Dynamic analysis of a fractional-order single-species model with diffusion[*]

Hong-Li Li[a,1]**, Long Zhang**[a]**, Cheng Hu**[a]**, Yao-Lin Jiang**[a,b]**, Zhidong Teng**[a]

[a]College of Mathematics and System Sciences, Xinjiang University,
Urumqi 830046, China
lihongli3800087@163.com

[b]Department of Mathematics, Xi'an Jiaotong University,
Xi'an 710049, China

Abstract. In this paper, we consider a fractional-order single-species model, which is composed of several patches connected by diffusion. We first prove the existence, uniqueness, non-negativity, and boundedness of solutions for the model, as desired in any population dynamics. Moreover, we also obtain some sufficient conditions ensuring the existence and uniform asymptotic stability of the positive equilibrium point for the investigated system. Finally, numerical simulations are presented to demonstrate the validity and feasibility of the theoretical results.

Keywords: uniform asymptotic stability, fractional-order, single-species model, diffusion, graph theory.

1 Introduction

In the natural world, dispersal often occurs among patches in ecological environments. In recent years, due to the ecological effects of local construction and development, such as the locations of manufacturing industries, the construction of dams and highways, as well as the development of tourism, habitats have been increasingly separated into isolated patches. Thus, realistic models should include dispersal processes that take into consideration the effects of spatial heterogeneity [11, 16, 18, 19, 29–31, 33]. In [30], Takeuchi considered the following integer-order single-species system:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = x_i(t)g_i\big(x_i(t)\big) + \sum_{j=1}^{n} D_{ij}\big(x_j(t) - x_i(t)\big),$$

$$x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, n,$$

(1)

[1]Corresponding author.

where x_i denotes population density in the ith patch, $g_i(x_i)$ denotes the density dependent growth rate in the ith patch, constant $D_{ij} \geqslant 0$ is the dispersal rate from patch j to patch i, and $D_{ii} = 0$. By applying cooperative system theory, Takeuchi showed that species can survive in all patches at a globally asymptotically stable equilibrium point for any dispersal rate whenever it can survive in each isolated patch. In [16], Li and Shuai proved the global asymptotic stability of the positive equilibrium point for system (1) by using the graph theoretic approach.

Many authors pointed out that fractional-order calculus is very suitable for the description of memory and hereditary properties of various materials and processes, which are neglected in classical integer-order models [9, 10, 13, 26]. Although many interesting and important results have been done in modelling the dynamics of population models, they have been restricted to integer-order differential equations. Recently, fractional-order differential equations have garnered a lot of attention and appreciation since they are naturally related to systems with memory, which exists in most biological systems. Also they are closely related to fractals, which are abundant in biological systems. For these reasons, many phenomena in mathematical biology [1, 2, 7, 12, 24, 27, 28] and some other interdisciplinary fields [5, 6, 8, 20–23] can be described very successfully by the models using fractional-order differential equations. Due to the limited theories for analyzing the dynamics of fractional-order systems, stability study of fractional-order population models is only the beginning. In [7], El-Sayed et al. considered the following fractional-order logistic system:

$$
\begin{aligned}
&{}_0^c D_t^q x(t) = r x(t) \big(1 - x(t) \big), \quad t > 0, \\
&x(0) = x_0,
\end{aligned}
\tag{2}
$$

where $0 < q < 1$ and ${}_0^c D_t^q$ is in the sense of the Caputo fractional derivative, $x(t)$ denotes population density at time t, and r is the intrinsic rate of population or the strength of intraspecific competition of population. They showed existence and uniqueness of solution and studied the asymptotical stability of the positive equilibrium point. However, to the best of the authors knowledge, to this day, still no scholar has investigated the dynamic behavior of fractional-order single-species model in a patchy environment. Motivated by the above considerations, in this paper, we consider the following fractional-order single-species model, which is composed of several patches connected by diffusion:

$$
\begin{aligned}
&{}_{t_0}^c D_t^q x_i(t) = x_i(t) \big(a_i - b_i x_i(t) \big) + \sum_{j=1}^{n} d_{ij} x_j(t) - \sum_{j=1}^{n} d_{ji} x_i(t), \\
&x_i(t_0) = x_{it_0}, \quad i = 1, 2, \ldots, n,
\end{aligned}
\tag{3}
$$

where $0 < q < 1$, $t_0 \geqslant 0$ is the initial time, and ${}_{t_0}^c D_t^q$ is in the sense of the Caputo fractional derivative, $x_i(t)$ denotes population density in patch i at time t, a_i is the intrinsic growth rate of population in patch i, b_i is the intraspecific competition coefficient of population in patch i, the dispersal rate of population from patch j to patch i is given by d_{ij}. Once an individual enters into a patch, it is counted as an individual in the patch and homogeneously mixed with other individuals in the patch, $\sum_{j=1}^{n} d_{ij} x_j$ denotes

the total immigration of population individuals from every other patch to patch i, and $\sum_{j=1}^{n} d_{ji} x_i$ denotes the total emigration of population individuals from patch i to every other patch. Since $\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ji} x_i$ always holds, the exchange of population individuals among all patches is always in balance. All parameter values are assumed to be nonnegative and $b_i > 0$ for all i.

The contributions of this paper are as follows. Firstly, we prove the non-negativity and boundedness of solutions for a fractional-order population model, as desired in any population dynamics. Secondly, combining Lyapunov method and graph theoretic approach, we obtain a uniform asymptotic stability principle, which has a close relation to the dispersal matrix. Finally, by numerical simulations, we show the effects of dispersal rates on the solutions of the considered system.

This paper is organized as follows. In Section 2, some definitions and lemmas are reminded. In Section 3, we prove the existence, uniqueness, non-negativity and boundedness of solutions for system (3). In addition, we consider the stability of the positive equilibrium point for the considered system, and some criteria are derived to ensure the uniform asymptotic stability of the positive equilibrium point. In Section 4, numerical simulations are presented to illustrate the effectiveness of our theoretical results. Finally, conclusions are given in Section 5.

2 Preliminaries

In this section, we will introduce some definitions and some useful lemmas. It is well known that the initial conditions of fractional differential equations with Caputo derivatives take on the same form as for integer-order ones, which have well-understood practical meanings and more applications in modeling and analysis. Throughout the paper, we use the Caputo fractional-order derivative.

Definition 1. (See [26].) The fractional integral with non-integer order $q > 0$ of function $f(t)$ is defined as follows:

$$t_0 I_t^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - \tau)^{q-1} f(\tau) \, \mathrm{d}\tau,$$

where $\Gamma(\cdot)$ is the Gamma function, $\Gamma(s) = \int_0^{\infty} t^{s-1} \mathrm{e}^{-t} \, \mathrm{d}t$.

Definition 2. (See [26].) The Caputo derivative of fractional order q of function $f(t)$ is defined as follows:

$$t_0^c D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t - \tau)^{q-n+1}} \, \mathrm{d}\tau,$$

where $\Gamma(\cdot)$ is the Gamma function, $\Gamma(s) = \int_0^{\infty} t^{s-1} \mathrm{e}^{-t} \, \mathrm{d}t$, $t \geqslant t_0$, and n is a positive integer such that $n - 1 < q < n$.

Definition 3. (See [17].) The constant x^* is an equilibrium point of Caputo fractional dynamic system

$$_{t_0}^{c}D_t^q x(t) = f(t, x), \quad 0 < q < 1,$$

with the initial condition $x(t_0) = x_{t_0}$ if and only if $f(t, x^*) = 0$.

Remark 1. When $0 < q < 1$, it follows from Definition 2 that the Caputo fractional-order system (3) has the same equilibrium points as the the integer-order system

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = x_i(a_i - b_i x_i) + \sum_{j=1}^{n} d_{ij}x_j - \sum_{j=1}^{n} d_{ji}x_i, \quad i = 1, 2, \dots, n. \tag{4}$$

To prove the non-negativity and boundedness of solutions of system (3), we need the following lemmas.

Lemma 1. (*See* [25].) *Let* $0 < q \leqslant 1$. *Suppose that* $f(t) \in C[a, b]$ *and* $_{t_0}^{c}D_t^q f(t) \in C[a, b]$. *If* $_{t_0}^{c}D_t^q f(t) \geqslant 0$ *for all* $t \in [a, b]$, *then* $f(t)$ *is a nondecreasing function for each* $t \in [a, b]$. *If* $_{t_0}^{c}D_t^q f(t) \leqslant 0$ *for all* $t \in (a, b)$, *then* $f(t)$ *is a non-increasing function for each* $t \in [a, b]$.

Lemma 2. (*See* [15].) *Let* $V(t)$ *be a continuous function on* $[t_0, +\infty)$ *and satisfying*

$$_{t_0}^{c}D_t^q V(t) \leqslant \lambda V(t), \qquad V(t_0) = V_{t_0},$$

where $0 < q < 1$, $\lambda \in \mathbb{R}$, *and* t_0 *is the initial time. Then*

$$V(t) \leqslant V_{t_0} E_q\big[\lambda(t - t_0)^q\big].$$

For stability analysis of fractional-order single-species model in a patchy environment, we need the following lemmas.

Lemma 3. (*See* [14].) *Suppose that the following assumptions are satisfied*:

(H1) *There exist positive definite function* $V_i(t, x_i)$ *and* k-*function* $b_i(\cdot)$ *such that*

$$V_i(t, x_i) \leqslant b_i\big(\|x_i\|\big) \quad \forall\, x_i \in \mathbb{R}^{m_i},\ i = 1, 2, \dots, n.$$

(H2) *For positive definite function* $V_i(t, x_i)$, *there exist* k-*function* $\varphi_i(\cdot)$, *function* $F_{ij}(t, x_i, x_j)$, *and a matrix* $A = (a_{ij})_{n \times n}$ *in which* $a_{ij} \geqslant 0$ *such that*

$$_{t_0}^{c}D_t^q V_i(t, x_i) \leqslant -\varphi_i\big(\|x_i\|\big) + \sum_{j=1}^{n} a_{ij}F_{ij}(t, x_i, x_j), \quad 0 < q < 1,$$

$x_i \in \mathbb{R}^{m_i}$, $x_j \in \mathbb{R}^{m_j}$.

(H3) *There exists function* $G_i(t, x_i)$ *such that* $F_{ij}(t, x_i, x_j) \leqslant G_i(t, x_i) - G_j(t, x_j)$.

(H4) *The weighted digraph* $(\mathcal{G}, A)$ *is strongly connected.*

Then the function $V(t, x) = \sum_{i=1}^{n} c_i V(t, x_i)$ *is a Lyapunov function, and the trivial solution of coupled system on network is uniformly asymptotically stable.*

Remark 2. If the weighted digraph $(\mathcal{G}, A)$ is strongly connected, then $c_i > 0$ for $1 \leqslant i \leqslant n$, where c_i is the cofactor of the ith diagonal element in the Laplacian matrix of $(\mathcal{G}, A)$, the basic concepts and notations on graph theory can be seen in [16].

Lemma 4. (*See* [32].) *Let $x(t) \in \mathbb{R}_+$ be a continuous and derivable function. Then, for any time instant $t \geqslant t_0$,*

$$
{}_{t_0}^{c}D_t^q\left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*}\right] \leqslant \left(1 - \frac{x^*}{x(t)}\right){}_{t_0}^{c}D_t^q x(t), \quad 0 < q < 1,\ x^* \in \mathbb{R}_+.
$$

3 Main results

In this section, we will prove the existence, uniqueness, non-negativity and boundedness of solutions for system (3). In addition, some sufficient conditions are derived to ensure the uniform asymptotic stability of system (3).

To prove the the existence and uniqueness of the solution for system (3), we need the following lemma.

Lemma 5. (*See* [17].) *Consider the system*

$$
{}_{t_0}^{c}D_t^q x(t) = f(t, x), \quad t > t_0, \tag{5}
$$

with initial condition $x(t_0)$, where $0 < q \leqslant 1$, $f : [t_0, \infty) \times \Omega \to \mathbb{R}^n$, $\Omega \in \mathbb{R}^n$, if $f(t, x)$ satisfies the locally Lipschitz condition with respect to x, then there exists a unique solution of (5) on $[t_0, \infty) \times \Omega$.

Definition 4. $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is a solution of problem (3) if

(i) $x(t) \in \Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$, where $\Omega_i = \{x_i \in \mathbb{R} \colon |x_i| \leqslant c_i\}$;

(ii) $x(t)$ satisfies system (3).

Theorem 1. *For each $x_{t_0} = (x_{1t_0}, x_{2t_0}, \ldots, x_{nt_0}) \in \Omega$, there exists a unique solution $x(t) \in \Omega$ of system (3) with initial condition $x(t_0) = x_{t_0}$.*

Proof. We denote $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ and consider a mapping $H(x) = (H_1(x), H_2(x), \ldots, H_n(x))$, where

$$
H_i(x) = x_i(a_i - b_i x_i) + \sum_{j=1}^{n} d_{ij} x_j - \sum_{j=1}^{n} d_{ji} x_i, \quad i = 1, 2, \ldots, n. \tag{6}
$$

For any $x, y \in \mathbb{R}^n$, we have

$$
\begin{aligned}
\big\|H(x) - H(y)\big\| &= \sum_{i=1}^{n} \big|H_i(x) - H_i(y)\big| \\
&= \sum_{i=1}^{n} \bigg| x_i(a_i - b_i x_i) + \sum_{j=1}^{n} d_{ij} x_j - \sum_{j=1}^{n} d_{ji} x_i - y_i(a_i - b_i y_i) \\
&\quad - \sum_{j=1}^{n} d_{ij} y_j + \sum_{j=1}^{n} d_{ji} y_i \bigg|
\end{aligned}
$$

$$
= \sum_{i=1}^{n} \left| a_i(x_i - y_i) - b_i\left(x_i^2 - y_i^2\right) + \sum_{j=1}^{n} d_{ij}(x_j - y_j) - \sum_{j=1}^{n} d_{ji}(x_i - y_i) \right|
$$

$$
\leqslant \sum_{i=1}^{n} \left(a_i + b_i\left(|x_i| + |y_i|\right) + 2\sum_{j=1}^{n} d_{ji} \right) |x_i - y_i|
$$

$$
\leqslant \sum_{i=1}^{n} \left(a_i + 2b_i c_i + 2\sum_{j=1}^{n} d_{ji} \right) |x_i - y_i| \leqslant L\|x - y\|, \tag{7}
$$

where $L = \max_{i \in \mathbb{L}}\{a_i + 2b_i c_i + 2\sum_{j=1}^{n} d_{ji}\}$, $\mathbb{L} = \{1, 2, \ldots, n\}$. Thus, $H(x)$ satisfies the Lipschitz condition with respect to x. It follows from Lemma 5 that there exists a unique solution $x(t)$ of system (3) with initial condition x_{t_0}. This completes the proof of Theorem 1. $\qquad\square$

Considering the biological significance, we are only interested in solutions that are nonnegative and bounded. The following result ensures the non-negativity and boundedness of solutions for system (3). Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and $\Omega_+ = \{(x_1, x_2, \ldots, x_n) \in \Omega \mid x_i \in \mathbb{R}_+, i = 1, 2, \ldots, n\}$.

Theorem 2. *All the solutions of system* (3)*, which start in* Ω_+*, are non-negative and uniformly bounded.*

Proof. Firstly, we prove that all the solutions of system (3), which start in Ω_+, are non-negative. That is, $x_i(t) \geqslant 0$ for all $i \in I = \{1, 2, \ldots, n\}$ and $t \geqslant t_0$. Suppose that is not true, then there exist a positive $t_1 > t_0$ and some $i \in I_1 \subseteq I$ such that

$$
\begin{aligned}
x_i(t) > 0, & \quad i \in I, \ t_0 \leqslant t < t_1, \\
x_j(t_1) > 0, & \quad j \in I/I_1, \\
x_i(t_1) = 0, & \quad i \in I_1, \\
x_i(t_1^+) < 0, & \quad i \in I_1.
\end{aligned} \tag{8}
$$

Based on (8) and system (3), we have

$$
{}_{t_0}^c D_{t_1}^q x_i(t_1)\big|_{x_i(t_1)=0} = \sum_{j=1}^{n} d_{ij} x_j(t_1) \geqslant 0, \quad i \in I_1.
$$

According to Lemma 1, we have $x_i(t_1^+) \geqslant 0$, which contradicts the fact $x_i(t_1^+) < 0$. Therefore, we have $x_i(t) \geqslant 0$ for all $i \in I$ and $t \geqslant t_0$.

Next, we show that all the solutions of system (3), which start in Ω_+, are uniformly bounded. Denote $W(t) = \sum_{i=1}^{n} x_i(t)$, then we have

$$
{}_{t_0}^c D_t^q W(t) + W(t)
$$
$$
= \sum_{i=1}^{n} \left[x_i(a_i - b_i x_i) + \sum_{j=1}^{n} d_{ij} x_j - \sum_{j=1}^{n} d_{ji} x_i \right] + \sum_{i=1}^{n} x_i
$$

$$= \sum_{i=1}^{n} x_i(a_i + 1 - b_i x_i) = \sum_{i=1}^{n} \left[-b_i \left(x_i - \frac{a_i + 1}{2b_i} \right)^2 + \frac{(a_i + 1)^2}{4b_i} \right]$$

$$\leqslant \sum_{i=1}^{n} \frac{(a_i + 1)^2}{4b_i}. \tag{9}$$

Let $U(t) = W(t) - \sum_{i=1}^{n}(a_i + 1)^2/(4b_i)$. It follows from (9) that

$$_{t_0}^{c}D_t^q U(t) \leqslant -U(t).$$

By Lemma 2, we have

$$W(t) \leqslant \left(W(t_0) - \sum_{i=1}^{n} \frac{(a_i + 1)^2}{4b_i} \right) E_q\left[-(t - t_0)^q \right] + \sum_{i=1}^{n} \frac{(a_i + 1)^2}{4b_i}$$

$$\rightarrow \sum_{i=1}^{n} \frac{(a_i + 1)^2}{4b_i}, \quad t \rightarrow \infty.$$

Therefore, all the solutions of system (3), which start in Ω_+, are confined to the region Γ, where

$$\Gamma = \left\{ (x_1, x_2, \ldots, x_n) \in \Omega_+ \;\middle|\; \sum_{i=1}^{n} x_i \leqslant \sum_{i=1}^{n} \frac{(a_i + 1)^2}{4b_i} + \epsilon, \; \epsilon > 0 \right\}.$$

This completes the proof of Theorem 2. □

For system (4), we consider a mapping $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))$, where $x = (x_1, x_2, \ldots, x_n)$ and

$$F_i(x) = x_i(a_i - b_i x_i) + \sum_{j=1}^{n} d_{ij}x_j - \sum_{j=1}^{n} d_{ji}x_i. \tag{10}$$

The Fréchet derivative of $F(x)$ is $DF(x) = [\partial f_i(x_1, x_2, \ldots, x_n)/\partial x_j]$ with

$$\frac{\partial F_i(x_1, x_2, \ldots, x_n)}{\partial x_j} = \begin{cases} a_i - \sum_{j \neq i} d_{ji} - 2b_i x_i & \text{for } j = i, \; i = 1, 2, \ldots, n, \\ d_{ij} & \text{for } j \neq i, \; i = 1, 2, \ldots, n. \end{cases}$$

Let

$$M_0 = DF(0) = \begin{pmatrix} a_1 - \sum_{j \neq 1} d_{j1} & d_{12} & \ldots & d_{1n} \\ d_{21} & a_2 - \sum_{j \neq 2} d_{j2} & \ldots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \ldots & a_n - \sum_{j \neq n} d_{jn} \end{pmatrix}.$$

Clearly, F is continuously differential, $F(0) = 0$, $F_i(x) \geqslant 0$ for all $x \in \mathbb{R}_+^n$ with $x_i = 0$, $i = 1, 2, \ldots, n$. For any $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))$ and $x = (x_1, x_2, \ldots, x_n)$, $\partial F_i/\partial x_j = d_{ij} \geqslant 0$, $i, j = 1, 2, \ldots, n$, and $i \neq j$, so F is cooperative. Since matrix $D = (d_{ij})_{n \times n}$ is irreducible, it follows that the matrix $DF(0)$ is also irreducible. In addition, it follows from (4) that

$$\sum_{i=1}^n \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \sum_{i=1}^n x_i(a_i - b_i x_i) < 0 \quad \text{for sufficiently large } x_i,$$

which shows the boundedness of solutions of system (4). Let $s(M_0)$ denote the stability modulus of an $n \times n$ matrix M_0, which is defined by $s(M_0) = \max\{\mathrm{Re}\,\lambda \mid \lambda \text{ is an}$ eigenvalue of $M_0\}$. Then if $s(M_0) > 0$, by [34, Cor. 3.1], system (4) admits a positive equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$. Since the Caputo fractional-order system (3) has the same equilibrium points as the integer-order system (4), we know that system (3) has a positive equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$.

We define a function

$$V_i(x_i) = x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}.$$

It can be vertified that $V_i(x_i) > 0$ for all $x_i > 0$ and $V_i(x_i) = 0$ if and only if $x_i = x_i^*$. Calculating the fractional-order derivatives of $V_i(x_i)$ along the solutions of system (3), we have

$$
\begin{aligned}
{}_{t_0}^c D_t^q V_i(x_i) = {}_{t_0}^c D_t^q \left(x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*} \right) &\leqslant \left(1 - \frac{x_i^*}{x_i} \right) {}_{t_0}^c D_t^q x_i \\
&= (x_i - x_i^*) \left(a_i - b_i x_i + \sum_{j=1}^n d_{ij} \frac{x_j}{x_i} - \sum_{j=1}^n d_{ji} \right) \\
&= (x_i - x_i^*) \left[-b_i(x_i - x_i^*) + \sum_{j=1}^n d_{ij} \left(\frac{x_j}{x_i} - \frac{x_j^*}{x_i^*} \right) \right] \\
&= -b_i(x_i - x_i^*)^2 + \sum_{j=1}^n a_{ij} F_{ij}(x_i, x_j),
\end{aligned}
\tag{11}
$$

where $a_{ij} = d_{ij} x_j^*$, $F_{ij}(x_i, x_j) = x_j/x_j^* - x_i/x_i^* - x_i^* x_j/(x_i x_j^*) + 1$. Let $G_i(x_i) = -x_i/x_i^* + \ln(x_i/x_i^*)$, then we have

$$
\begin{aligned}
F_{ij}(x_i, x_j) &= G_i(x_i) - G_j(x_j) + 1 - \frac{x_i^* x_j}{x_i x_j^*} + \ln \frac{x_i^* x_j}{x_i x_j^*} \\
&\leqslant G_i(x_i) - G_j(x_j).
\end{aligned}
$$

In fact, $1 - a + \ln a \leqslant 0$ for $a > 0$ and $1 - a + \ln a = 0$ for $a = 1$. If $D = (d_{ij})_{n \times n}$ is irreducible, we know that $A = (a_{ij})_{n \times n}$ is irreducible. That is to say, the weighted digraph $(\mathcal{G}, A)$ is strongly connected, it follows from Remark 2 that $c_i > 0$ for $1 \leqslant i \leqslant n$.

Clearly, we have shown that V_i, F_{ij}, G_i, and a_{ij} satisfy the conditions of Lemma 3. Therefore, $V(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i V_i(x_i)$ is a Lyapunov function for system (3), and ${}_{t_0}^{c} D_t^q V(x) = 0$ implies that $x_i = x_i^*$ for all $1 \leqslant i \leqslant n$. It follows from Lemma 3 that the positive equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ of system (3) is uniformly asymptotically stable. Therefore, we thus have the following result.

Theorem 3. *Suppose that the matrix $D = (d_{ij})_{n \times n}$ is irreducible. If $s(M_0) > 0$, then system (3) has a positive equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$, which is uniformly asymptotically stable.*

Remark 3. The dispersal matrix $D = (d_{ij})_{n \times n}$ is not required to be symmetric, namely, the dispersal rate from patch i to patch j may not be the same as that from patch j to patch i. A typical assumption we impose on dispersal matrix D is that it is irreducible. From biological point of view, this means that population individuals in each patch can disperse between any two patches directly or indirectly.

Remark 4. Evidently, the derived result in this paper is still true for $q = 1$ although the above discussion is based on $0 < q < 1$. However, the case $q > 1$ is not considered in this paper since Lemma 3 is not solved for $q > 1$, this deserves our further consideration.

4 Numerical simulations

An effcient method for solving fractional-order differential equations is the predictor–correctors scheme or more precisely, PECE (Predict, Evaluate, Correct, Evaluate) [3, 4], which represents a generalization of Adams–Bashforth–Moulton algorithm. In this section, we use the PECE method for the numerical simulations of the considered model.

In the following, we consider a fractional-order single-species model, which is composed of two patches connected by diffusion as follows:

$$
{}_0^c D_t^q x_i(t) = x_i(t) \big(a_i - b_i x_i(t) \big) + \sum_{j=1}^{2} d_{ij} x_j(t) - \sum_{j=1}^{2} d_{ji} x_i(t),
$$

$$
x_i(0) = x_{i0}, \quad i = 1, 2,
$$

(12)

where the parameters are chosen in Table 1.

We keep all parameter values in Table 1 unchanged. By virtue of Matlab, we can compute that $x^* = (10.48, 6.47)$. Since $a_{ij} = d_{ij} x_j^*$, $i, j = 1, 2, 3$, we have

$$
A = (a_{ij})_{2 \times 2} = \begin{pmatrix} 0 & 2.588 \\ 2.096 & 0 \end{pmatrix},
$$

$$
L = (p_{ij})_{2 \times 2} = \begin{pmatrix} 2.588 & -2.588 \\ -2.096 & 2.096 \end{pmatrix}.
$$

Table 1. Parameter values used in the simulations of system (12).

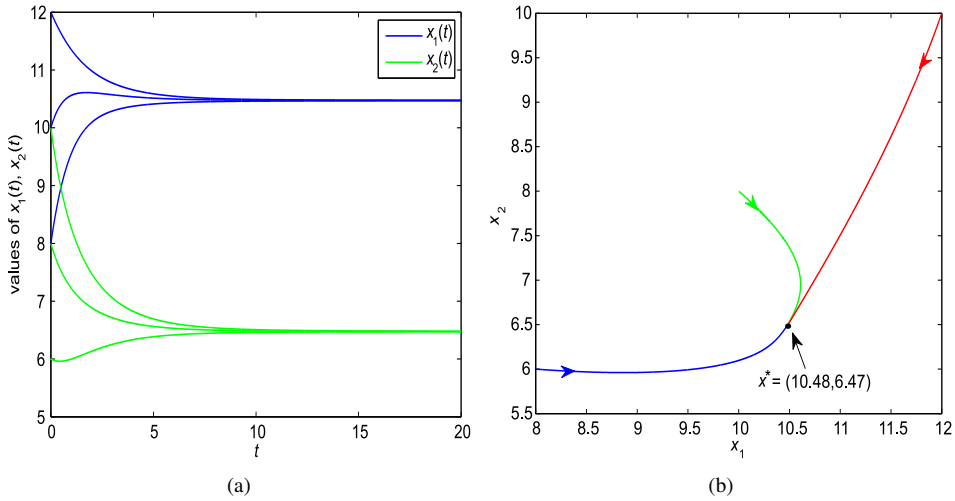| Parameter	a_1	a_2	b_1	b_2	d_{11}	d_{12}	d_{21}	d_{22}	q
Value	1	0.4	0.1	0.05	0	0.4	0.2	0	0.98

Figure 1. Uniform asymptotic stability of system (12) with parameter values as given in Table 1. (a) Time series of x_1 and x_2. (b) Phase portrait of x_1 and x_2.
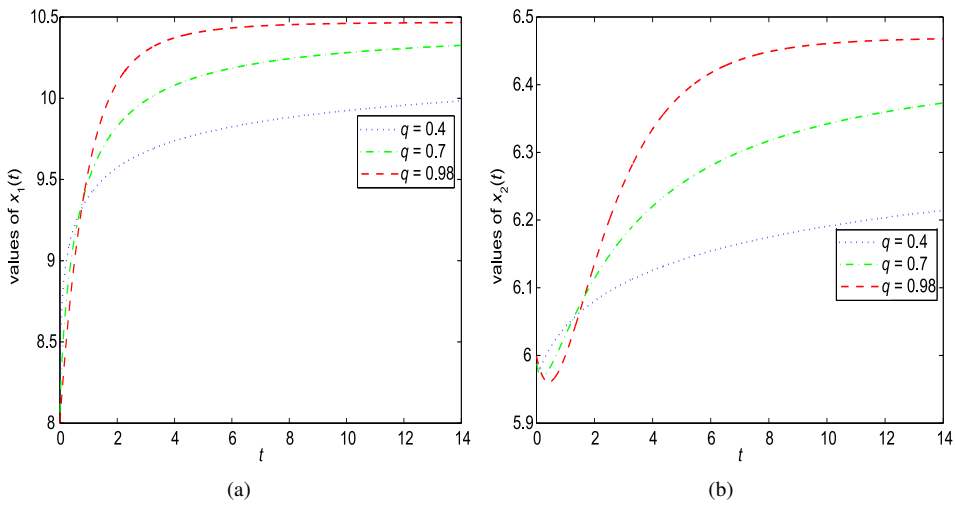

Figure 2. State trajectories of system (12) with parameter values as given in Table 1. (a) Time series of x_1. (b) Time series of x_2.

By simple calculation, $c_1 = 2.096$, $c_2 = 2.588$. Obviously, $(\mathcal{G}, A)$ is strongly connected, it is very easy to verify that condition $s(M_0) = 0.8899 > 0$ holds. Then it follows from Theorem 3 that system (12) is uniformly asymptotically stable, numerical simulations can be seen in Fig. 1, where the initial values are taken as $(8 + 2k, 6 + 2k)$ $(k = 1, 2, 3)$. From Fig. 1 we find that all solutions of system (12), which through these initial values will converge to the positive equilibrium point x^*.
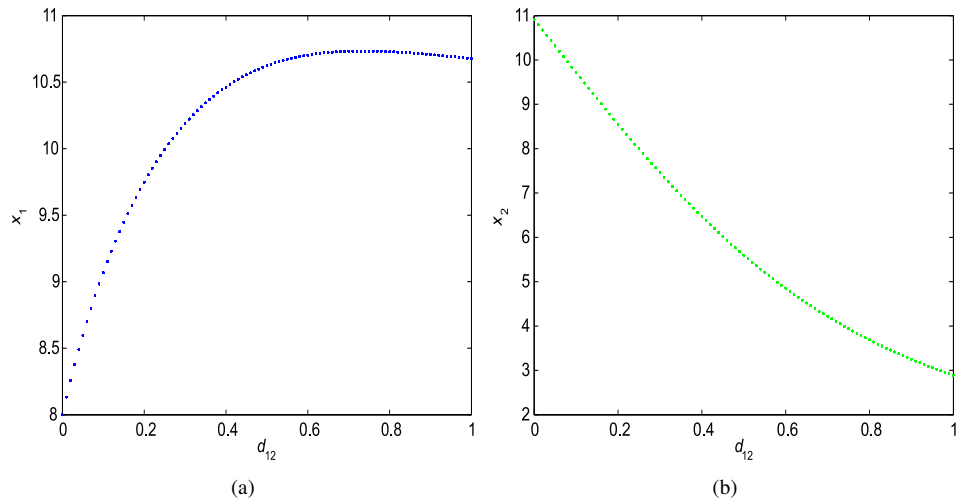
Figure 3. Influence of dispersal rate d_{12} on system (12). Here we vary the parameter d_{12}, while keeping the other parameters fixed as mentioned in Table 1. (a) Population density in patch 1. (b) Population density in patch 2.
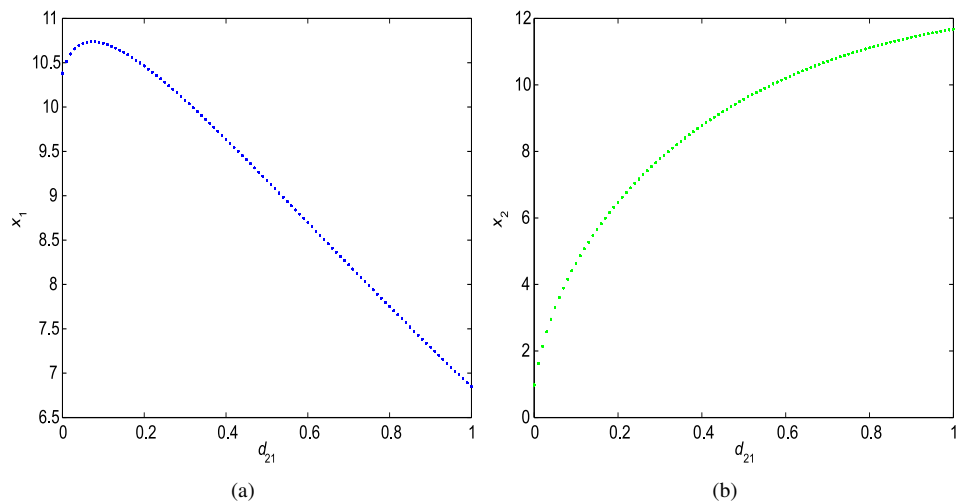

Figure 4. Influence of dispersal rate d_{21} on system (12). Here we vary the parameter d_{21}, while keeping the other parameters fixed as mentioned in Table 1. (a) Population density in patch 1. (b) Population density in patch 2.

Next, we will show the effect of fractional order on the solutions of system (12). We vary the parameter q, while keeping the other parameters fixed as mentioned in Table 1. Figure 2 shows the state trajectories of system (12) with initial value $(8, 6)$ when $q = 0.4$, $q = 0.7$, and $q = 0.98$. We can see from Fig. 2 that q is a significant factor, which affects the convergence speed of solutions.

Remark 5. We can observe from Fig. 2 that the convergence velocities of solutions of system (12) becomes fast as the increase of the fractional-order q ($0 < q < 1$).

Finally, we will show the effects of dispersal rates on the solutions of system (12). On the one hand, we vary the parameter d_{12}, while keeping the other parameters fixed as mentioned in Table 1. Numerical simulations can be seen in Fig. 3, where the initial value is taken as $(8, 6)$. On the other hand, we vary the parameter d_{21}, while keeping the other parameters fixed as mentioned in Table 1. Numerical simulations can be seen in Fig. 4, where the initial value is taken as $(8, 6)$.

Remark 6. We can observe from Figs. 3 and 4 that dispersal rates have no influence on the uniform asymptotic stability of system (12), that is, dispersal rates only change the position of the positive equilibrium point of system (12) and retain its stability, which is in accord with the integer-order single-species model with diffusion [16, 30].

5   Conclusions

This paper investigates a fractional-order single-species model, which is composed of n ($n \geqslant 2$) patches connected by diffusion. By using some useful inequality techniques, Lyapunov method and graph theoretic approach, we prove that existence, uniqueness, non-negativity and boundedness of solutions for system (3), and some criteria are established to ensure the uniform asymptotic stability of the positive equilibrium point of system (3). These critaria have a close relation to the dispersal matrix. Finally, numerical simulations are presented to demonstrate the validity of the theoretical results and show the effects of fractional order and dispersal rates on the solutions of system (12).

References

1. E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator–prey and rabies models, *J. Math. Anal. Appl.*, **325**(1):542–553, 2007.

2. S. Das, P.K. Gupta, A mathematical model on fractional Lotka–Volterra equations, *J. Theor. Biol.*, **277**(1):1–6, 2011.

3. K. Diethelm, N.J. Ford, A.D. Freed, A predictor-corrector approach for the numerical solution of fractional dierential equations, *Nonlinear Dyn.*, **29**(1–4):3–22, 2002.

4. K. Diethelm, N.J. Ford, A.D. Freed, Detailed error analysis for a fractional Adams method, *Numer. Algorithms*, **36**(1):31–52, 2004.

5. A.E.M. El-Mesiry, E. Ahmed, On a factional model for earthquakes, *Appl. Math. Comput.*, **178**(2):207–211, 2006.

6. A.M.A. El-Sayed, Fractional-order diffusion-wave equation, *Int. J. Theor. Phys.*, **35**(2):311–322, 1996.

7. A.M.A. El-Sayed, A.E.M. El-Mesiry, H.A.A. El-Saka, On the fractional-order logistic equation, *Appl. Math. Lett.*, **20**(7):817–823, 2007.

8. A.M.A. El-sayed, H.M. Nour, A. Elsaid, A.E. Matouk, A. Elsonbaty, Dynamical behaviors, circuit realization, chaos control, and synchronization of a new fractional order hyperchaotic system, *Appl. Math. Modelling*, **40**(5):3516–3534, 2016.

9. E.M. Elsayed, Qualitative behavior of difference equation of order two, *Math. Comput. Modelling*, **50**(7):1130–1141, 2009.

10. E.M. Elsayed, On the solutions and periodic nature of some systems of difference equations, *Int. J. Biomath.*, **7**(6):1450067, 2014.

11. H.I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.

12. M. Javidi, N. Nyamoradi, A mathematical model on fractional Lotka–Volterra equations, *Appl. Math. Modelling*, **37**(20):8946–8956, 2013.

13. A.A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, New York, 2006.

14. H.L. Li, Y.L. Jiang, Z.L. Wang, X.M. Feng, Z.D. Teng, Stability analysis for coupled systems of fractional differential equations on networks, *Int. J. Comput. Math.*, **94**(2):263–274, 2017.

15. H.L. Li, Y.L. Jiang, Z.L. Wang, L. Zhang, Z.D. Teng, Mittag–Leffler stability of coupled system of fractional-order differential equations on network, *Appl. Math. Comput.*, **270**:269–277, 2015.

16. M.Y. Li, Z.S. Shuai, Global-stability problem for coupled systems of differential equations on networks, *J. Differ. Equ.*, **248**(1):1–20, 2010.

17. Y. Li, Y.Q. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability, *Comput. Math. Appl.*, **59**(5):1810–1821, 2010.

18. Z.J. Liu, S.M. Zhong, An impulsive periodic predator–prey system with Holling type III functional response and diffusion, *Appl. Math. Modelling*, **36**(2):5976–5990, 2012.

19. Z.Y. Lu, Y. Takeuchi, Global asymptotic behavior in single-species discrete diffusion systems, *J. Math. Biol.*, **32**(1):67–77, 1993.

20. D. Matignon, Stability results for fractional differential equations with applications to control processing, *Comput. Eng. Syst. Appl.*, **2**:963–968, 1996.

21. A.E. Matouk, Stability conditions, hyperchaos and control in a novel fractional order hyperchaotic system, *Phys. Lett. A*, **373**(25):2166–2173, 2009.

22. A.E. Matouk, Chaos, feedback control and synchronization of a fractional-order modified autonomous Van der Pol–Duffing circuit, *Commun. Nonlinear Sci. Numer. Simul.*, **16**(2):975–986, 2011.

23. A.E. Matouk, Chaos synchronization of a fractional-order modified Van der Pol–Duffing system via new linear control, backstepping control and Takagi–Sugeno fuzzy approaches, *Complexity*, **21**(S1):116–124, 2016.

24. A.E. Matouk, A.A. Elsadany, E. Ahmed, H.N. Agiza, Dynamical behavior of fractional-order Hastings–Powell food chain model and its discretization, *Commun. Nonlinear Sci. Numer. Simul.*, **27**(1):153–167, 2015.

25. Z.M. Odibat, N.T. Shawagfeh, Generalized Taylors formula, *Appl. Math. Comput.*, **186**(1):286–293, 2007.

26. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

27. F.A. Rihan, S. Lakshmanan, A.H. Hashish, R. Rakkiyappan, E. Ahmed, Fractional-order delayed predator-prey systems with Holling type-II functional response, *Nonlinear Dyn.*, **80**(1–2):777–789, 2015.

28. M. Rivero, J.J. Trujillo, L. Vázquez, M.P. Velasco, Fractional dynamics of populations, *Appl. Math. Comput.*, **218**(3):1089–1095, 2011.

29. X.Y. Shi, G.H. Song, Z.Y. Li, The effect of diffusion on giant pandas that live in complex patchy environments, *Nonlinear Anal. Model. Control*, **20**(1):56–71, 2015.

30. Y. Takeuchi, Cooperative systems theory and global stability of diffusion models, *Acta. Appl. Math.*, **14**(1):49–57, 1989.

31. Y. Takeuchi, W.D. Wang, Y. Satio, Global stability of population models with patch structure, *Nonlinear Anal., Real World Appl.*, **7**(2):235–247, 2006.

32. C. Vargas-De-León, Volterra-type Lyapunov functions for fractional-order epidemic systems, *Commun. Nonlinear Sci. Numer. Simul.*, **24**(1):75–85, 2015.

33. L. Zhang, Z.D. Teng, D.L. DeAngelis, S.G. Ruan, Single species models with logistic growth and dissymmetric impulse dispersal, *Math. Biosci.*, **241**(2):188–197, 2013.

34. X.Q. Zhao, Z.J. Jing, Global asymptotic behavior in some cooperative systems of functional differential equations, *Can. Appl. Math. Q.*, **4**(4):421–444, 1996.