Nonlinear Analysis: Modelling and Control, Vol. 22, No. 4, 545–565 ISSN 1392-5113
https://doi.org/10.15388/NA.2017.4.9

Some coincidence point results for T-contraction mappings on partially ordered b-metric spaces and applications to integral equations

Stojan Radenović[a,b], **Tran Van An**[c], **Le Thanh Quan**[c]

[a]Nonlinear Analysis Research Group, Ton Duc Thang University,
Ho Chi Minh City, Vietnam

[b]Faculty of Mathematics and Statistics, Ton Duc Thang University,
Ho Chi Minh City, Vietnam
stojan.radenovic@tdt.edu.vn

[c]Department of Mathematics, Vinh University,
Vinh City, Nghe An Province, Vietnam
tvandhv@gmail.com; lethanhquan82@gmail.com

Received: October 3, 2016 / **Revised:** February 11, 2017 / **Published online:** June 19, 2017

Abstract. In this paper, we prove some fixed point results for T-contraction mappings in partially ordered b-metric spaces that generalize the main results of [H. Huang, S. Radenović, J. Vujaković, On some recent coincidence and immediate consequences in partially ordered b-metric spaces, *Fixed Point Theory Appl.*, 2015, Paper No. 63]. As an application, we discuss the existence for a solution of a nonlinear integral equation.

Keywords: b-metric, T-contraction, fixed point, integral equation.

1 Introduction and preliminaries

The Banach contraction principle is one of the most important results in mathematical analysis. It is the most widely applied fixed point result in many branches of mathematics and it was also generalized in many different directions.

In 1989, Bakhtin [6] introduced the concept of a b-metric space as a generalization of a metric space. In 1993, Czerwik [7] extended many results related to the b-metric spaces. Fixed point results in partially ordered metric spaces were firstly obtained by Ran and Reurings (see [18]) and then by Nieto and López (see [15, 16]). Subsequently, many authors presented numerous interesting and significant results in ordered metric and ordered b-metric spaces (see [2–5, 10, 17, 20–22]). In 2010, S. Moradi and M. Omid [13] proposed the concept of a T-contraction and obtained some fixed point results in metric spaces. After that several interesting results of the existence of fixed points for T-Kannan and T-Chatterjea contractive mappings are introduced (see [9, 14, 19]).

The purpose of this paper is to state some fixed point results for T-contractive mappings in partially ordered b-metric spaces that generalize the main results of [10]. As an application, we discuss the existence for a solution of a nonlinear integral equation.

First, we recall some notions and properties that will be needed throughout the paper.

Definition 1. (See [8].) Let X be a nonempty set, $s \geqslant 1$ be a real number, and let $d : X \times X \to [0, \infty)$ be a function such that for all $x, y, z \in X$,

(i) $d(x, y) = 0$ if only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

Then d is called a *b-metric* on X, and (X, d, s) is called a *b-metric space*. If $(X, \preccurlyeq)$ is still a partially ordered set, then $(X, d, s, \preccurlyeq)$ is called a partially ordered b-metric space.

Obviously, for $s = 1$, b-metric space is a metric space.

Otherwise, for more concepts such as b-convergence, b-Cauchy sequence, b-completeness, b-closed set in b-metric spaces, a regular b-metric space, etc., we refer the reader to [2–5, 10, 17, 20–22] and the references mentioned therein.

Following is an example of a b-metric space.

Example 1. (See [11].) Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}^+$ be defined by

$$d(x, y) = |x - y|^2 \quad \text{for any } x, y \in X.$$

Then (X, d, s) is a b-metric space with the coefficient $s = 2$.

Lemma 1. (*See* [12].) *Let (X, d, s) be a b-metric space and $\{x_n\}$ be a sequence such that*

$$d(x_n, x_{n+1}) \leqslant \alpha d(x_{n-1}, x_n) \quad \textit{for all } n \geqslant 1,$$

where $\alpha \in (0, 1/s)$. Then $\{x_n\}$ is b-Cauchy sequence in X.

Lemma 2. (*See* [3].) *Let (X, d, s) be a b-metric space with $s \geqslant 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2} d(x, y) \leqslant \liminf_{x\to\infty} d(x_n, y_n) \leqslant \limsup_{x\to\infty} d(x_n, y_n) \leqslant s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{x\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leqslant \liminf_{x\to\infty} d(x_n, z) \leqslant \limsup_{x\to\infty} d(x_n, z) \leqslant s d(x, y).$$

Now we shall give some notions used in the next section. These notions are generalizations of notions introduced in [10].

Definition 2. Let $(X, \preccurlyeq)$ be a partially ordered set and $f, g, h, k : X \to X$ be four mappings such that $hf(X) \cup hg(X) \subseteq k(X)$. Then

(i) The pair (f, g) is called *h-compatible* if $\lim_{n\to\infty} d(fhgx_n, ghfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} hfx_n = \lim_{n\to\infty} hgx_n = t$ for some $t \in X$. If we choose $hx = x$ for all $x \in X$, then the pair (f, g) is called *compatible* [10]. If we choose $gx = x$ for all $x \in X$, then f is called *h-compatible*.

(ii) The pair (f, g) is called *h-weakly compatible* if $fhgx = ghfx$ whenever $hgx = hfx$. If we choose $hx = x$ for all $x \in X$, then the pair (f, g) is called *weakly compatible* [10]. If we choose $gx = x$ for all $x \in X$, then f is called *h-weakly compatible*.

(iii) The pair (f, g) is called *h-weakly increasing* with respect to k if, for all $x \in X$, $hfx \preccurlyeq hgy$ for all $y \in (hk)^{-1}(hfx)$ and $hgx \preccurlyeq hfy$ for all $y \in (hk)^{-1}(hgx)$. If we choose $hx = x$ for all $x \in X$, then the pair (f, g) is called *weakly increasing* with respect to k [10]. If we choose $kx = x$ for all $x \in X$, then the pair (f, g) is called *h-weakly increasing*.

(iv) The pair (f, g) is called *h-partially weakly increasing* with respect to k if $hfx \preccurlyeq hgy$ for all $y \in (hk)^{-1}(hfx)$. If we choose $hx = x$ for all $x \in X$, then the pair (f, g) is called *partially weakly increasing* with respect to k [10]. If we choose $kx = x$ for all $x \in X$, then the pair (f, g) is called *h-partially weakly increasing*.

(v) f is called *monotone g-non-decreasing* with respect to $(h, \preccurlyeq)$ if $hgx \preccurlyeq hgy$ implies $hfx \preccurlyeq hfy$. If we choose $hx = x$ for all $x \in X$, then f is called *monotone g-non-decreasing* with respect to "$\preccurlyeq$" [10]. If we choose $gx = x$ for all $x \in X$, then f is called *monotone non-decreasing* with respect to $(h, \preccurlyeq)$.

(vi) If $fx = gx = kx = hx$, then x is called the *coincidence point* of f, g, k, h.

Recall that, for f and g, there are two self-mappings on a nonempty set X. If $w = fx = gx$ for some x in X, then x is called a *coincidence point* of the pair (f, g), and w is called a *point of coincidence* of the pair (f, g) (see, e.g., [1]).

In [10], authors introduced and proved the results as follows.

Theorem 1. (*See* [10].) *Let $(X, d, s, \preccurlyeq)$ be a partially ordered complete b-metric space with $s > 1$ and $f, g, S, R : X \to X$ be four mappings satisfying the following*:

(i) $f(X) \subseteq R(X)$ *and* $g(X) \subseteq S(X)$.

(ii) *For every two elements $x, y \in X$ such that Sx, Ry are comparable, we have*

$$s^i d(fx, gy) \leqslant M_s(x, y), \tag{1}$$

where $i > 1$ is a constant and

$$M_s(x, y) = \max\left\{d(Sx, Ry), d(Sx, fx), d(Ry, gy), \frac{d(Sx, gy) + d(Ry, fx)}{2s}\right\}.$$

(iii) *f, g, R and S are continuous.*

(iv) *The pairs (f, S) and (g, R) are compatible.*

(v) *The pairs (f, g) and (g, f) are partially weakly increasing with respect to R and S, respectively.*

Then the pairs (f, S) and (g, R) have a coincidence point z in X. Moreover, if Rz and Sz are comparable, then z is a coincidence point of f, g, R and S.

In the following theorem, the authors omit the assumption of continuity of f, g, R, S by adding the regularity of the space $(X, d, s, \preccurlyeq)$ and replace the compatibility of the pairs (f, S) and (g, R) by the weak compatibility of these pairs.

Theorem 2. (*See* [10].) *Let $(X, d, s, \preccurlyeq)$ be a regular partially ordered complete b-metric space with $s > 1$ and $f, g, S, R : X \to X$ be four mappings satisfying the following*:

(i) *$f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$.*
(ii) *For every two elements $x, y \in X$ such that Sx, Ry are comparable, we have*

$$s^i d(fx, gy) \leqslant M_s(x, y), \tag{2}$$

where $i > 1$ is a constant and

$$M_s(x, y) = \max\left\{d(Sx, Ry), d(Sx, fx), d(Ry, gy), \frac{d(Sx, gy) + d(Ry, fx)}{2s}\right\}.$$

(iii) *$R(X)$ and $S(X)$ are b-closed subsets of X.*
(iv) *The pairs (f, S) and (g, R) are weakly compatible.*
(v) *The pairs (f, g) and (g, f) are partially weakly increasing with respect to R and S, respectively.*

Then the pairs (f, S) and (g, R) have a coincidence point z in X. Moreover, if Rz and Sz are comparable, then z is a coincidence point of f, g, R and S.

By choosing $Rx = Sx = x$ for all $x \in X$ in Theorems 1 and 2, the authors obtained result.

Corollary 1. (*See* [10].) *Let $(X, d, s, \preccurlyeq)$ be a partially ordered complete b-metric space with $s > 1$ and $f, g : X \to X$ be two mappings satisfying the following*:

(i) *For every two comparable elements $x, y \in X$, we have*

$$s^i d(fx, gy) \leqslant M_s(x, y), \tag{3}$$

where $i > 1$ is a constant and

$$M_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s}\right\}.$$

(ii) *The pair (f, g) is partially weakly increasing.*
(iii) *f and g are continuous, or $(X, d, \preccurlyeq)$ is regular.*

Then the pair (f, g) has a common fixed point z in X.

2 Main results

First, we present the notion of a T-contraction on partially ordered b-metric spaces. Same as in [13], by replacing the metric space by a b-metric space we have

Definition 3. Let (X, d, s) be a b-metric space with $s > 1$ and $T, S : X \to X$ be two mappings. Then S is called a *T-contraction* if there exists $k \in [0, 1)$ such that, for all $x, y \in X$,

$$d(TSx, TSy) \leqslant kd(Tx, Ty). \tag{4}$$

If we choose $Tx = x$ for all $x \in X$, then S is a Banach contraction. The following example gives a T-contraction, but not a Banach contraction.

Example 2. Let $X = [1, \infty)$ be endowed with the b-metric $d(x, y) = |x-y|^2$ with $s = 2$. Define $T, S : X \to X$ such that $Tx = 2 - 1/x$ and $Sx = 4x$, then S is not a contractive mapping. But, for all $x, y \in X$, we have

$$\begin{aligned} d(TSx, TSy) &= \left|2 - \frac{1}{4x} - 2 + \frac{1}{4y}\right|^2 = \frac{1}{16}\left|\left(2 - \frac{1}{x}\right) - \left(2 - \frac{1}{y}\right)\right|^2 \\ &= \frac{1}{16} d(Tx, Ty). \end{aligned}$$

So, S is a T-contraction.

Now, we establish some results.

Theorem 3. *Let $(X, d, s, \preccurlyeq)$ be a partially ordered complete b-metric space with $s > 1$ and $T, f, g, S, R : X \to X$ be five mappings satisfying the following*:

(i) *$f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$.*

(ii) *T is one-to-one.*

(iii) *For every two elements $x, y \in X$ such that TSx, TRy are comparable, we have*

$$s^i d\bigl(Tfx, Tgy)\bigr) \leqslant M_s^T(x, y), \tag{5}$$

where $i > 1$ is a constant and

$$\begin{aligned} M_s^T(x, y) = \max\biggl\{ & d(TSx, TRy), d(TSx, Tfx), d(TRy, Tgy), \\ & \frac{d(TSx, Tgy) + d(TRy, Tfx)}{2s} \biggr\}. \end{aligned}$$

(iv) *f, g, R and S are continuous.*

(v) *The pairs (f, S) and (g, R) are T-compatible.*

(vi) *The pairs (f, g) and (g, f) are T-partially weakly increasing with respect to R and S, respectively.*

Then the pairs (f, S) and (g, R) have a coincidence point z in X. Moreover, if TRz and TSz are comparable, then z is a coincidence point of f, g, R and S.

Proof. Let x_0 be an arbitrary point of X. Since $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$, we have $Tf(X) \subseteq TR(X)$ and $Tg(X) \subseteq TS(X)$. Now, we can define $x_1 \in X$ such that $fx_0 = Rx_1$ and $x_2 \in X$ such that $gx_1 = Sx_2$. It implies that $Tfx_0 = TRx_1$ and $Tgx_1 = TSx_2$. Continuing this way, we construct a sequence $\{z_n\}$ defined by:

$$z_{2n+1} = TRx_{2n+1} = Tfx_{2n} \quad \text{and} \quad z_{2n+2} = TSx_{2n+2} = Tgx_{2n+1} \quad \text{for all } n \in \mathbb{N}.$$

As $x_1 \in (TR)^{-1}(Tfx_0)$ and $x_2 \in (TS)^{-1}(Tgx_1)$, and the pairs (f, g) and (g, f) are T-partially weakly increasing with respect to R and S, respectively, so we have

$$z_1 = TRx_1 = Tfx_0 \preccurlyeq Tgx_1 = z_2 = TSx_2 \preccurlyeq Tfx_2 = TRx_3 = z_3.$$

Repeating this process, we obtain $z_n \preccurlyeq z_{n+1}$ for all $n \geqslant 1$.

First, we prove that

$$d(z_{n+1}, z_{n+2}) \leqslant \lambda d(z_n, z_{n+1}) \quad \text{for all } n \geqslant 1, \tag{6}$$

where $\lambda \in [0, 1/s)$. We consider two cases.

Case 1. Assume that $z_n \neq z_{n+1}$ for all $n \geqslant 1$. Since $TSx_{2n} = z_{2n}$ and $TRx_{2n-1} = z_{2n-1}$ are comparable, by (5) we have

$$\begin{aligned}
&s^i d(z_{2n+1}, z_{2n}) \\
&\quad = s^i d(Tfx_{2n}, Tgx_{2n-1}) \\
&\quad \leqslant \max\biggl\{ d(TSx_{2n}, TRx_{2n-1}), d(TSx_{2n}, Tfx_{2n}), d(TRx_{2n-1}, Tgx_{2n-1}), \\
&\qquad\qquad \frac{d(TSx_{2n}, Tgx_{2n-1}) + d(TRx_{2n-1}, Tfx_{2n})}{2s} \biggr\} \\
&\quad = \max\biggl\{ d(z_{2n}, z_{2n-1}), d(z_{2n}, z_{2n+1}), \frac{d(z_{2n-1}, z_{2n+1})}{2s} \biggr\} \\
&\quad \leqslant \max\biggl\{ d(z_{2n}, z_{2n-1}), d(z_{2n}, z_{2n+1}), \frac{s[d(z_{2n-1}, z_{2n}) + d(z_{2n}, z_{2n+1})]}{2s} \biggr\} \\
&\quad = \max\bigl\{ d(z_{2n}, z_{2n-1}), d(z_{2n}, z_{2n+1}) \bigr\}.
\end{aligned}$$

Therefore, we have

$$s^i d(z_{2n+1}, z_{2n}) \leqslant \max\bigl\{ d(z_{2n}, z_{2n-1}), d(z_{2n}, z_{2n+1}) \bigr\}. \tag{7}$$

If $d(z_{2n}, z_{2n+1}) \geqslant d(z_{2n-1}, z_{2n}) > 0$ for some $n \geqslant 1$, then from (7) we have

$$s^i d(z_{2n}, z_{2n+1}) \leqslant d(z_{2n}, z_{2n+1}).$$

This is a contradiction because $s^i > 1$. Thus, from (7) it follows that

$$s^i d(z_{2n}, z_{2n+1}) \leqslant d(z_{2n-1}, z_{2n}) \quad \text{for all } n \geqslant 1. \tag{8}$$

Since $TSx_{2n} = z_{2n}$ and $TRx_{2n+1} = z_{2n+1}$ are comparable, by (5) we have

$$\begin{aligned}
&s^i d(z_{2n+1}, z_{2n+2}) \\
&\quad = s^i d(Tfx_{2n}, Tgx_{2n+1}) \\
&\quad \leqslant \max\biggl\{ d(TSx_{2n}, TRx_{2n+1}), d(TSx_{2n}, Tfx_{2n}), d(TRx_{2n+1}, Tgx_{2n+1}), \\
&\qquad\qquad \frac{d(TSx_{2n}, Tgx_{2n+1}) + d(TRx_{2n+1}, Tfx_{2n})}{2s} \biggr\} \\
&\quad = \max\biggl\{ d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2})}{2s} \biggr\} \\
&\quad \leqslant \max\biggl\{ d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{s[d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})]}{2s} \biggr\} \\
&\quad = \max\bigl\{ d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}) \bigr\}.
\end{aligned}$$

Therefore, we have

$$s^i d(z_{2n+1}, z_{2n+2}) \leqslant \max\bigl\{ d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}) \bigr\}. \tag{9}$$

If $d(z_{2n+1}, z_{2n+2}) \geqslant d(z_{2n}, z_{2n+1}) > 0$ for some $n \geqslant 1$, then from (9) we have

$$s^i d(z_{2n+1}, z_{2n+2}) \leqslant d(z_{2n+1}, z_{2n+2}).$$

This is a contradiction because $s^i > 1$. Thus, from (9) it follows that

$$s^i d(z_{2n+1}, z_{2n+2}) \leqslant d(z_{2n}, z_{2n+1}) \quad \text{for all } n \geqslant 1. \tag{10}$$

Now, by combining (8) with (10) we get (6), where $\lambda = 1/s^i \in [0, 1/s)$.

Case 2. Assume now that $z_{n_0} = z_{n_0+1}$ for some $n_0 \geqslant 1$. If $n_0 = 2k - 1$, then $z_{2k-1} = z_{2k}$ follows that $z_{2k} = z_{2k+1}$. Indeed, since $TSx_{2k} = z_{2k}$ and $TRx_{2k-1} = z_{2k-1}$ are comparable, by (7) we have

$$\begin{aligned}
s^i d(z_{2k+1}, z_{2k}) &\leqslant \max\bigl\{ d(z_{2k}, z_{2k-1}), d(z_{2k}, z_{2k+1}) \bigr\} \\
&= \max\bigl\{ 0, d(z_{2k}, z_{2k+1}) \bigr\},
\end{aligned}$$

which establishes that $d(z_{2k}, z_{2k+1}) = 0$, that is, $z_{2k} = z_{2k+1}$. If $n_0 = 2k$, then $z_{2k} = z_{2k+1}$ gives that $z_{2k+1} = z_{2k+2}$. Actually, since $TSx_{2k} = z_{2k}$ and $TRx_{2k+1} = z_{2k+1}$ are comparable, then by (9) we have that

$$\begin{aligned}
s^i d(z_{2k+2}, z_{2k+1}) &\leqslant \max\bigl\{ d(z_{2k+1}, z_{2k}), d(z_{2k+1}, z_{2k+2}) \bigr\} \\
&= \max\bigl\{ 0, d(z_{2k+1}, z_{2k+2}) \bigr\}.
\end{aligned}$$

This implies that $d(z_{2k+1}, z_{2k+2}) = 0$, that is, $z_{2k+1} = z_{2k+2}$. Consequently, the sequence $\{z_n\}$ in both cases becomes constant for $n \geqslant n_0$, and hence (6) holds for $n \geqslant n_0$.

So, by making the most of (6) and Lemma 1 we obtain that $\{z_n\}$ is a b-Cauchy sequence in X. Since $(X, d, s, \preccurlyeq)$ is a complete b-metric space, then there exists $z \in X$ such that $\lim_{n\to\infty} z_n = z$. Therefore,

$$\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} z_{2n+1} = z \quad \text{and} \quad \lim_{n\to\infty} TSx_{2n+2} = \lim_{n\to\infty} z_{2n+2} = z.$$

By the continuity of f and S we have

$$\lim_{n\to\infty} STfx_{2n} = Sz, \qquad \lim_{n\to\infty} fTSx_{2n} = fz. \tag{11}$$

Since (f, S) is T-compatible, we have

$$\lim_{n\to\infty} d(STfx_n, fTSx_n) = 0. \tag{12}$$

Now, we have

$$\begin{aligned} d(Sz, fz) &\leqslant sd(Sz, STfx_{2n}) + sd(STfx_{2n}, fz) \\ &\leqslant sd(Sz, STfx_{2n}) + s^2\big[d(STfx_{2n}, fTSx_{2n}) + d(fTSx_{2n}, fz)\big]. \end{aligned} \tag{13}$$

Letting $n \to \infty$ in (13), from (11) and (12) we get $d(Sz, fz) = 0$, that is, $Sz = fz$. By similar arguments we obtain $gz = Rz$. Thus, the pairs (f, S) and (g, R) have a coincidence point z in X.

Now, if TSz and TRz are comparable, then by (5) we have

$$\begin{aligned} s^i d(Tfz, Tgz) &\leqslant \max\biggl\{d(TSz, TRz), d(TSz, Tfz), d(TRz, Tgz), \\ &\qquad \frac{d(TSz, Tgz) + d(TRz, Tfz)}{2s}\biggr\} \\ &= \max\biggl\{d(TSz, TRz), d(TSz, TSz), d(TRz, TRz), \\ &\qquad \frac{d(TSz, TRz) + d(TRz, TSz)}{2s}\biggr\} \\ &= d(TSz, TRz) = d(Tfz, Tgz). \end{aligned}$$

Because $s^i > 1$, this implies that $Tfz = Tgz$. Since T is one-to-one, we have that $fz = gz$. From the above, we have $fz = gz = Sz = Rz$. So, z is a coincidence point of f, g, R, S. □

If we choose $Tx = x$ for all $x \in X$ in Theorem 3, then we have Theorem 1.

The following example shows that Theorem 3 is a proper generalization of Theorem 1.

Example 3. Let $X = [0, 1]$ with $d(x, y) = |x - y|^2$ for all $x, y \in X$ and order "$\preccurlyeq$" on X as follows:

$$x \preccurlyeq y \quad \text{if and only if} \quad x \geqslant y \quad \text{for all } x, y \in X.$$

Define mappings $g, S, f, R : X \to X$ by

$$Sx = Rx = \frac{x}{\sqrt{2}} \quad \text{and} \quad fx = gx = \frac{x}{\sqrt{2^{i+1/2}}} \quad \text{for all } x \in X,\ i > 1.$$

Then

(i) $(X, d, 2, \preccurlyeq)$ is a partially ordered complete b-metric space.
(ii) $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$.
(iii) S, g, f, R are continuous.
(iv) The pairs (f, S) and (g, R) are compatible.
(v) The pairs (f, g) and (g, f) are partially weakly increasing with respect to R and S, respectively.
(vi) Theorem 1 is not applicable to S, g, f and R.
(vii) There exists $T : X \to X$ such that T, f, g, R, S satisfy for all conditions of Theorem 3.

Proof. It is easy to check that conclusions (i)–(iii) hold.

(iv) We prove that (f, S) is compatible. Indeed, let $\{x_n\}$ be a sequence in X such that for some $t \in X$, $\lim_{n\to\infty} d(fx_n, t) = \lim_{n\to\infty} d(Sx_n, t) = 0$. Then we have

$$\lim_{n\to\infty} |fx_n - t|^2 = \lim_{n\to\infty} |Sx_n - t|^2 = 0.$$

It is equivalent to

$$\lim_{n\to\infty} \left| \frac{x_n}{\sqrt{2^{i+1/2}}} - t \right|^2 = \lim_{n\to\infty} \left| \frac{x_n}{\sqrt{2}} - t \right|^2 = 0.$$

That is,

$$\lim_{n\to\infty} \left| x_n - t\sqrt{2^{i+1/2}} \right|^2 = \lim_{n\to\infty} \left| x_n - t\sqrt{2} \right|^2 = 0.$$

The uniqueness of the limit gives that $t\sqrt{2^{i+1/2}} = t\sqrt{2}$. This implies that $t = 0$. By the continuity of f and S we have

$$\lim_{n\to\infty} Sfx_n = St = S0 = 0, \qquad \lim_{n\to\infty} fSx_n = ft = f0 = 0.$$

So, from Lemma 2 we get that $\lim_{n\to\infty} d(Sfx_n, fSx_n) = 0$.

Similarly, we have that (g, R) is compatible.

(v) Let $x, y \in X$ such that $y \in R^{-1}fx$, that is, $Ry = fx$. By the definition of f and R we have

$$\frac{y}{\sqrt{2}} = \frac{x}{\sqrt{2^{i+1/2}}} \quad \text{or} \quad y = \frac{x\sqrt{2}}{\sqrt{2^{i+1/2}}}.$$

It implies that

$$gy = g\left(\frac{x\sqrt{2}}{\sqrt{2^{i+1/2}}} \right) = \frac{x\sqrt{2}}{2^{i+1/2}} \leqslant \frac{x}{\sqrt{2^{i+1/2}}} = fx.$$

Therefore, $fx \preccurlyeq gy$. Hence, (f,g) is partially weakly increasing with respect to R. Similarly, we have that (g,f) is partially weakly increasing with respect to S.

(vi) Let $x \in (0,1]$ and $y = 0$. Then

$$2^i d(fx, g0) = 2^i d\left(\frac{x}{\sqrt{2^{i+1/2}}}, 0\right) = 2^i \left|\frac{x}{\sqrt{2^{i+1/2}}} - 0\right|^2 = \frac{x^2}{\sqrt{2}}$$

and

$$\begin{aligned} M_2(x,0) &= \max\left\{d(Sx,R0), d(Sx,fx), d(R0,g0), \frac{d(Sx,g0)+d(R0,fx)}{4}\right\} \\ &= \max\left\{d\left(\frac{x}{\sqrt{2}},0\right), d\left(\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2^{i+1/2}}}\right), 0, \frac{d(\frac{x}{\sqrt{2}},0)+d(0,\frac{x}{\sqrt{2^{i+1/2}}})}{4}\right\} \\ &= d\left(\frac{x}{\sqrt{2}},0\right) = \frac{x^2}{2}. \end{aligned}$$

So, we have $2^i d(fx,g0) > M_2(x,0)$. Therefore, Theorem 1 is not applicable to S, g, f and R.

(vii) Now, let $Tx = x^3$ for all $x \in X$. We shall show that g, S, R, f, T satisfy all assumptions of Theorem 3.

First, we prove that (f,S) is T-compatible. Indeed, let $\{x_n\}$ be a sequence in X such that for some $t \in X$, $\lim_{n\to\infty} d(Tfx_n, t) = \lim_{n\to\infty} d(TSx_n, t) = 0$. Then we have

$$\lim_{n\to\infty} |Tfx_n - t|^2 = \lim_{n\to\infty} |TSx_n - t|^2 = 0.$$

This is equivalent to

$$\lim_{n\to\infty} \left|\left(\frac{x_n}{\sqrt{2^{i+1/2}}}\right)^3 - t\right|^2 = \lim_{n\to\infty} \left|\left(\frac{x_n}{\sqrt{2}}\right)^3 - t\right|^2 = 0.$$

That is,

$$\lim_{n\to\infty} \left|x_n^3 - t\left(\sqrt{2^{i+1/2}}\right)^3\right|^2 = \lim_{n\to\infty} \left|x_n^3 - t(\sqrt{2})^3\right|^2 = 0.$$

The uniqueness of the limit gives that $t(\sqrt{2^{i+1/2}})^3 = t(\sqrt{2})^3$. This implies that $t = 0$. By the continuity of f and S we have

$$\lim_{n\to\infty} STfx_n = St = S0 = 0, \qquad \lim_{n\to\infty} fTSx_n = ft = f0 = 0.$$

So, from Lemma 2 we obtain that $\lim_{n\to\infty} d(STfx_n, fTSx_n) = 0$.

Similarly, we have that (g,R) is T-compatible.

Next, we prove that (f,g) is T-partially weakly increasing with respect to R. Let $x, y \in X$ such that $y \in (TR)^{-1}(Tfx)$, that is, $TRy = Tfx$ or $Ry = fx$. By the definition of f and R, we have

$$\frac{y}{\sqrt{2}} = \frac{x}{\sqrt{2^{i+1/2}}} \quad \text{or} \quad y = \frac{x\sqrt{2}}{\sqrt{2^{i+1/2}}}.$$

It implies that

$$Tgy = Tg\left(\frac{x\sqrt{2}}{\sqrt{2^{i+1/2}}}\right) = \left(\frac{x\sqrt{2}}{2^{i+1/2}}\right)^3 \leqslant \left(\frac{x}{\sqrt{2^{i+1/2}}}\right)^3 = Tfx.$$

It is equivalent to $Tfx \preccurlyeq Tgy$. Hence, (f,g) is T-partially weakly increasing with respect to R. Similarly, we get that (g,f) is T-partially weakly increasing with respect to S.

Next, we consider the following cases.

Case 1. $x = y \in [0,1]$. Then $2^i d(Tfx, Tgy) = 0 \leqslant M_2^T(x,y)$.

Case 2. $x, y \in [0,1]$ and $x \neq y$. Then

$$2^i d(Tfx, Tgy) = 2^i \left|\left(\frac{x}{\sqrt{2^{i+1/2}}}\right)^3 - \left(\frac{y}{\sqrt{2^{i+1/2}}}\right)^3\right|^2 = \frac{(x^3-y^3)^2}{2^{3i+3/2}}$$

and

$$\begin{aligned} M_2^T(x,y) &\geqslant d(TSx, TRy) = d\left(T\left(\frac{x}{\sqrt{2}}\right), T\left(\frac{y}{\sqrt{2}}\right)\right) = \left|\frac{x^3}{2^{3/2}} - \frac{y^3}{2^{3/2}}\right|^2 \\ &= \frac{(x^3-y^3)^2}{2^3} > \frac{(x^3-y^3)^2}{2^{3i+3/2}} = 2^i d(Tfx, Tgy), \end{aligned}$$

It follows that (5) is satisfied for all $x, y \in X$. Therefore, all assumptions of Theorem 3 are satisfied. □

Now, if we choose $Rx = Sx = x$ and $gx = fx$ for all $x \in X$ in Theorem 3, then we obtain the following result.

Corollary 2. *Let $(X, d, s, \preccurlyeq)$ be a partially ordered complete b-metric space with $s > 1$ and $T, g : X \to X$ be two mappings satisfying the following*:

(i) *T is one-to-one.*

(ii) *For every two elements $x, y \in X$ such that Tx, Ty are comparable, we have*

$$s^i d(Tgx, Tgy) \leqslant M_s^T(x,y), \tag{14}$$

where $i > 1$ is a constant and

$$M_s^T(x,y) = \max\left\{d(Tx,Ty), d(Tx,Tgx), d(Ty,Tgy), \frac{d(Tx,Tgy)+d(Ty,Tgx)}{2s}\right\}.$$

(iii) *g is continuous.*

(iv) *g is T-compatible.*

(v) *The pair (g,g) is T-partially weakly increasing.*

Then g have a fixed point z in X.

Supposing "g and T are commute" instead of "g is T-compatible" in Corollary 2, we obtain that

Corollary 3. *Let $(X, d, s, \preccurlyeq)$ be a partially ordered complete b-metric space with $s > 1$ and $T, g : X \to X$ be two mappings satisfying the following*:

(i) *T is one-to-one.*
(ii) *For every two elements $x, y \in X$ such that Tx, Ty are comparable, we have*

$$s^i d(Tgx, Tgy) \leqslant M_s^T(x, y), \tag{15}$$

where $i > 1$ is a constant and

$$M_s^T(x, y) = \max\left\{d(Tx, Ty), d(Tx, Tgx), d(Ty, Tgy), \frac{d(Tx, Tgy) + d(Ty, Tgx)}{2s}\right\}.$$

(iii) *g is continuous.*
(iv) *g and T are commute.*
(v) *The pair (g, g) is T-partially weakly increasing.*

Then g have a fixed point z in X.

Now, replacing "the pair (g, g) is T-partially weakly increasing" by "g is monotone non-decreasing with respect to $(T, \preccurlyeq)$, and there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Tgx_0$" and the contractive condition "$s^i d(Tgx, Tgy) \leqslant M_s^T(x, y)$" by "$s^i d(Tgx, Tgy) \leqslant d(Tx, Ty)$" in Corollary 3, we get

Theorem 4. *Let $(X, d, s, \preccurlyeq)$ be a partially ordered complete b-metric space with $s > 1$, and $T, g : X \to X$ be two mappings satisfying the following*:

(i) *T is one-to-one.*
(ii) *For every two elements $x, y \in X$ such that Tx, Ty are comparable, we have*

$$s^i d(Tgx, Tgy) \leqslant d(Tx, Ty), \tag{16}$$

where $i > 1$ is a constant.
(iii) *g is continuous.*
(iv) *g and T are commute.*
(v) *g is monotone non-decreasing with respect to $(T, \preccurlyeq)$.*
(vi) *There exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Tgx_0$.*

Then g have a fixed point z in X.

Proof. By the given assumptions there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Tgx_0$. Putting $x_n = gx_{n-1}$, for every $n \geqslant 1$, we construct the sequence $\{x_n\} \subset X$. Since g is monotone non-decreasing with respect to $(T, \preccurlyeq)$, we get a sequence $\{Tx_n\} \subset X$ such that

$Tx_{n+1} = Tgx_n$ for $n = 0, 1, 2, \dots$ and

$$Tx_0 \preccurlyeq Tx_1 \preccurlyeq Tx_2 \preccurlyeq \cdots \preccurlyeq Tx_{n+1} \preccurlyeq \cdots .$$

If there exists $k_0 \in \mathbb{N}$ such that $Tx_{k_0+1} = Tx_{k_0}$, then $Tgx_{k_0} = Tx_{k_0}$. Since T is one-to-one, we have $gx_{k_0} = x_{k_0}$, that is, x_{k_0} is a fixed point of g, and the proof is finished. If $Tx_{n+1} \neq Tx_n$ for all $n \in \mathbb{N}$, then from (16) we have

$$s^i d(Tx_{n+1}, Tx_n) = s^i d(Tgx_n, Tgx_{n-1}) \leqslant d(Tx_n, Tx_{n-1}).$$

Since $s^i > 1$, we obtain $d(Tx_{n+1}, Tx_n) \leqslant \lambda d(Tx_n, Tx_{n-1})$, where $\lambda = 1/s^i \in (0,1)$. So, from Lemma 1 we get that $\{Tx_n\}$ is a b-Cauchy sequence in X. Since $(X, d, s, \preccurlyeq)$ is a complete b-metric space, there exists $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$. By the continuity of g we have $\lim_{n\to\infty} gTx_n = gz$. Since g and T are commute, we have

$$\lim_{n\to\infty} gTx_n = \lim_{n\to\infty} Tgx_n = \lim_{n\to\infty} Tx_{n+1} = z.$$

The uniqueness of the limit gives that $gz = z$. □

In the following theorem, we omit the assumption of continuity of f, g, R, S by adding the regularity of the space $(X, d, s, \preccurlyeq)$ and replace the compatibility of the pairs (f, S) and (g, R) by the weak compatibility of these pairs.

Theorem 5. *Let $(X, d, s, \preccurlyeq)$ be a regular partially ordered complete b-metric space with $s > 1$ and $T, f, g, S, R : X \to X$ be five mappings satisfying the following*:

(i) *$f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$.*
(ii) *T is one-to-one.*
(iii) *For every two elements $x, y \in X$ such that TSx, TRy are comparable, we have*

$$s^i d(Tfx, Tgy) \leqslant M_s^T(x, y), \tag{17}$$

where $i > 1$ is a constant and

$$M_s^T(x,y) = \max\left\{d(TSx,TRy), d(TSx,Tfx), d(TRy,Tgy), \frac{d(TSx,Tgy) + d(TRy,Tfx)}{2s}\right\}.$$

(iv) *$TR(X)$ and $TS(X)$ are b-closed subsets of X.*
(v) *The pairs (f, S) and (g, R) are T-weakly compatible.*
(vi) *The pairs (f, g) and (g, f) are T-partially weakly increasing with respect to R and S, respectively.*

Then the pairs (f, S) and (g, R) have a coincidence point z in X. Moreover, if TRz and TSz are comparable, then z is a coincidence point of f, g, R and S.

Proof. Similar to the proof of Theorem 3, we can construct the sequence $\{z_n\}$ defined by

$$z_{2n+1} = TRx_{2n+1} = Tfx_{2n} \quad \text{and} \quad z_{2n+2} = TSx_{2n+2} = Tgx_{2n+1}$$

and obtain that there exists $z \in X$ such that $\lim_{n\to\infty} z_n = z$. Since $TR(X)$ and $TS(X)$ are b-closed, $\{z_{2n+1}\} \subseteq TR(X)$ and $\{z_{2n+2}\} \subseteq TS(X)$, then there exist $u, v \in X$ such that $z = TRu$, $z = TSv$ and

$$\lim_{n\to\infty} TRx_{2n+1} = \lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} z_{2n+1} = z = TSv,$$
$$\lim_{n\to\infty} TSx_{2n+2} = \lim_{n\to\infty} Tgx_{2n+1} = \lim_{n\to\infty} z_{2n+2} = z = TSv.$$

We now prove that z is a coincidence point of the pair (f, S).

By using $\lim_{n\to\infty} TRx_{2n+1} = TSv$ and the regularity of $(X, d, s, \preccurlyeq)$ it follows that $TRx_{2n+1} \preccurlyeq TSv$. As a consequence, by (17) we get that

$$d(Tfv, Tgx_{2n+1}) \leqslant \frac{1}{s^i} M_s^T(v, x_{2n+1})$$

and we have

$$\begin{aligned}
\frac{1}{s} d(TSv, Tfv) &\leqslant d(TSv, Tgx_{2n+1}) + d(Tgx_{2n+1}, Tfv) \\
&\leqslant d(TSv, Tgx_{2n+1}) + \frac{1}{s^i} M_s^T(v, x_{2n+1}) \\
&= d(TSv, Tgx_{2n+1}) + \frac{1}{s^i} \max\biggl\{ d(TSv, TRx_{2n+1}), d(TSv, Tfv), \\
&\quad d(TRx_{2n+1}, Tgx_{2n+1}), \frac{d(TSv, Tgx_{2n+1}) + d(TRx_{2n+1}, Tfv)}{2s} \biggr\} \\
&\leqslant d(TSv, Tgx_{2n+1}) + \frac{1}{s^i} \max\biggl\{ d(TSv, TRx_{2n+1}), d(TSv, Tfv), \\
&\quad s\bigl[d(TRx_{2n+1}, TSv) + d(TSv, Tgx_{2n+1})\bigr], \\
&\quad \frac{d(TSv, Tgx_{2n+1})}{2s} + \frac{d(TRx_{2n+1}, TSv) + d(TSv, Tfv)}{2} \biggr\}. \qquad (18)
\end{aligned}$$

Letting $n \to \infty$ in (18), we arrive at

$$\frac{1}{s} d(TSv, Tfv) \leqslant \frac{1}{s^i} d(TSv, Tfv).$$

Because $s^i > s > 1$, we have $d(TSv, Tfv) = 0$, that is, $Tfv = TSv = z$. Since the pair (f, S) is T-weakly compatible, we have that $fz = fTSv = STfv = Sz$. So, z is a coincidence point of the pair (f, S). Similarly, it can be shown that z is a coincidence point of the pair (g, R).

Now, suppose that TSz and TRz are comparable. Since $fz = Sz$ and $gz = Rz$, we have that

$$Tfz = TSz \quad \text{and} \quad Tgz = TRz. \qquad (19)$$

Therefore, by (17) and (19), we obtain that

$$\begin{aligned} s^i d(TSz, TRz) &= s^i d(Tfz, Tgz) \\ &\leqslant \max\biggl\{ d(TSz, TRz), d(TSz, Tfz), d(TRz, Tgz), \\ &\qquad \frac{d(TSz, Tgz) + d(TRz, Tfz)}{2s} \biggr\} \\ &= \max\biggl\{ d(TSz, TRz), 0, 0, \frac{d(TSz, TRz)}{s} \biggr\} \\ &= d(TSz, TRz). \end{aligned}$$

Because $s^i > 1$, this implies that $d(TSz, TRz) = 0$, that is, $TSz = TRz$. Since T is one-to-one, we get $Sz = Rz$. Therefore, $Sz = Rz = fz = gz$. So, z is a coincidence point of f, g, R, S. □

If we choose $Tx = x$ for all $x \in X$ in Theorem 5, then we have Theorem 2.

The following example shows that Theorem 5 is a proper generalization of Theorem 2.

Example 4. In Example 3, if $\{x_n\}$ is a increasing sequence in X, then we have $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$. This is equivalent to $x_n \geqslant x_{n+1}$ for all $n \in \mathbb{N}$. Therefore, if $\{x_n\}$ is a increasing sequence and $\lim_{n\to\infty} x_n = x$, then $x_n \geqslant x$ for all $n \in \mathbb{N}$, that is, $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$. Similarly, if $\{y_n\}$ is a decreasing sequence and $\lim_{n\to\infty} y_n = y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$. Hence, we have that $(X, d, s, \preccurlyeq)$ is a regular partially ordered complete b-metric space with $s = 2$. It is easy to check that $R(X), S(X)$ are b-closed subsets of X, the pairs (f, S) and (g, R) are weakly compatible, the pairs (f, g) and (g, f) are partially weakly increasing with respect to R and S, respectively. Similar to the proof of Example 3, if we choose $x \in (0, 1]$ and $y = 0$, then condition (2) in Theorem 2 is not true. Thus, Theorem 2 is not applicable to S, g, f and R. However, if we choose $Tx = x^3$ for all $x \in X$, then T, f, g, R, S satisfy all assumptions of Theorem 5. Therefore, Theorem 5 is applicable to T, f, g, S and R.

If we choose $Rx = Sx = Tx = x$ for all $x \in X$ in Theorems 3 and 5, then we have Corollary 1, and if we choose $Rx = Sx = x$ and $fx = gx$ for all $x \in X$ in Theorem 5, we obtain that

Corollary 4. *Let $(X, d, s, \preccurlyeq)$ be a regular partially ordered complete b-metric space with $s > 1$ and $T, g : X \to X$ be two mappings satisfying the following*:

(i) *T is one-to-one.*

(ii) *For every two elements $x, y \in X$ such that Tx, Ty are comparable, we have*

$$s^i d(Tgx, Tgy) \leqslant M_s^T(x, y), \quad i = \mathrm{const} > 1, \tag{20}$$

$$\begin{aligned} M_s^T(x, y) = \max\biggl\{ & d(Tx, Ty), d(Tx, Tgx), d(Ty, Tgy), \\ & \frac{d(Tx, Tgy) + d(Ty, Tgx)}{2s} \biggr\}. \end{aligned}$$

(iii) *$T(X)$ is b-closed subset of X.*
(iv) *g is T-weakly compatible.*
(v) *The pair (g, g) is T-partially weakly increasing.*

Then g have a fixed point z in X.

3 Application to integral equations

In this section, we apply Theorem 4 to study the existence of a solution to the integral equation

$$x(t) = \eta(t) + \lambda \int_0^{e^{-t}} K(t,r) f\bigl(r, x(r)\bigr)\,\mathrm{d}r, \tag{21}$$

where $t \in I = [0,1]$ and $\lambda > 0$. Let Γ be the family of all functions $\gamma : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) γ is non-decreasing and $(\gamma(t))^p \leqslant \gamma(t^p)$ for all $p \geqslant 1$.
(ii) $\gamma(t) \leqslant t$ for all $t \in [0,\infty)$.

For example, $\gamma_1(t) = kt$, where $0 \leqslant k < 1$, and $\gamma_2(t) = t/(t+1)$ belong to Γ. We will analyze equation (21) under the following assumptions:

(A1) $\eta : I \to \mathbb{R}$ is continuous function.
(A2) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a constant $L \geqslant 0$ and γ belongs to Γ such that, for all $t \in I$, for all $u, v \in \mathbb{R}$ with $u \geqslant v$,
$$\bigl|f(t,u) - f(t,v)\bigr| \leqslant L\gamma(u-v).$$
(A3) $f(t,x)$ is monotone non-decreasing with respect to "$\leqslant$" in x, that is, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \leqslant x_2$, then we have $f(t,x_1) \leqslant f(t,x_2)$ for all $t \in I$.
(A4) $K : I \times I \to \mathbb{R}^+$ is continuous on $I \times I$ and
$$\int_0^{e^{-t}} K(t,r)\,\mathrm{d}r \leqslant A \quad \text{for all } t \in I.$$
(A5) There exists a map $T : C(I) \to C(I)$ such that T is one-to-one, and for all $x \in C(I)$, $t \in I$, we have
$$\begin{aligned} &T\left(\eta(t) + \lambda \int_0^{e^{-t}} K(t,r) f\bigl(r, x(r)\bigr)\,\mathrm{d}r\right) \\ &\quad = \eta(t) + \lambda \int_0^{e^{-t}} K(t,r) f\bigl(r, Tx(r)\bigr)\,\mathrm{d}r, \end{aligned} \tag{22}$$

and there exists $x_0 \in C(I)$ such that

$$T\left(\eta(t) + \lambda \int\limits_0^{\mathrm{e}^{-t}} K(t,r) f\bigl(r, x_0(r)\bigr)\,\mathrm{d}r\right) \leqslant Tx_0(t) \quad \text{for all } t \in I. \tag{23}$$

(A6) $(\lambda AL)^p \leqslant 1/2^{i(p-1)}$.

Consider the space $X = C(I)$ of all continuous functions defined on $I = [0,1]$ with the standard metric given by

$$\rho(x,y) = \sup_{t\in I}\bigl|x(t) - y(t)\bigr| \quad \text{for all } x, y \in X.$$

This space can also be equipped with a partial order given by

$$x, y \in X, \quad x \preccurlyeq y \quad \text{if and only if } x(t) \geqslant y(t) \text{ for all } t \in I.$$

Now, for $p \geqslant 1$, we define

$$d(x,y) = \bigl(\rho(x,y)\bigr)^p = \Bigl(\sup_{t\in I}\bigl|x(t) - y(t)\bigr|\Bigr)^p = \sup_{t\in I}\bigl|x(t) - y(t)\bigr|^p$$

for all $x, y \in X$.

It is easy to see that (X, d) is a complete b-metric space with $s = 2^{p-1}$ [3]. Now, we will prove the following result.

Theorem 6. *Under assumptions* (A1)–(A6), *equation* (21) *has a solution in* X.

Proof. Let $g : X \to X$ defined by

$$gx(t) = \eta(t) + \lambda \int\limits_0^{\mathrm{e}^{-t}} K(t,r) f\bigl(r, x(r)\bigr)\,\mathrm{d}r$$

and $T : X \to X$ satisfy condition (A5).

(i) It is easy to see that $(X, d, s, \preccurlyeq)$ is a partially ordered complete b-metric space. By virtue of our assumptions, T and g are well defined (this means that if $x \in X$, then $Tx, gx \in X$) and T is one-to-one.

(ii) By (22) it implies that g and T are commute.

(iii) Now, for any $x, y \in X$ with $Tx \preccurlyeq Ty$, this implies that $Tx(t) \geqslant Ty(t)$ for all $t \in I$. Then, by the condition (A2), we get

$$\begin{aligned}
\bigl|Tgx(t) - Tgy(t)\bigr| &= \bigl|gTx(t) - gTy(t)\bigr| \\
&= \left|\lambda \int\limits_0^{\mathrm{e}^{-t}} K(t,r)\bigl[f\bigl(r, Tx(r)\bigr) - f\bigl(r, Ty(r)\bigr)\bigr]\,\mathrm{d}r\right| \\
&\leqslant \lambda \int\limits_0^{\mathrm{e}^{-t}} K(t,r)\bigl|L\gamma\bigl(Tx(r) - Ty(r)\bigr)\bigr|\,\mathrm{d}r.
\end{aligned}$$

Since the function γ is non-decreasing, we have

$$\gamma\big(Tx(r) - Ty(r)\big) \leqslant \gamma\Big(\sup_{r\in I}\big|Tx(r) - Ty(r)\big|\Big) = \gamma\big(\rho(Tx, Ty)\big).$$

It follows that

$$\begin{aligned}&\big|Tgx(t) - Tgy(t)\big| \\ &\quad \leqslant \lambda \int_0^{\mathrm{e}^{-t}} K(t,r)L\gamma\big(\rho(Tx,Ty)\big)\,\mathrm{d}r \leqslant \lambda AL\gamma\big(\rho(Tx,Ty)\big).\end{aligned}$$

Hence, we have

$$\begin{aligned}&2^{i(p-1)}d(Tgx, Tgy) \\ &\quad = 2^{i(p-1)}\sup_{t\in I}\big|Tgx(t) - Tgy(t)\big|^p \leqslant 2^{i(p-1)}(\lambda AL)^p\big(\gamma\big(\rho(Tx,Ty)\big)\big)^p \\ &\quad \leqslant 2^{i(p-1)}(\lambda AL)^p\big(\gamma\big(\rho(Tx,Ty)\big)^p\big) = 2^{i(p-1)}(\lambda AL)^p\gamma\big(d(Tx,Ty)\big) \\ &\quad \leqslant 2^{i(p-1)}(\lambda AL)^p d(Tx,Ty) \leqslant 2^{i(p-1)}\frac{1}{2^{i(p-1)}}d(Tx,Ty) \\ &\quad = d(Tx,Ty).\end{aligned}$$

(iv) Let $x_1, x_2 \in X$ such that $Tx_1(t) \preccurlyeq Tx_2(t)$ for all $t \in I$, that is, $Tx_1(t) \geqslant Tx_2(t)$ for all $t \in I$. Then, by the conditions (A3) and (A4), we have

$$\begin{aligned}Tgx_1(t) - Tgx_2(t) &= gTx_1(t) - gTx_2(t) \\ &= \lambda\int_0^{\mathrm{e}^{-t}} K(t,r)\big[f\big(r, Tx_1(r)\big) - f\big(r, Tx_2(r)\big)\big]\,\mathrm{d}r \geqslant 0.\end{aligned}$$

This implies that $Tgx_1(t) \preccurlyeq Tgx_2(t)$. Therefore, g is monotone non-decreasing with respect to $(T, \preccurlyeq)$.

(v) By (23) it implies that there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Tgx_0$.

From the above, all assumptions of Theorem 4 hold. Therefore, there exists $x \in C(I)$, and x is a fixed point of g. Hence, the integral equation (21) has a solution $x \in C(I)$. $\square$

The following example guarantees the existence of the functions T, K, g, f that satisfy all assumptions of Theorem 6.

Example 5. Let $X = C(I)$ be the set of all continuous functions on $I = [0, 1]$, the b-metric with $s = 2$ defined by $d(x,y) = \sup_{t\in I}|x(t) - y(t)|^2$ for all $x, y \in X$, and the partial order "$\preccurlyeq$" given by $x \preccurlyeq y$ if $x(t) \geqslant y(t)$ for all $t \in I$. Consider the integral equation

$$x(t) = \left(t^2 - \frac{t^2\mathrm{e}^{-4t}}{5.2^i}\right) + \frac{1}{2^{i+1}}\int_0^{\mathrm{e}^{-t}} t^2\mathrm{e}^t r^2\big(x(r) + r^2\big)\,\mathrm{d}r.$$

Put

$$gx(t) = \left(t^2 - \frac{t^2\mathrm{e}^{-4t}}{5.2^i}\right) + \frac{1}{2^{i+1}} \int_0^{\mathrm{e}^{-t}} t^2\mathrm{e}^t r^2\big(x(r) + r^2\big)\,\mathrm{d}r,$$

$$f(t,x) = t\big(x + t^2\big),$$

$$K(t,r) = t^2\mathrm{e}^t r \quad \text{and} \quad Tx(t) = -t^2 + 2x(t) \quad \text{for all } x \in X,\ r,t \in I.$$

Then

(i) The functions g and f are continuous, and the function T is one-to-one.

(ii) Let $L = 2$ and $\gamma(t) = t/2$ for all $t \in I$, then for every $t \in I$ and for every pair $x, y \in \mathbb{R}$ satisfying $x \geqslant y$, we have

$$\begin{aligned}\big|f(t,x) - f(t,y)\big| &= \big|t\big(x + t^2 - y - t^2\big)\big| = \big|t(x-y)\big| \\ &\leqslant |x-y| = 2\frac{1}{2}(x-y) = L\gamma(x-y).\end{aligned}$$

(iii) $K(t,r)$ is continuous on $I \times I$, and for all $t \in I$, we have

$$0 \leqslant \int_0^{\mathrm{e}^{-t}} K(t,r)\,\mathrm{d}r = \int_0^{\mathrm{e}^{-t}} t^2\mathrm{e}^t r\,\mathrm{d}r \leqslant \frac{1}{2}.$$

(iv) For any pair $x_1, x_2 \in \mathbb{R}$ satisfying $x_1 \leqslant x_2$, we have $t(x_1 + t^2) \leqslant t(x_2 + t^2)$ for all $t \in I$. This implies that $f(t, x_1) \leqslant f(t, x_2)$ for all $t \in I$. So, $f(t,x)$ is monotone non-decreasing with respect to "$\leqslant$" in x.

(v) Let $x \in X$ and for any $t, r \in I$, we have

$$\begin{aligned}gTx(t) &= \left(t^2 - \frac{t^2\mathrm{e}^{-4t}}{5.2^i}\right) + \frac{1}{2^{i+1}} \int_0^{\mathrm{e}^{-t}} t^2\mathrm{e}^t r^2\big(Tx(r) + r^2\big)\,\mathrm{d}r \\ &= \left(t^2 - \frac{t^2\mathrm{e}^{-4t}}{5.2^i}\right) + \frac{1}{2^{i+1}} \int_0^{\mathrm{e}^{-t}} t^2\mathrm{e}^t r^2\big(-r^2 + 2x(r) + r^2\big)\,\mathrm{d}r \\ &= \left(t^2 - \frac{t^2\mathrm{e}^{-4t}}{5.2^i}\right) + \frac{1}{2^{i+1}} \int_0^{\mathrm{e}^{-t}} t^2\mathrm{e}^t r^2 2x(r)\,\mathrm{d}r \\ &= 2\left[\left(t^2 - \frac{t^2\mathrm{e}^{-4t}}{5.2^i}\right) + \frac{1}{2^{i+1}} \int_0^{\mathrm{e}^{-t}} t^2\mathrm{e}^t r^2\big(x(r) + r^2\big)\,\mathrm{d}r\right] \\ &= -t^2 + 2gx(t) = Tgx(t).\end{aligned}$$

It implies that g and T are commute.

(vi) Put $x_0(t) = 3t^2$ for all $t \in I$. Since

$$-t^2 + 2\left[\left(t^2 - \frac{t^2 \mathrm{e}^{-4t}}{5.2^i}\right) + \frac{1}{2^{i+1}} \int\limits_0^{\mathrm{e}^{-t}} t^2 \mathrm{e}^t r^2 \bigl(3r^2 + r^2\bigr)\right] \leqslant 5t^2,$$

we get that

$$T\left(\eta(t) + \lambda \int\limits_0^{\mathrm{e}^{-t}} K(t,r) f\bigl(r, x_0(r)\bigr)\,\mathrm{d}r\right) \leqslant T x_0(t).$$

(vii) We have

$$(\lambda A L)^p = \left(\frac{1}{2^{i+1}} \cdot \frac{1}{2} \cdot 2\right)^p = \left(\frac{1}{2^{i+1}}\right)^p = \frac{1}{2^{(i+1)p}} < \frac{1}{2^{i(p-1)}}.$$

From the above, all assumptions to T, K, g, f in Theorem 6 are satisfied. In fact, it is easy to see that $x(t) = t^2$ for all $t \in [0,1]$ is a solution of the equation

$$x(t) = \left(t^2 - \frac{t^2 \mathrm{e}^{-4t}}{5.2^i}\right) + \frac{1}{2^{i+1}} \int\limits_0^{\mathrm{e}^{-t}} t^2 \mathrm{e}^t r^2 \bigl(x(r) + r^2\bigr)\,\mathrm{d}r.$$

Acknowledgment. The author are greatly indebted to anonymous reviewers for their helpful comments to revise the paper.

References

1. M. Abbas, G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.*, **341**(1):416–420, 2008.
2. R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, **87**(1):109–116, 2008.
3. A. Aghajani, M. Abbas, J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Math. Slovaca*, **64**(4):941–960, 2014.
4. A. Aghajani, R. Arab, Fixed points of (ψ, ϕ, θ)-contractive mappings in partially ordered b-metric spaces and applications to quadratic integral equations, *Fixed Point Theory Appl.*, **2013**:245, 2013.
5. R. Allahyari, R. Arab, A.S. Haghighi, A generalization on weak contractions in partially ordered b-metric spaces and its applications to quadratic integral equations, *J. Inequal. Appl.*, **2014**:355, 2014.
6. I.A. Bakhtin, The contraction principle in quasi metric spaces, *Funkts. Anal.*, **30**:26–37, 1989 (in Russian).
7. S. Czerwik, Contractionmappings in b-metric spaces, *Acta Math. Univ. Ostrav.*, **1**:5–11, 1993.

8. S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia*, **46**:263–276, 1998.
9. M. Filipović, L. Paunović, S. Radenović, M. Rajović, Remarks on "Cone metric spaces and fixed point theorems of T-Kannan and T-Chatterjea contractive mappings", *Math. Comput. Modelling*, **54**(5–6):1467–1472, 2011.
10. H. Huang, S. Radenović, J. Vujaković, On some recent coincidence and immediate consequences in partially ordered b-metric spaces, *Fixed Point Theory Appl.*, **2015**:63, 2015.
11. N. Hussain, R. Saadati, R.P. Agrawal, On the topology and wt-distance on metric type spaces, *Fixed Point Theory Appl.*, **2014**:88, 2014.
12. M. Jovanović, Z. Kadelburg, S. Radenović, Common fixed point results in metric-type spaces, *Fixed Point Theory Appl.*, **2010**:978121, 2010.
13. S. Moradi, M. Omid, A fixed point theorem for integral type inequality depending on another function, *Int. J. Math. Anal.*, **4**(29–32):1491–1499, 2010.
14. Z. Mustafa, J.R. Roshan, V. Parvaneh, Z. Kadelburg, Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spaces, *J. Inequal. Appl.*, **2014**(46):1–14, 2014.
15. J.J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**(3):223–239, 2005.
16. J.J. Nieto, R. Rodríguez-López, Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sin., Engl. Ser.*, **23**:2205–2212, 2007.
17. V. Parvaneh, J.R. Roshan, S. Radenović, Existence of tripled coincidence points in ordered b-metric spaces and an application to a system of integral equations, *Fixed Point Theory Appl.*, **2013**:130, 2013.
18. A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, *Proc. Am. Math. Soc.*, **132**:1435–1443, 2004.
19. A. Razani, V. Parvaneh, Some fixed point theorems for weakly T-Chatterjea and weakly T-Kannan-contractive mappings in complete metric spaces, *Russ. Math.*, **57**(3):38–45, 2013.
20. J.R. Roshan, V. Parvaneh, I. Altun, Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations, *Appl. Math. Comput.*, **226**:725–737, 2014.
21. J.R. Roshan, V. Parvaneh, Z. Kadelburg, Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces, *J. Nonlinear Sci. Appl.*, **7**:229–245, 2014.
22. J.R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, W. Shatanawi, Common fixed points of almost gen-eralized $(\psi;\varphi)_s$ contractive mappings in ordered b-metric spaces, *Fixed Point Theory Appl.*, **2013**:159, 2013.