbb{R}_0^+$ and a simple arc Γ_0, excluding both end points $(\lambda_*, 0)$ and $(0, \mu_*)$, such that $\Gamma_0 \subset \mathbb{R}_0^+ \times \mathbb{R}_0^+$ separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint subset $\mathscr{O}_1$ and $\mathscr{O}_2$ such that system* (5) *has no solutions, at least one or at least two nontrivial positive solutions according to $(\lambda, \mu) \in \mathscr{O}_2, \Gamma_0$ or $\mathscr{O}_1$, respectively.*

(ii) *There exist $\lambda^* \geqslant \lambda_*$ and $\mu^* \geqslant \mu_*$ such that system* (5) *has no solutions for $(\lambda, \mu) \in \{(\lambda, 0) \mid \lambda > \lambda^*\} \cup \{(0, \mu) \mid \mu > \mu^*\}$, at least one semi-trivial positive solution for $(\lambda, \mu) \in \{(\lambda^*, 0), (0, \mu^*)\}$ and at least two semi-trivial positive solutions for $(\lambda, \mu) \in \{(\lambda, 0) \mid \lambda \in (0, \lambda^*)\} \cup \{(0, \mu) \mid \mu \in (0, \mu^*)\}$.*

Now, for any given $\tau \in [0, 1]$, we define

$$\begin{aligned} g_{1,\tau}\bigl(t, u(t), v(t)\bigr) &= \tau f_1\bigl(t, u(t), v(t)\bigr) + (1-\tau) f_1\bigl(t, u(t), 0\bigr), \\ g_{2,\tau}\bigl(t, u(t), v(t)\bigr) &= \tau f_2\bigl(t, u(t), v(t)\bigr) + (1-\tau) f_2\bigl(t, 0, v(t)\bigr). \end{aligned} \tag{26}$$

For any $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$, we define the mappings $A_{\lambda,v}, B_{\mu,u}$ and $T_{\lambda,\mu}$ by

$$\bigl(A^\tau_{\lambda,v} u\bigr)(t) = \lambda \int_0^1 H_1(t,s) c_1(s) g_{1,\tau}\bigl(s, u(s), v(s)\bigr)\, \mathrm{d}s, \tag{27}$$

$$\bigl(B^\tau_{\mu,u} v\bigr)(t) = \mu \int_0^1 H_2(t,s) c_2(s) g_{2,\tau}\bigl(s, v(s), u(s)\bigr)\, \mathrm{d}s \tag{28}$$

and
$$T^\tau_{\lambda,\mu}(u,v)(t) = \big(\big(A^\tau_{\lambda,v}u\big)(t), \big(B^\tau_{\mu,u}v\big)(t)\big). \tag{29}$$

By Lemma 5, we know that, for any $u, v \in K$ and $\lambda, \mu \geqslant 0$, $\tau \in [0,1]$, the operators $A^\tau_{\lambda,v}, B^\tau_{\mu,u} : K \to K$, $T^\tau_{\lambda,\mu} : K \times K \to K \times K$ are well defined and $T^\tau_{\lambda,\mu} : K \times K \to K \times K$ are completely continuous.

It is obvious that the existence of positive solutions of system (5) is equivalent to the existence of nontrivial fixed points of $T^1_{\lambda,\mu}$ in $K \times K$.

Lemma 12. *Assume that* (H1)–(H4) *and* (H7) *hold, then, for any* $r > 0$ *there exists a* $(\lambda_r, \mu_r) \in \mathbb{R}^+_0 \times \mathbb{R}^+_0$ *such that, for any* $(\lambda, \mu) \in ([0, \lambda_r] \times [0, \mu_r]) \setminus \{(0,0)\}$, $T^1_{\lambda,\mu}$ *has a nontrivial fixed point in* $K_r \times K_r$.

Proof. By (H3) and (H7), for any given $r > 0$, we can denote

$$\alpha = \sup_{u \in \partial K_r} \|A_v u\|, \qquad \beta = \sup_{v \in \partial K_r} \|B_u v\|.$$

It is easy to see that α and β are both positive. Set $(\lambda_r, \mu_r) = (r/(2\alpha), r/(2\beta))$, then, for any $(\lambda, \mu) \in [0, \lambda_r] \times [0, \mu_r]$, we have

$$\big\|A^1_{\lambda,v}(u)\big\| \leqslant \lambda\alpha < r = \|u\| \quad \forall (u,v) \in \partial K_r \times K$$

and

$$\big\|B^1_{\mu,u}(v)\big\| \leqslant \mu\beta < r = \|v\| \quad \forall (u,v) \in K \times \partial K_r.$$

It follows from Theorem 6 and Lemma 9 that

$$i\big(T^1_{\lambda,\mu},\, K_r \times K_r,\, K \times K\big) = i\big(A^1_{\lambda,v}, K_r, K\big) \times i\big(B^1_{\mu,u}, K_r, K\big) = 1,$$

together with (H7), which implies that $T^1_{\lambda,\mu}$ has a nontrivial fixed point in $K_r \times K_r$ for all $(\lambda, \mu) \in ([0, \lambda_r] \times [0, \mu_r]) \setminus \{(0,0)\}$. □

Lemma 13. *Assume that* (H1)–(H4), (H6′) *and* (H7) *hold. Denote*

$$S_u \equiv \big\{u \mid A^\tau_{\lambda,v}(u) = u,\ \lambda \in I,\ \tau \in [0,1] \text{ and } (u,v) \in K \times K\big\}$$

and

$$S_v \equiv \big\{v \mid B^\tau_{\mu,u}(v) = v,\ \mu \in I,\ \tau \in [0,1] \text{ and } (u,v) \in K \times K\big\},$$

where $I \subset [p, +\infty)$ *for some constant* $p > 0$. *Then there exists a constant* C_I *such that* $\|u\| \leqslant C_I$ *for all* $u \in S_u$ *and* $\|v\| \leqslant C_I$ *for all* $v \in S_v$.

Proof. First, we prove that there exists a constant C'_I such that $\|u\| \leqslant C'_I$ for all $u \in S_u$. Suppose, by the contradiction, that there exist sequences $\{(\lambda_n, \tau_n)\} \subset I \times [0,1]$ and $\{(u_n, v_n)\} \subset K \times K$ such that $A^{\tau_n}_{\lambda_n, v_n}(u_n) = u_n$ and $\lim_{n\to\infty} \|u_n\| = +\infty$. By (H6′), for

$$k > \left(p\gamma_\delta \int\limits_\delta^{1-\delta} H_1\left(\frac{1}{2}, s\right) c_1(s)\,\mathrm{d}s\right)^{-1},$$

there exists an $R > 0$ such that

$$f_1(t,u,0) \geqslant ku \quad \forall (t,u) \in [0,1] \times [R,+\infty).$$

Now choose $u_N \in \{u_n\}$ such that $u_N(t) \geqslant \gamma_\delta \|u_N\| \geqslant R$ for all $t \in [\delta, 1-\delta]$, by (H4), (H7) and the above inequality, we obtain that

$$\begin{aligned}\|u_N\| &\geqslant A^{\tau_N}_{\lambda_N,v_N} u_N\left(\frac{1}{2}\right) \geqslant \lambda_N \int_0^1 H_1\left(\frac{1}{2}, s\right) c_1(s) f_1\big(s, u_N(s), 0\big)\,\mathrm{d}s \\ &\geqslant k\lambda_N \int_\delta^{1-\delta} H_1\left(\frac{1}{2}, s\right) c_1(s) u_N(s)\,\mathrm{d}s \\ &\geqslant kp\gamma_\delta \int_\delta^{1-\delta} H_1\left(\frac{1}{2}, s\right) c_1(s)\,\mathrm{d}s \cdot \|u_N\| > \|u_N\|,\end{aligned}$$

which is a contradiction.

Similarly, we can show that there exists a constant C''_I such that $\|v\| \leqslant C''_I$ for all $v \in S_v$. Let $C_I = \max\{C'_I, C''_I\}$, then the proof is completed. □

Lemma 14. *Assume that* (H1)–(H4), (H6′) *and* (H7) *hold and* $T^1_{\bar\lambda,\bar\mu}$ *has a nontrivial fixed point* $(\bar u, \bar v) \in K \times K$ *for some* $(\bar\lambda, \bar\mu) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \{(0,0)\}$, *then* $T^1_{\lambda,\mu}$ *has a nontrivial fixed point in* $K \times K$ *for any* $(\lambda,\mu) \in ([0,\bar\lambda] \times [0,\bar\mu]) \setminus \{(0,0)\}$.

Proof. For any given $(\lambda,\mu) \in ([0,\bar\lambda] \times [0,\bar\mu]) \setminus \{(0,0)\}$, it is easy to verify that $(\alpha_u, \alpha_v) = (0,0)$ and $(\beta_u, \beta_v) = (\bar u, \bar v)$ are a pair of lower and upper solutions to system (5). From Theorem 2 and conditions (H4), (H7), we know that system (5) has at least one positive solution for $(\lambda,\mu) \in ([0,\bar\lambda] \times [0,\bar\mu]) \setminus \{(0,0)\}$, that is, $T^1_{\lambda,\mu}$ has a nontrivial fixed point in $K \times K$ for any $(\lambda,\mu) \in ([0,\bar\lambda] \times [0,\bar\mu]) \setminus \{(0,0)\}$. □

In what follows, denote

$$\begin{aligned}&\mathscr{S} = \big\{(\lambda,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+ \;\big|\; T^1_{\lambda,\mu} \text{ has a fixed point in } K \times K\big\}, \\ &\operatorname{int}\mathscr{S} = \{\text{the interior of } \mathscr{S}\}.\end{aligned} \tag{30}$$

We will discuss the properties and the structures of sets $\mathscr{S}$ and $\operatorname{int}\mathscr{S}$ in the subsequent lemmas.

Lemma 15. *Assume that* (H1)–(H4), (H6′) *and* (H7) *hold, then*

(i) $\{(0,0)\} \not\subseteq \mathscr{S}$;
(ii) $\mathscr{S}$ *is closed and bounded*;
(iii) $\operatorname{int}\mathscr{S}$ *is nonempty, open and bounded.*

Proof. (i) Clearly, $(0,0) \in \mathscr{S}$. By Lemma 12, $\mathscr{S} \setminus \{(0,0)\}$ is nonempty.

(ii) From Lemmas 13 and 14, the compactness of $T^1_{\lambda,\mu}$ and Lebesgue dominated convergence theorem, $\mathscr{S}$ is closed.

Next, we prove that $\mathscr{S}$ is bounded. If $\mathscr{S}$ is bounded, then there exist sequences $\{(u_n,v_n)\}_{n=1}^{\infty} \subset K \times K$ and $\{(\lambda_n,\mu_n)\}_{n=1}^{\infty} \subset \mathbb{R}^+ \times \mathbb{R}^+$ such that $T^1_{\lambda_n,\mu_n}(u_n,v_n) = (u_n,v_n)$ and $\lim_{n\to\infty} \lambda_n = +\infty$ or $\lim_{n\to\infty} \mu_n = +\infty$.

Without loss of generality, suppose that $\lim_{n\to\infty} \lambda_n = +\infty$. From Lemma 13, we know that there exists a constant $M > 0$ such that $\|u_n\| \leqslant M$, $n = 1,2,\dots$. Let $q = \min\{f_1(t,u,0) \mid (t,u) \in [0,1] \times [0,M]\}$, then q is positive and

$$\begin{aligned}\|u_n\| &\geqslant A^1_{\lambda_n,v_n} u_n\left(\frac{1}{2}\right) \geqslant \lambda_n \int\limits_0^1 H_1\left(\frac{1}{2},s\right) c_1(s) f_1\big(s,u(s),0\big)\,\mathrm{d}s \\ &\geqslant q\lambda_n \int\limits_{\delta}^{1-\delta} H_1\left(\frac{1}{2},s\right) c_1(s)\,\mathrm{d}s \to \infty \quad \text{as } n \to +\infty,\end{aligned}$$

which is a contradiction. Here $\mathscr{S}$ is bounded.

(iii) From Lemma 12, it is easy to see that $\operatorname{int}\mathscr{S}$ is nonempty. It is clear that $\operatorname{int}\mathscr{S}$ is open and bounded. □

For convenience, we introduce the following notations:

$$\begin{aligned}\partial(\operatorname{int}\mathscr{S}) &:= \{\text{the boundary of } \operatorname{int}\mathscr{S}\}, \\ d(\operatorname{int}\mathscr{S}) &:= \{\text{the derived set of } \operatorname{int}\mathscr{S}\}, \\ \overline{\operatorname{int}\mathscr{S}} &:= \{\text{the closure of } \operatorname{int}\mathscr{S}\}.\end{aligned}$$

Based on the work of [3–6], we can easily get the following result.

Lemma 16. (*See* [3].) *Assume that* (H1)–(H4), (H6′) *and* (H7) *hold, then there exists a* $(\lambda_*,\mu_*) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *such that* $\{(\lambda,0) \mid \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) \mid \mu \in [0,\mu_*]\} \subset \partial(\operatorname{int}\mathscr{S})$ *and* $\overline{\operatorname{int}\mathscr{S}} \subset [0,\lambda_*] \times [0,\mu_*]$.

Now define a family of straight lines

$$L(t) = \big\{(\lambda,\mu) \in \mathbb{R}^2 \bigm| \mu = \lambda - t,\ t \in [-\mu_*,\lambda_*]\big\}. \tag{31}$$

Moreover, by view of Lemma 16, for all $t \in [-\mu_*,\lambda_*]$,

$$\begin{aligned}&\lambda(t) = \sup\big\{\lambda \bigm| (\lambda,\mu) \in L(t) \cap \overline{\operatorname{int}\mathscr{S}}\big\}, \\ &\mu(t) = \lambda(t) - t \quad \text{and} \quad \Gamma(t) = \big(\lambda(t),\mu(t)\big)\end{aligned} \tag{32}$$

are well defined. Then we have

Lemma 17. (*See* [3].) *Assume that* (H1)–(H4), (H6′) *and* (H7) *hold, then* $\Gamma(t) \in L(t) \cap \partial(\operatorname{int}\mathscr{S})$, $t \in [-\mu_*,\lambda_*]$.

Lemma 18. (*See* [3].) *Assume that* (H1)–(H4), (H6′) *and* (H7) *hold, then the following conclusions are valid*:

(i) $\lambda(t)$ *is monotone nondecreasing and* $\mu(t)$ *is monotone nondecreasing, which implies that* $\{\Gamma(t) \mid t \in [-\mu_*, \lambda_*]\}$ *is a simple arc*;

(ii) $\{\Gamma(t) \mid t \in [-\mu_*, \lambda_*]\} \cap \{(\lambda, \mu) \mid \lambda\mu = 0\} = \{(\lambda_*, 0), (0, \mu_*)\}$;

(iii) $\partial(\operatorname{int}\mathscr{S})$ *is a simple closed curve, and* $\partial(\operatorname{int}\mathscr{S}) = \{\Gamma(t) \mid t \in [-\mu_*, \lambda_*]\} \cup \{(\lambda, 0) \mid \lambda \in [0, \lambda_*]\} \cup \{(0, \mu) \mid \mu \in [0, \mu_*]\}$;

(iv) $\overline{\operatorname{int}\mathscr{S}} = \bigcup_{t\in[-\mu_*,\lambda_*]}\{(\lambda, \mu) \in L(t) \mid 0 \leqslant \lambda \leqslant \lambda(t)$ *and* $0 \leqslant \mu \leqslant \mu(t)\}$.

Lemma 19. (*See* [3].) *Assume that* (H1)–(H4), (H6′) *and* (H7) *hold, then there exist* $\lambda^* \leqslant \lambda_*$ *and* $\mu^* \leqslant \mu_*$ *such that* $\partial(\operatorname{int}\mathscr{S}) = \{\Gamma(t) \mid t \in [-\mu_*, \lambda_*]\} \cup \{(\lambda, 0) \mid \lambda \in [0, \lambda^*]\} \cup \{(0, \mu) \mid \mu \in [0, \mu^*]\}$.

Proof of Theorem 4. (i) From Lemma 18, $\{\Gamma(t) \mid t \in [-\mu_*, \lambda_*]\}$ is a simple arc with both end points $(\lambda_*, 0)$ and $(0, \mu_*)$ and separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint subsets $\operatorname{int}\mathscr{S}$ and $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \setminus \overline{\operatorname{int}\mathscr{S}}$. Denote

$$\Gamma_0 = \bigl\{\Gamma(t) \bigm| t \in [-\mu_*, \lambda_*]\bigr\}, \quad \mathscr{O}_1 = \operatorname{int}\mathscr{S} \quad \text{and} \quad \mathscr{O}_2 = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \setminus \overline{\operatorname{int}\mathscr{S}}. \tag{33}$$

From Lemmas 18 and 19 and the fact that system (5) has zero solution iff $(\lambda, \mu) = (0, 0)$, we obtain that system (5) has no solutions for $(\lambda, \mu) \in \mathscr{O}_2$ and at least one positive solution for $(\lambda, \mu) \in \Gamma_0 \cup \mathscr{O}_1$. Next, it is sufficient to show that system (5) has at least two nontrivial positive solutions for $(\lambda, \mu) \in \mathscr{O}_1$.

By $(\lambda, \mu) \in \mathscr{O}_1$, there is a $(\bar\lambda, \bar\mu) \in \mathscr{O}_1$ with $\lambda < \bar\lambda$ and $\mu < \bar\mu$. First, we show that there exists an open bounded set $W \subset C^2([0, 1], \mathbb{R}) \times C^2([0, 1], \mathbb{R})$ such that

$$i\bigl(T^1_{\lambda,\mu}, W \cap (K \times K), K \times K\bigr) = 1. \tag{34}$$

Assume that $(\bar u, \bar v)$ is a nontrivial positive solution to system (5) with $(\lambda, \mu) = (\bar\lambda, \bar\mu)$. Let

$$\Phi_i(t) = k_{3,i}\varphi_{3,i}(t) + k_{4,i}\varphi_{2,i}(t) + k_{1,i}\varphi_{1,i}(t) + k_{2,i}\varphi_{2,i}(t), \quad i = 1, 2.$$

From the uniform continuity of f_1, f_2 on compact subsets, there exists an $\epsilon \in (0, 1)$ such that, for all $\varepsilon \in (0, \epsilon]$,

$$\lambda\bigl[f_1\bigl(t, \bar u(t) + \varepsilon\Phi_1(t), \bar v(t) + \varepsilon\Phi_2(t)\bigr) - f_1\bigl(t, \bar u(t), \bar v(t)\bigr)\bigr] < (\bar\lambda - \lambda)q_1, \quad t \in [0, 1],$$
$$\mu\bigl[f_2\bigl(t, \bar u(t) + \varepsilon\Phi_1(t), \bar v(t) + \varepsilon\Phi_2(t)\bigr) - f_2\bigl(t, \bar u(t), \bar v(t)\bigr)\bigr] < (\bar\mu - \mu)q_2, \quad t \in [0, 1],$$

where $q_1 = \min\{f_1(t, u, 0) \mid (t, u) \in [0, 1] \times [0, \|\bar u\|]\} > 0$, $q_2 = \min\{f_2(t, 0, v) \mid (t, v) \in [0, 1] \times [0, \|\bar v\|]\} > 0$. Then

$$\begin{aligned}
&\lambda f_1\bigl(t, \bar u(t) + \varepsilon\Phi_1(t), \bar v(t) + \varepsilon\Phi_2(t)\bigr) - \bar\lambda f_1\bigl(t, \bar u(t), \bar v(t)\bigr)\\
&\quad < (\bar\lambda - \lambda)\bigl[q_1 - f_1\bigl(t, \bar u(t), 0\bigr)\bigr] \leqslant 0, \quad t \in [0, 1],\\
&\mu f_2\bigl(t, \bar u(t) + \varepsilon\Phi_1(t), \bar v(t) + \varepsilon\Phi_2(t)\bigr) - \bar\mu f_2\bigl(t, \bar u(t), \bar v(t)\bigr)\\
&\quad < \bigl(\bar\mu - \mu\bigr)\bigl[q_2 - f_2\bigl(t, 0, \bar v(t)\bigr)\bigr] \leqslant 0, \quad t \in [0, 1].
\end{aligned}$$

By Lemma 1, we have that

$$\begin{aligned}
&-\big(\bar{u}(t)+\varepsilon\Phi_1(t)\big)'' - a_1(t)\big(\bar{u}(t)+\varepsilon\Phi_1(t)\big)' - b_1(t)\big(\bar{u}(t)+\varepsilon\Phi_1(t)\big)\\
&\quad > \lambda c_1(t) f_1\big(t, \bar{u}(t)+\varepsilon\Phi_1(t), \bar{v}(t)+\varepsilon\Phi_2(t)\big), \quad t\in(0,1),\\
&-\big(\bar{v}(t)+\varepsilon\Phi_2(t)\big)'' - a_2(t)\big(\bar{v}(t)+\varepsilon\Phi_2(t)\big)' - b_2(t)\big(\bar{v}(t)+\varepsilon\Phi_2(t)\big)\\
&\quad > \mu c_2(t) f_2\big(t, \bar{u}(t)+\varepsilon\Phi_1(t), \bar{v}(t)+\varepsilon\Phi_2(t)\big), \quad t\in(0,1).
\end{aligned}$$

By (H2) ad Definition 1, it is easy to see that $(\beta_u(t),\beta_v(t)) = (\bar{u}(t)+\varepsilon\Phi_1(t), \bar{v}(t)+\varepsilon\Phi_2(t))$ is a strict upper solution of (5). From Theorem 3, we know that system (5) has at least two nontrivial positive solutions as $(\lambda,\mu)\in\mathscr{O}_1$.

(ii) For convenience, denote

$$\begin{aligned}
L &= \big\{(\lambda,0) \bigm| \lambda\in[0,\lambda^*)\big\} \cup \big\{(0,\mu) \bigm| \mu\in[0,\mu^*)\big\},\\
L' &= \big\{(\lambda,0) \bigm| \lambda>\lambda^*\big\} \cup \big\{(0,\mu) \bigm| \mu>\mu^*\big\}.
\end{aligned}$$

By Lemma 19, we have that

$$\big\{(\lambda^*,0),(0,\mu^*)\big\}\cup L\subset\partial\mathscr{S} \quad \text{and} \quad L'\cap\mathscr{S}=\emptyset.$$

From item (ii) of Lemma 15 and the fact that system (5) has a zero solution iff $(\lambda,\mu) = (0,0)$, we obtain that system (5) has no solutions for $(\lambda,\mu)\in L'$ and at least two semi-trivial positive solution for $(\lambda,\mu)\in\{(\lambda^*,0),(0,\mu^*)\}\cup L\setminus\{(0,0)\}$. Now it is sufficient to show that system (5) has at least two semi-trivial positive solutions for $(\lambda,\mu)\in L\setminus\{(0,0)\}$.

Without loss of generality, assume that $\lambda\in(0,\lambda^*)$, $\mu=0$ and $(u^*,0)$ is a semi-trivial positive solution of system (5) with $(\lambda,\mu)=(\lambda^*,0)$. Since f_1 is uniformly continuous on closed intervals, then there exists an $\epsilon\in(0,1)$ such that, for all $\varepsilon\in(0,\epsilon]$,

$$\lambda\big[f_1\big(t,u^*+\varepsilon\Phi_1(t),0\big) - f_1(t,u^*,0)\big] < (\lambda^*-\lambda)q, \quad t\in[0,1],$$

where $q=\min\{f_1(t,u,0) \mid (t,u)\in[0,1]\times[0,\|u^*\|]\}>0$. Thus,

$$\lambda f_1\big(t,u^*+\varepsilon\Phi_1(t),0\big) - \lambda^* f_1(t,u^*,0) < (\lambda^*-\lambda)\big[q - f_1(t,u^*,0)\big] \leqslant 0, \quad t\in[0,1].$$

Furthermore, by Lemma 1, we have that

$$\begin{aligned}
&-\big(u^*(t)+\varepsilon\Phi_1(t)\big)'' - a_1\big(t\big(u^*(t)+\varepsilon\Phi_1(t)\big)' - b_1(t)\big(u^*(t)+\varepsilon\Phi_1(t)\big)\\
&\quad > \lambda f_1(t,u^*(t)+\varepsilon\Phi_1(t),0), \quad t\in(0,1).
\end{aligned}$$

By (H2) and Definition 1, we get that $u^*(t)+\varepsilon\varphi_{2,1}(t)$ is a strict upper solution to the first equation (with $v=0$) of system (5).

Hence, by Theorem 3, the first equation (with $v=0$) of system (5) has at least two positive solutions u_1, u_2 for $\lambda\in(0,\lambda^*)$. Thus, system (5) has at least two semi-trivial positive solutions $(u_1,0)$, $(u_2,0)$ for $\lambda\in(0,\lambda^*)$ for $\mu=0$. □

Example 1. Consider the following integral boundary value system:

$$
\begin{aligned}
&u''(t) - u(t) + \frac{\lambda}{t}\bigl(1 + u^5(t) + v^3(t)\bigr) = 0, \quad t \in (0,1),\\
&v''(t) - v(t) + \frac{\mu}{t}\bigl(1 + \mathrm{e}^{u(t)} + \mathrm{e}^{v(t)}\bigr) = 0, \quad t \in (0,1),\\
&u(0) = \int_0^1 u(s)\,\mathrm{d}s, \qquad u(1) = \int_0^1 u(s)\,\mathrm{d}s,\\
&v(0) = \int_0^1 v(s)\,\mathrm{d}s, \qquad v(1) = \int_0^1 v(s)\,\mathrm{d}s.
\end{aligned}
\tag{35}
$$

System (35) is a special case of form (5), where $a_1(t) = a_2(t) \equiv 0$, $b_1(t) = b_2(t) \equiv -1$, $c_1(t) = c_2(t) = 1/t$, $h_1(t) = h_2(t) = g_1(t) = g_2(t) \equiv 1$, $f_1(t,x,y) = 1 + x^5 + y^3$, $f_2(t,x,y) = 1 + \mathrm{e}^x + \mathrm{e}^y$. Obviously, $c_1(t)$, $c_2(t)$ are singular at $t = 0$.

Based on Lemma 1, let $\varphi_{1,i}$ and $\varphi_{2,i}$ be the unique solutions of the following two boundary value problems, respectively:

$$
\begin{aligned}
&\varphi''_{1,i}(t) - \varphi_{1,i}(t) = 0, \quad t \in (0,1),\\
&\varphi_{1,i}(0) = 0, \quad \varphi_{1,i}(1) = 1, \quad i = 1,2,
\end{aligned}
$$

$$
\begin{aligned}
&\varphi''_{2,i}(t) - \varphi_{2,i}(t) = 0, \quad t \in (0,1),\\
&\varphi_{2,i}(0) = 1, \quad \varphi_{2,i}(1) = 0, \quad i = 1,2.
\end{aligned}
$$

Then it is easy to verify that

$$
\begin{gathered}
\varphi_{1,i}(t) = \frac{\mathrm{e}}{\mathrm{e}^2 - 1}\bigl(\mathrm{e}^t - \mathrm{e}^{-t}\bigr), \qquad \varphi_{2,i}(t) = \frac{1}{\mathrm{e}^2 - 1}\bigl(\mathrm{e}^{2-t} - \mathrm{e}^t\bigr),\\
k_{1,i} = k_{4,i} = \frac{2}{\mathrm{e}+1}, \qquad k_{2,i} = k_{3,i} = \frac{\mathrm{e}-1}{\mathrm{e}+1}, \qquad k_i = \frac{4 - (\mathrm{e}-1)^2}{(\mathrm{e}+1)^2},\\
\rho_i = \varphi'_{1,i}(0) = \frac{2\mathrm{e}}{\mathrm{e}^2 - 1}, \qquad p_i(t) = 1,\\
G_i(t,s) = \frac{1}{2(\mathrm{e}^2 - 1)}
\begin{cases}
(\mathrm{e}^t - \mathrm{e}^{-t}(\mathrm{e}^{2-s} - \mathrm{e}^s), & 0 \leqslant t \leqslant s \leqslant 1,\\
(\mathrm{e}^s - \mathrm{e}^{-s}(\mathrm{e}^{2-t} - \mathrm{e}^t), & 0 \leqslant s \leqslant t \leqslant 1.
\end{cases}
\end{gathered}
$$

By computation, we know that $0 \leqslant G_i(t,s) \leqslant 2s$, $t, s \in [0,1]$ and

$$
\mathscr{H}_i(s) = G_i(s,s) + \frac{1}{k}\int_0^1 G_i(s,\tau)\,\mathrm{d}\tau + \frac{1}{k}\int_0^1 G_i(s,\tau)\sigma\tau \leqslant 2s + \frac{4s}{k} \leqslant 80s, \quad s \in [0,1],
$$

then $\int_0^1 \mathscr{H}_i(s)c_i(s)\,\mathrm{d}s < +\infty$. It is obvious that f_1, f_2 satisfy (H4), (H7) and (H6′). Therefore, the conclusions of Theorem 4 are hold.

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