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Stability analysis of fractional-order delayed neural networks*

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Abstract. At the beginning, a class of fractional-order delayed neural networks were employed. It is known that the active functions in a target model may be Lipschitz continuous, while some others may also possessing inverse Lipschitz properties. Based upon the topological degree theory, nonsmooth analysis, as well as nonlinear measure method, several novel sufficient conditions are established towards the existence as well as uniqueness of the equilibrium point, which are voiced in terms of linear matrix inequalities (LMIs). Furthermore, the stability analysis is also attached. One numerical example and its simulations are presented to illustrate the theoretical findings.

Keywords: fractional-order neural network, inverse Lipschitz neuron activations, topological degree theory, stability analysis.

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1 Introduction

Fractional operators have been mentioned by Leibnitz in a letter to L'Hôspital in 1695. Although it possess a long mathematical history, the applications of fractional calculus to physics and engineering are only a recent focus of interest when referring to its unique advantage.

Compared with the classical integer-order model, fractional-order system contains infinite memory. Taking into account this fact, it is easy to see that the incorporation of a memory term into a neural network is an extremely important improvement, which is more suitable to describe the memory and hereditary properties of various materials and processes, thus, a system described by fractional-order calculation is even more closely to the real world problems. Admittedly, classical integer system fails in this aspect.

In the past few decades, a great progress in studying fractional evolution model has been made. Indeed, fractional-order system plays a crucial role in many fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, etc. Also, fractional calculus has been successfully incorporated into the neural networks and some interesting results have been reported in [1,5,9,10,21,26,29,30]. Among which, [9,10] pay attention to the synchronization control of fractional-order neural networks, some stability analysis are derived in [26, 29, 30].

Very recently, considerable efforts have been devoted to the analysis of neural networks due to its extensive applications in control, signal processing, pattern recognition, image processing, and associative memory [3, 4, 6, 7, 12, 13, 19]. For most of these successful applications, stability is usually a prerequisite, and fruitful results are available in [14, 16, 18, 24, 25, 33, 35, 36]. Based on the free-matrix-based integral inequality, the corresponding exponential stability of delayed neural networks is addressed in [14]. [16] and [25] investigated the impulsive and Markov switching neural networks, respectively. [33] addressed the Mittag–Leffler stability of fractional-order Hopfield system.

Time delays were first introduced by Marcus and Westervelt in 1989 [22]. Subsequently, it was found that time delays often occur in the neural networks due to the finite switching speed of the neuron amplifiers as well as the finite speed of signal propagation. Naturally, time delays are regarded as a main source of poor performance and instability of dynamic systems. Thus, this contributed to the increased attention of the stability analysis of neural networks.

Although significant successes have been derived for the stability analysis of delayed neural networks, what should be noteworthy is that, when tackle with the similar problems, a basic assumption is frequently used, which require the neuron activation functions subject to the strictness Lipschitz continuous, and/or monotonic increasing. It is still unknown what will happen when the hypotheses on the activation functions are replaced by an opposite one, the equilibrium point in this system will exist or not. In consequence, this led to significant attraction of researchers, and some noticeable results are delivered in [11,23,34]. Among which, the stability analysis for the Cohen–Grossberg neural networks with inverse Lipschitz neuron activations are addressed in [23], and some other dynamic behavior analysis of the model with inverse Lipschitz functions are based on the integer-order system, up to now, there are no existing works focus on the fractional-order system that possess inverse Lipschitz functions, which still a challenging problem from both a theoretical and practical point of view. Thus formulated the motivation of this paper.

Motivated by the above discussion, the objective of this paper is to construct the stability analysis of the fractional-order delayed neural networks. Based on the topological degree theory, nonsmooth analysis approach, as well as nonlinear measure method, several novel sufficient conditions are established towards the existence as well as the uniqueness of the equilibrium point, which are delivered in the forms of LMI. What should be emphasized is that the derived conclusions contains both the active functions are Lipschitz as well as inverse Lipschitz continuous. Furthermore, the stability analysis is also attached in the end.

The structure of this paper is outlined in the following manner. In Section 2, some preliminaries are introduced. Section 3 contains several new sufficient conditions to check the stability problems of delayed neural networks. In Section 4, one numerical example is given to substantiate the theoretical results. Conclusions are drawn in Section 5.

For the readers' convenience, some useful notations are recalled: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $[\cdot,\cdot]$ represents the interval. For any matrix A, A^{T} and A^{-1} stand for the transpose and inverse of A. Set $\mathcal{LP}(\epsilon)$, $\mathcal{L}(\epsilon)$, $\mathcal{G}(\epsilon)$ and $\mathcal{GL}(\epsilon)$ implies locally partial ϵ-inverse Lipschitz functions, locally ϵ-inverse Lipschitz functions, globally ϵ-inverse Lipschitz functions and globally Lipschitz functions, respectively. Denote the space of continuous functions mapping $[-\tau,0]$ into $\mathbb{R}^n$ by $\mathcal{C}$. For a class of all continuous column vector function $y(t)$, its norm is given by $\|y(t)\| = \sum_{i=1}^n \sup_{t\in[0,T]}\{\mathrm{e}^{-t}|y_i(t)|\}$. Moreover, by $\langle x,y\rangle = x^{\mathrm{T}}y$ we mean the inner product of x, y.

2 Problem formulation and preliminaries

Consider a class of fractional-order delayed neural networks given by

$$D^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^n b_{ij} f_j\big(x_j(t-\tau)\big) + I_i, \quad i = 1,2,\dots,n, \tag{1}$$

where $0 < \alpha < 1$, D^α is chosen as the Caputo fractional derivative operator $D^\alpha_{0,t}$, n corresponds to the number of units in a system, $x_i(t)$ implies the state of the ith unit at time t, $c_i > 0$ stands for the self-regulating parameters of the neuron, a_{ij} and b_{ij} are the connection strength of the jth neuron on the ith neuron, respectively; $f_j(\cdot)$ denotes the activation functions; I_i signifies the external bias on the ith neuron, and τ represents the transmission delays.

The initial conditions of the given system (1) are listed as

$$x(s) = \phi(s), \quad s \in [-\tau, 0]. \tag{2}$$

In this paper, we need the following definitions.

Definition 1. The equilibrium point of system (1) is said to be stable if for any $\varepsilon > 0$, there exists $\delta(t_0, \varepsilon) > 0$ such that $\|\psi - x^*\| < \delta$ implies $\|x(t) - x^*\| < \varepsilon$ holds for any solution with $t \geqslant t_0 \geqslant 0$. It is uniformly stable if δ is independent of t_0.

Definition 2. A continuous function $h : \mathbb{R} \to \mathbb{R}$ is said to be locally partial ϵ-inverse Lipschitz if h is a monotonic increasing function and for any $v \in \mathbb{R}$, there exist two positive constants q_v, r_v that depend on v such that

$$\big|h(u) - h(v)\big| \geqslant q_v |u - v|^{\epsilon} \quad \forall |u - v| \leqslant r_v$$

holds, where $\epsilon > 0$ is a positive constants.

If q and r are independent of v, i.e., for any $v \in \mathbb{R}$, there have two fixed constants $q > 0$, $r > 0$ such that

$$\big|h(u) - h(v)\big| \geqslant q |u - v|^{\epsilon} \quad \forall |u - v| \leqslant r,$$

in this case, $h(\cdot)$ is said to be locally ϵ-inverse Lipschitz. Moreover, if $r = +\infty$, then $h(\cdot)$ is said to be globally ϵ-inverse Lipschitz.

Definition 3. A function $h : \mathbb{R} \to \mathbb{R}$ is said to be globally Lipschitz if there has a constant $p > 0$ such that

$$\big|h(u) - h(v)\big| \leqslant p |u - v|^{\epsilon} \quad \forall u, v \in \mathbb{R}.$$

Having given these useful definitions, we now return to the following two preliminary assumptions, which will be frequently used in analyzing the existence and uniqueness of the solution for system (1):

(H1) The activation functions $f_i(\cdot) \in \mathcal{LP}(\epsilon)$ for $i = 1, 2, \ldots, n$.

(H2) The activation functions $f_i(\cdot)$ are all belong to $\mathcal{G}(1) \cap \mathcal{GL}$, $i = 1, 2, \ldots, n$, which implies that there exist positive constants f_i', f_i'' such that

$$f_i' \leqslant \frac{f_i(u) - f_i(v)}{u - v} \leqslant f_i''$$

holds for any $u, v \in \mathbb{R}$, $u \neq v$.

We now attempt to give a basic understanding of nonsmooth analysis. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a class of locally Lipschitz continuous functions. Then, according to Rademacher's theorem [27], F is differentiable almost everywhere. Let D_F signify the set of those points, where F is differentiable and $F'(x)$ stand for the Jacobian of F at $x \in D_F$. Then the set D_F is dense in $\mathbb{R}^n$. In the following line, we will ready to introduce the definition of generalized Jacobian [20, 31, 32].

Definition 4. For any $x \in \mathbb{R}^n$, the generalized Jacobian $\partial F(x)$ of a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^n$ is a set of matrices defined by

$$\partial F(x) = \mathrm{co}\Big\{ W \;\Big|\; \text{there exists a sequence } \{x^k\} \subset D_F \text{ with } \lim_{x^k \to x} F'(x^k) = W \Big\},$$

in which $\mathrm{co}(\cdot)$ stand for the convex hull of a set.

The generalized Jacobian is a natural generalization of the Jacobian for continuously differential functions, at those points x, where F is continuously differentiable, $\partial F(x)$ reduces to a single matrix, which is the Jacobian of F, while at those points, where F is not differential or not continuously differentiable, $\partial F(x)$ may contain more than one matrices.

Definition 5. (See [17].) Assume that Ξ is an open set, which belong to $\mathbb{R}^n$, and $G : \Xi \to \mathbb{R}^n$ is an operator. $m_\Xi(G)$ signifies the nonlinear measure of G on Ξ with the following form:

$$m_\Xi(G) = \sup_{\substack{v_1, v_2 \in \Xi \\ v_1 \neq v_2}} \frac{\langle G(v_1) - G(v_2), v_1 - v_2 \rangle}{\|v_1 - v_2\|_2^2}.$$

Lemma 1. (*See* [17].) *G can be treated as an injective mapping on Ξ if it follows that $m_\Xi(G) < 0$. Moreover, if Ξ is selected as $\mathbb{R}^n$, then G is a homeomorphism of $\mathbb{R}^n$.*

Remark 1. From Lemma 1 we can easily conclude that the system $G(x) = 0$ has one unique solution if we properly choose $m_\Xi(G) < 0$ and $\Xi = \mathbb{R}^n$.

Lemma 2. (*See* [2].) *Let scalar $s > 0$, $x, y \in \mathbb{R}^n$, and $Q \in \mathbb{R}^{n\times n}$, then*

$$2x^{\mathrm{T}}Qy \leqslant sx^{\mathrm{T}}QQ^{\mathrm{T}}x + s^{-1}y^{\mathrm{T}}y.$$

Lemma 3. (*See* [28].) *If $h(\cdot) \in \mathcal{LP}(\epsilon)$ and $h(0) = 0$, then there exist constants $q_0 > 0$ and $r_0 > 0$ such that*

$$|h(s)| \geqslant q_0|s|^\epsilon \quad \forall |s| \leqslant r_0.$$

Moreover,

$$|h(s)| \geqslant q_0 r_0^\epsilon \quad \forall |s| \geqslant r_0.$$

Set Θ be a nonempty, bounded and open subset of $\mathbb{R}^n$. The closure and boundary of Θ are denoted by $\bar{\Theta}$ and $\partial\Theta$, respectively.

Lemma 4. (*See* [8].) *Let $H(\lambda, x) : [0,1]\times\bar{\Theta} \to \mathbb{R}^n$ be a continuous homotopic mapping. If $H(\lambda, x) = y$ has no solutions in $\partial\Theta$ for $\lambda \in [0, 1]$ and $y \in \mathbb{R}^n \setminus H(\lambda, \partial\Theta)$, then the topological degree $\deg(H(\lambda, x), \Theta, y)$ of $H(\lambda, x)$ is a constant, which is independent of λ. In this case, $\deg(H(0, x), \Theta, y) = \deg(H(1, x), \Theta, y)$.*

Lemma 5. (*See* [8].) *Let $H(x) : \bar{\Theta} \to \mathbb{R}^n$ be a continuous mapping. If $\deg(H(x), \Theta, y) \neq 0$, then there exists at least one solution of $H(x) = y$ in Θ.*

Lemma 6. (*See* [15].) *If $x(t) \in \mathcal{C}^n[0,\infty)$ and $n - 1 < \alpha < n \in Z^+$, then*

$$D^{-\alpha}D^{-\beta}x(t) = D^{-(\alpha+\beta)}x(t), \quad \alpha, \beta \geqslant 0,$$
$$D^{\alpha}D^{-\beta}x(t) = x(t), \quad \alpha = \beta \geqslant 0,$$
$$D^{-\alpha}D^{\beta}x(t) = x(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} x^i(0), \quad \alpha = \beta \geqslant 0.$$

3 Main results

This section will furnish sufficient conditions for characterizing the existence and uniqueness of the equilibrium point for a class of fractional-order delayed neural networks, the stability properties of the target model is also attached. Then we will discuss our main contributions to this problem in some details.

3.1 The uniqueness of equilibrium point

Theorem 1. *Under assumptions* (H1), *if there exist an arbitrary positive constant* σ *and a diagonal matrix* $P = \operatorname{diag}\{p_1, \dots, p_n\} > 0$ *such that*

$$\begin{pmatrix} PA + (PA)^{\mathrm{T}} + \sigma I & PB \\ * & -\sigma I \end{pmatrix} < 0 \tag{3}$$

holds, then the neural networks (1) *has a unique equilibrium point.*

Proof. If x^* is an equilibrium point of (1) with $x^* = (x_1^*, x_2^*, \dots, x_n^*)^{\mathrm{T}}$, then one can read that

$$-c_i x_i^* + \sum_{j=1}^{n} a_{ij} f_j(x_j^*) + \sum_{j=1}^{n} b_{ij} f_j(x_j^*) + I_i = 0, \quad i = 1, 2, \dots, n. \tag{4}$$

Defining the following map associated with model (1):

$$F(x) = Cx - (A + B) f(x) - I, \tag{5}$$

where

$$x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}, \qquad I = (I_1, I_2, \dots, I_n)^{\mathrm{T}},$$
$$C = \operatorname{diag}(c_1, c_2, \dots, c_n), \qquad A = (a_{ij})_{n \times n},$$
$$B = (b_{ij})_{n \times n}, \qquad f(x) = \bigl(f_1(x), f_2(x), \dots, f_n(x)\bigr)^{\mathrm{T}}.$$

Obviously, the equilibrium point of model (1) is the solution of the equation $F(x) = 0$. It follows from the definition of the mapping F presented in (5) that

$$F(x) - F(0) = Cx - (A + B)\bigl(f(x) - f(0)\bigr),$$

which then yields that

$$F(x) = Cx - (A + B)\hat{f}(x) + F(0) \tag{6}$$

with

$$\hat{f}(x) = f(x) - f(0).$$

Furthermore, referring to assumption (H1) and its definitions, one can read that $\hat{f}_i \in \mathcal{LP}(\epsilon)$ with $\hat{f}_i(0) = 0$, $\hat{f}_i(x_i)x_i > 0$ $(x_i \neq 0)$.

Set

$$\Theta = \{(x_1, \dots, x_n)^{\mathrm{T}}\colon |x_i| < R,\ i = 1, 2, \dots, n\}, \quad R > 0,$$

and

$$H(\lambda, x) = Cx - \lambda(A + B)\hat{f}(x) + \lambda F(0), \tag{7}$$

where $x \in \bar{\Theta} = \{(x_1, \dots, x_n)^{\mathrm{T}}\colon |x_i| \leqslant R,\ i = 1, 2, \dots, n\}$ and $\lambda \in [0, 1]$.

Referring to Lemma 2 and the assumptions that injecting to the active functions, one has

$$\begin{aligned}
\hat{f}^{\mathrm{T}}(x)PH(\lambda, x) &= \hat{f}^{\mathrm{T}}(x)P\big(Cx - \lambda(A + B)\hat{f}(x) + \lambda F(0)\big) \\
&\geqslant \hat{f}^{\mathrm{T}}(x)PCx + \lambda\hat{f}^{\mathrm{T}}(x)PF(0) \\
&\quad - \lambda\hat{f}^{\mathrm{T}}(x)\left\{\frac{PA + A^{\mathrm{T}}P}{2} + \frac{(PB)(PB)^{\mathrm{T}}}{2\sigma} + \frac{\sigma}{2}I\right\}\hat{f}(x) \\
&\geqslant \hat{f}^{\mathrm{T}}(x)PCx + \lambda\hat{f}^{\mathrm{T}}(x)PF(0) \\
&\geqslant \sum_{i=1}^{n} \big[\big|\hat{f}_i(x_i)\big|p_i c_i|x_i| - \big|\hat{f}_i(x_i)\big|p_i\big|F_i(0)\big|\big] \\
&= \sum_{i=1}^{n} \big|\hat{f}_i(x_i)\big|p_i c_i\left[|x_i| - \frac{|F_i(0)|}{c_i}\right].
\end{aligned} \tag{8}$$

On the strength of Lemma 3, we know that there exist some positive constants q_k and r_k, $k = 1, 2, \dots, n$, such that

$$\big|\hat{f}_i(x_i)\big| \geqslant q_i r_i^{\epsilon} \quad \forall |x_i| \geqslant r_i,\ i = 1, 2, \dots, n. \tag{9}$$

Define $r = \max_{1 \leqslant k \leqslant n}\{r_k\}$, $u = \max\{\max_{1 \leqslant i \leqslant n}|F_i(0)|/c_i\}$, $\mathbb{N}_k = \{n_1, n_2, \dots, n_k\} \subset \{1, 2, \dots, n\}$ for all $n > k$. Moreover, selecting

$$\Theta_{\mathbb{N}_k} = \{x \in \mathbb{R}^k\colon |x_i| \leqslant u,\ i \in \mathbb{N}_k\},$$

as well as

$$h_{\mathbb{N}_k}(x) = \sum_{i \in \mathbb{N}_k} p_i c_i\big|\hat{f}_i(x_i)\big|\big(|x_i| - u\big).$$

Realizing that $\Theta_{\mathbb{N}_k}$ is a compact subset of $\mathbb{R}^k$, $h_{\mathbb{N}_k}(x)$ is continuous on $\Theta_{\mathbb{N}_k}$. Thus, $\Theta_{\mathbb{N}_k}$ can reach its minimum value $\min_{(x) \in \Theta_{\mathbb{N}_k}} h_{\mathbb{N}_k}(x)$ on $\Theta_{\mathbb{N}_k}$.

Set $\zeta = \min_{1 \leqslant i \leqslant n}\{p_i c_i q_i r_i^{\epsilon}\}$, $\mathcal{M}_{\mathbb{N}_k} = \min_{x \in \Theta_{\mathbb{N}_k}} h_{\mathbb{N}_k}(x)$, $\mathcal{M} = \min\{\mathcal{M}_{\mathbb{N}_k}\colon \mathbb{N}_k \subset \{1, 2, \dots, n\}\}$ and $R > \max\{\sqrt{n}(u - \mathcal{M}/\zeta), \sqrt{n}r\}$, $x \in \partial\Theta$. Then there exist two index sets $\mathbb{N}_1$, $\bar{\mathbb{N}}_1$ render

$$|x_i| \leqslant u, \quad i \in \mathbb{N}_1,\ |x_i| > u,\ i \in \bar{\mathbb{N}}_1,$$

where $\mathbb{N}_1 \bigcup \bar{\mathbb{N}}_1 = \{1, 2, \dots, n\}$. Without loss of generality, we presuming that $\bar{\mathbb{N}}_1 \neq \emptyset$, namely, there possess an index $i_0 \in \bar{\mathbb{N}}_1$ such that

$$|x_{i_0}| \geqslant \frac{R}{\sqrt{n}} \geqslant \max\{u, r\}. \tag{10}$$

From (8) along with (10), for any $x \in \partial\Theta$ and $\lambda \in [0,1]$, one can read that

$$\begin{aligned}\hat{f}^{\mathrm{T}}(x)PH(\lambda,x) &\geqslant \sum_{i\in\mathbb{N}_1}\big|\hat{f}_i(x_i)\big|p_ic_i\big[|x_i|-u\big]+\sum_{i\in\bar{\mathbb{N}}_1}\big|\hat{f}_i(x_i)\big|p_ic_i\big[|x_i|-u\big]\\&\geqslant \mathcal{M}+q_{i_0}r_{i_0}^{\epsilon}p_{i_0}c_{i_0}\big(|x_{i_0}|-u\big)\\&\geqslant q_{i_0}r_{i_0}^{\epsilon}p_{i_0}c_{i_0}\left(|x_{i_0}|-u+\frac{\mathcal{M}}{\zeta}\right)\\&\geqslant q_{i_0}r_{i_0}^{\epsilon}p_{i_0}c_{i_0}\left(\frac{R}{\sqrt{n}}-u+\frac{\mathcal{M}}{\zeta}\right)>0.\end{aligned}\tag{11}$$

Now, one can safely conclude that $H(\lambda,x)\neq 0$ holds for all $x\in\partial\Theta$ and $\lambda\in[0,1]$. Thus, an immediate consequence from Lemma 4 reads

$$\deg\big(H(0,x),\Theta,0\big)=\deg\big(H(1,x),\Theta,0\big),$$

which also contains that

$$\deg\big(F(x),\Theta,0\big)=\deg(Cx,\Theta,0)=\prod_{i=1}^{n}c_i\neq 0.$$

By now, all the conditions in Lemma 4 are satisfied, so we can confidently assert that $F(x)=0$ has at least one solution in Θ, which implies that (1) contains at least one equilibrium point. This concludes the first part of the proof.

For what regards the second part of the proof, i.e., the uniqueness of the equilibrium point, we will employ the technique of contradiction to test this fact.

Presuming that x_1^* and x_2^* are two different equilibrium points of system (1), which then leads to

$$C(x_2^*-x_1^*)=(A+B)\big(f(x_2^*)-f(x_1^*)\big).\tag{12}$$

Once again applying Lemma 2, the following estimation is true on considering the LMI conditions that derived in (3):

$$\begin{aligned}0&<\big(f(x_2^*)-f(x_1^*)\big)^{\mathrm{T}}PC\big(x_2^*-x_1^*\big)\\&=\big(f(x_2^*)-f(x_1^*)\big)^{\mathrm{T}}P(A+B)\big(f(x_2^*)-f(x_1^*)\big)\\&\leqslant\frac{1}{2}\big(f(x_2^*)-f(x_1^*)\big)^{\mathrm{T}}\left(\frac{PA+A^{\mathrm{T}}P}{2}+\frac{(PB)(PB)^{\mathrm{T}}}{2\sigma}+\frac{\sigma}{2}I\right)\big(f(x_2^*)-f(x_1^*)\big)\\&<0.\end{aligned}\tag{13}$$

Obviously, this possess a contradiction. Thus, system (1) has a uniqueness equilibrium point. □

Before concluding the discussion on the inverse Lipschitz conditions case, some additional works are required on the system that possess Lipschitz active functions, which is reported by the following statement.

Theorem 2. *Suppose that the active functions subjected to assumption* (H2) *if there exist positive definite matrices $Q > 0$ and two positive definite diagonal matrices R_1, R_2 such that*

$$\begin{bmatrix} -C & QA & QB & LR_1 & LR_2 \\ * & -2R_1 & 0 & 0 & 0 \\ * & * & -2R_2 & 0 & 0 \\ * & * & * & -2R_1 & 0 \\ * & * & * & * & -2R_2 \end{bmatrix} < 0 \tag{14}$$

holds, where

$$L = \operatorname{diag}(l_1, l_2, \dots, l_n), \quad l_i = \max\{|f_i'|, |f_i''|\}.$$

Then the neural networks modeled by (1) *has a unique equilibrium point.*

Proof. The statement will be proven by using the technique of nonlinear measure. To this end, define the operator ω as

$$\omega(\alpha) = -C\alpha + Af(\alpha) + Bf(\alpha) + I$$

and then construct a differential system, which given below

$$\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = Q\omega(\alpha(t)). \tag{15}$$

Considering that the matrix Q is invertible, thus, systems (1) and (15) sharing the same equilibrium points sets.

If we want to testify that system (1) has a unique equilibrium point, then $m_{\mathbb{R}^n}(\omega) < 0$ must be hold, while it is convenient to drop the dependency on Definition 5, from which we can safely reached that $\langle \omega(\alpha) - \omega(\beta), \alpha - \beta \rangle < 0$ can guarantee $m_{\mathbb{R}^n}(\omega) < 0$. Here, attention will be focused on the fact that $\langle \omega(\alpha) - \omega(\beta), \alpha - \beta \rangle < 0$ holds.

By taking advantage of the definition of inner product, it can be shown that

$$\begin{aligned}
&\langle Q\omega(\alpha) - Q\omega(\beta), \alpha - \beta \rangle \\
&\quad = (\alpha - \beta)^{\mathrm{T}} Q(\omega(\alpha) - \omega(\beta)) \\
&\quad = -(\alpha - \beta)^{\mathrm{T}} QC(\alpha - \beta) + (\alpha - \beta)^{\mathrm{T}} QA(f(\alpha) - f(\beta)) \\
&\qquad + (\alpha - \beta)^{\mathrm{T}} QB(f(\alpha) - f(\beta)) \\
&\quad \leqslant -(\alpha - \beta)^{\mathrm{T}} QC(\alpha - \beta) + \frac{1}{2}(\alpha - \beta)^{\mathrm{T}} QAR_1^{-1}A^{\mathrm{T}}Q(\alpha - \beta) \\
&\qquad + \frac{1}{2}(\alpha - \beta)^{\mathrm{T}} LR_1L(\alpha - \beta) + \frac{1}{2}(\alpha - \beta)^{\mathrm{T}} QBR_2^{-1}B^{\mathrm{T}}Q(\alpha - \beta) \\
&\qquad + \frac{1}{2}(\alpha - \beta)^{\mathrm{T}} LR_2L(\alpha - \beta) \\
&\quad = (\alpha - \beta)^{\mathrm{T}} \left(-C + \frac{1}{2}QAR_1^{-1}A^{\mathrm{T}}Q + \frac{1}{2}LR_1L + \frac{1}{2}QBR_2^{-1}B^{\mathrm{T}}Q + \frac{1}{2}LR_2L \right) \\
&\qquad \times (\alpha - \beta).
\end{aligned} \tag{16}$$

As an immediate consequence of Schur complement lemma, we can conclude that (14) can guarantee $-C + QAR_1^{-1}A^{\mathrm{T}}Q/2 + LR_1L/2 + QBR_2^{-1}B^{\mathrm{T}}Q/2 + LR_2L/2 < 0$. Considering the fact that $\alpha \neq \beta$, one has $(\alpha - \beta)^{\mathrm{T}}Q(\omega(\alpha) - \omega(\beta)) < 0$.

Now, we can safely draw a conclusion that system (15) or system (1) has a unique equilibrium point. We now drop the main result again for compactness and continue with our discussion. □

3.2 Stability analysis

The main contribution in the following line is to prove the stability analysis for a fractional-order system in form (1). Defining that x_i^* is the uniqueness equilibrium point of (1), then making a coordinate transformation $y_i(t) = x_i(t) - x_i^*$, system (1) can be equivalently modified as

$$D^{\alpha}y_i(t) = -c_iy_i(t) + \sum_{j=1}^{n} a_{ij}g_j\big(y_j(t)\big) + \sum_{j=1}^{n} b_{ij}g_j\big(y_j(t-\tau)\big), \quad i = 1, 2, \dots, n, \tag{17}$$

where $g_j(y_j(t)) = f_j(x_j(t)) - f_j(x_j^*)$.

The initial conditions of the given system (17) is given by

$$x(s) = \psi(s), \quad s \in [-\tau, 0]. \tag{18}$$

Thus, to investigate the stability analysis of (1), we can turn to study its equivalent system (17). Then we have the following conclusion.

Theorem 3. *On the basis of Theorem* 2*, the unique equilibrium point of neural networks* (1) *is uniformly stable if the following inequality holds*:

$$1 - \left(\max_i(c_i) + \sum_{i=1}^{n} \max_j\big(|a_{ij}|l_j\big) + \sum_{i=1}^{n} \max_j\big(|b_{ij}|l_j\big)\right) > 0. \tag{19}$$

Proof. By taking advantage of Lemma 6, one can read that

$$\begin{aligned} y_i(t) &= D^{-\alpha}\left\{-c_iy_i(t) + \sum_{j=1}^{n} a_{ij}g_j\big(y_j(t)\big) + \sum_{j=1}^{n} b_{ij}g_j\big(y_j(t-\tau)\big)\right\} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(-c_iy_i(s) + \sum_{j=1}^{n} a_{ij}g_j\big(y_j(s)\big) + \sum_{j=1}^{n} b_{ij}g_j\big(y_j(s-\tau)\big)\right)\mathrm{d}s. \end{aligned} \tag{20}$$

The future estimation can be deduced by multiplying e^{-t} on both sides of equation (20):

$$\begin{aligned} &\mathrm{e}^{-t}y_i(t) \\ &\quad \leqslant \frac{\mathrm{e}^{-t}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(c_i\big|y_i(s)\big| + \sum_{j=1}^{n} |a_{ij}|\big|g_j\big(y_j(s)\big)\big| + \sum_{j=1}^{n} |b_{ij}|\big|g_j\big(y_j(s-\tau)\big)\big|\right)\mathrm{d}s, \end{aligned}$$

$$
\begin{aligned}
\mathrm{e}^{-t}y_i(t) \leqslant{}& c_i\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\mathrm{e}^{-(t-s)}\mathrm{e}^{-s}\bigl|y_i(s)\bigr|\,\mathrm{d}s \\
&+\sum_{j=1}^n |a_{ij}|l_j\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\mathrm{e}^{-(t-s)}\mathrm{e}^{-s}\bigl|y_j(s)\bigr|\,\mathrm{d}s \\
&+\sum_{j=1}^n |b_{ij}|l_j\frac{1}{\Gamma(\alpha)}\int\limits_0^\tau (t-s)^{\alpha-1}\mathrm{e}^{-(t-s+\tau)}\mathrm{e}^{-(s-\tau)}\bigl|\psi_j(s)\bigr|\,\mathrm{d}s \\
&+\sum_{j=1}^n |b_{ij}|l_j\frac{1}{\Gamma(\alpha)}\int\limits_\tau^t (t-s)^{\alpha-1}\mathrm{e}^{-(t-s+\tau)}\mathrm{e}^{-(s-\tau)}\bigl|y_j(s)\bigr|\,\mathrm{d}s \\
\leqslant{}& c_i\sup_t\bigl(\mathrm{e}^{-t}\bigl|y_i(t)\bigr|\bigr)\frac{1}{\Gamma(\alpha)}\int\limits_0^t u^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u \\
&+\max_j\bigl(|a_{ij}|l_j\bigr)\sum_{j=1}^n\sup_t\bigl(\mathrm{e}^{-t}\bigl|y_i(t)\bigr|\bigr)\frac{1}{\Gamma(\alpha)}\int\limits_0^t u^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u \\
&+\max_j\bigl(|b_{ij}|l_j\bigr)\frac{1}{\Gamma(\alpha)}\int\limits_{-\tau}^0 (t-\rho-\tau)^{\alpha-1}\mathrm{e}^{-(t-\rho)}\mathrm{e}^{-\rho}\bigl|\psi_j(\rho)\bigr|\,\mathrm{d}\rho \\
&+\max_j\bigl(|b_{ij}|l_j\bigr)\frac{1}{\Gamma(\alpha)}\int\limits_0^{t-\tau} (t-\rho-\tau)^{\alpha-1}\mathrm{e}^{-(t-\rho)}\mathrm{e}^{-\rho}\bigl|y_j(\rho)\bigr|\,\mathrm{d}\rho \\
\leqslant{}& c_i\sup_t\bigl(\mathrm{e}^{-t}\bigl|y_i(t)\bigr|\bigr)\frac{1}{\Gamma(\alpha)}\int\limits_0^t u^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u \\
&+\max_j\bigl(|a_{ij}|l_j\bigr)\sum_{j=1}^n\sup_t\bigl(\mathrm{e}^{-t}\bigl|y_j(t)\bigr|\bigr)\frac{1}{\Gamma(\alpha)}\int\limits_0^t u^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u \\
&+\max_j\bigl(|b_{ij}|l_j\bigr)\sum_{j=1}^n\sup_t\bigl(\mathrm{e}^{-t}\bigl|\psi_j(t)\bigr|\bigr)\mathrm{e}^{-\tau}\frac{1}{\Gamma(\alpha)}\int\limits_{t-\tau}^t \zeta^{\alpha-1}\mathrm{e}^{-\zeta}\,\mathrm{d}\zeta \\
&+\max_j\bigl(|b_{ij}|l_j\bigr)\sum_{j=1}^n\sup_t\bigl(\mathrm{e}^{-t}\bigl|y_j(t)\bigr|\bigr)\mathrm{e}^{-\tau}\frac{1}{\Gamma(\alpha)}\int\limits_{t-\tau}^t \zeta^{\alpha-1}\mathrm{e}^{-\zeta}\,\mathrm{d}\zeta \\
\leqslant{}& c_i\sup_t\bigl(\mathrm{e}^{-t}\bigl|y_i(t)\bigr|\bigr)+\max_j\bigl(|a_{ij}|l_j\bigr)\sum_{j=1}^n\sup_t\bigl(\mathrm{e}^{-t}\bigl|y_j(t)\bigr|\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&+ \max_j\big(|b_{ij}|l_j\big)\sum_{j=1}^{n}\sup_t\big(\mathrm{e}^{-t}\big|\psi_j(t)\big|\big)\mathrm{e}^{-\tau} \\
&+ \max_j\big(|b_{ij}|l_j\big)\sum_{j=1}^{n}\sup_t\big(\mathrm{e}^{-t}\big|y_j(t)\big|\big)\mathrm{e}^{-\tau} \\
\leqslant{}& c_i\sup_t\big(\mathrm{e}^{-t}\big|y_i(t)\big|\big) + \max_j\big(|a_{ij}|l_j\big)\big\|y(t)\big\| + \max_j\big(|b_{ij}|l_j\big)\big\|\psi(t)\big\| \\
&+ \max_j\big(|b_{ij}|l_j\big)\big\|y(t)\big\|, \qquad (21)
\end{aligned}
$$

which develops further as

$$
\begin{aligned}
\big\|y(t)\big\| \leqslant{}& \left(\max_i(c_i) + \sum_{i=1}^{n}\max_j\big(|a_{ij}|l_j\big) + \max_j\big(|b_{ij}|l_j\big)\right)\big\|y(t)\big\| \\
&+ \sum_{i=1}^{n}\max_j\big(|b_{ij}|l_j\big)\big\|\psi(t)\big\|, \qquad (22)
\end{aligned}
$$

this implies that

$$
\|y(t)\| \leqslant \frac{\sum_{i=1}^{n}\max_j(|b_{ij}|l_j)}{1-(\max_i(c_i)+\sum_{i=1}^{n}\max_j(|a_{ij}|l_j)+\sum_{i=1}^{n}\max_j(|b_{ij}|l_j))}\|\psi(t)\|. \qquad (23)
$$

Hence, resorting from the expression voiced in Definition 1, one can read that for all $\varepsilon > 0$, there exist

$$
\delta = \frac{1-(\max_i(c_i)+\sum_{i=1}^{n}\max_j(|a_{ij}|l_j)+\sum_{i=1}^{n}\max_j(|b_{ij}|l_j))}{\sum_{i=1}^{n}\max_j(|b_{ij}|l_j)}\varepsilon > 0
$$

such that when $\|\psi(t)\| < \delta$, $\|y(t)\| < \varepsilon$ is also true. Thus, the unique equilibrium point of (1) is uniformly stable, which ends the proof. □

Remark 2. If $\alpha = 1$, then the given fractional-order delayed neural networks will turn into an integer-order system. In analogy with the proof techniques that employed above, its stability statement can also be obtained.

Remark 3. As a final note, in the derived results, the unique equilibrium point of a fractional-order system with inverse Lipschitz active function is proposed, of equal importance is that the conclusions concentrated on the Lipschitz conditions are also considered. Subsequently, the stability analysis of the system, which possesses Lipschitz functions, is also considered, while, for a fractional-order system that contains inverse Lipschitz active functions, its dynamic behaviors are an open problem.

4 Numerical examples

This section provides a relevant example of fractional-order delayed systems that fall into the target system, which can well describe the merits of the proposed conditions.

Example. To present our techniques, a three-dimension neural networks will be presented in this section, the dynamics of individual nodes can be described by

$$D^{\alpha}x_i(t) = -c_i x_i(t) + \sum_{i=1}^{3} a_{ij} f_j\bigl(x_j(t)\bigr) + \sum_{i=1}^{3} b_{ij} f_j\bigl(x_j(t-\tau)\bigr) + I_i, \quad i = 1,2,3. \tag{24}$$

By carefully looking for the parameters for the above equation, the following initial values are selected:

$$C = \begin{pmatrix} 0.02 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.09 \end{pmatrix},$$

$$A = \begin{pmatrix} -0.1 & 0.5 & 0.3 \\ -0.2 & 0.3 & 0.2 \\ 0.4 & -0.2 & -0.1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.1 & -0.1 & -0.2 \\ 0.3 & 0.2 & -0.1 \\ -0.2 & 0.5 & 0.3 \end{pmatrix}.$$

Provided that the activation functions are taken as $f(s) = 0.5 \tanh s$, the delays are chosen as $\tau = 1$. Thus, one can read that this functions meet with the restrictions appeared in (H2) with $l_j = 0.5$. Moreover, the α is taken by 0.5.

To inspect the uniqueness of equilibrium point, applying the above initial parameters to the restrictions given in (14), one can arrive at the following feasible solutions:

$$Q = \begin{pmatrix} 0.1360 & -0.0102 & 0.0569 \\ -0.0102 & 0.0298 & -0.0302 \\ 0.0569 & -0.0302 & 0.0727 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 0.0669 & 0 & 0 \\ 0 & 0.0669 & 0 \\ 0 & 0 & 0.0669 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0.0640 & 0 & 0 \\ 0 & 0.0640 & 0 \\ 0 & 0 & 0.0640 \end{pmatrix}.$$

By now, all the restrictions that imposed on Theorem 2 are demonstrated, then one can safely read that model (24) with the above imposed parameters have a unique equilibrium point. The following lines are committed to test and verify the stability of the equilibrium point. Via refereing to the design algorithm as introduced in (19), one can arrive at

$$\begin{aligned} &1 - \left(\max_i(c_i) + \sum_{i=1}^{n} \max_j\bigl(|a_{ij}|l_j\bigr) + \sum_{i=1}^{n} \max_j\bigl(|b_{ij}|l_j\bigr) \right) \\ &\quad = 1 - 0.09 - \bigl(|0.5| + |0.3| + |-0.2|\bigr) \cdot 0.5 - \bigl(|-0.1| + |0.2| + |0.5|\bigr) \cdot 0.5 \\ &\quad = 0.01 > 0, \end{aligned}$$

which indirect clarify that he unique equilibrium point of neural networks (24) is uniformly stable. To give a more acceptable way for our finds, the corresponding simulation figures are performed in Figs. 1–4. Figure 1 depicts the phase portrait of the states in its state space, the state trajectories of the target model are plotted in Figs. 2–4, from which one can read that the unique equilibrium point is uniformly stable. Evidently, this consequences are coincident with the results acquired in the above section.

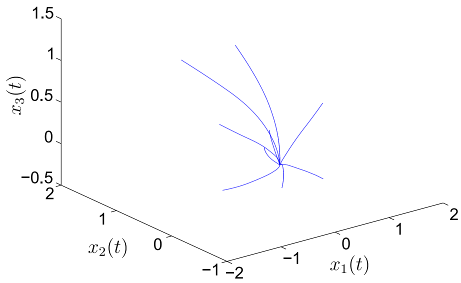

Figure 1. Phase portrait for the state variables $x_i(t)$, $i = 1, 2, 3$, of the target model in (24).

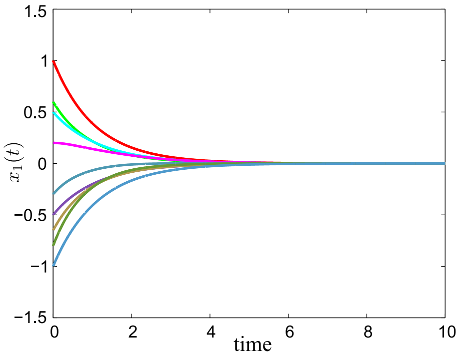

Figure 2. Time response of the desired trajectories $x_1(t)$ for $\alpha = 0.5$ in (24).

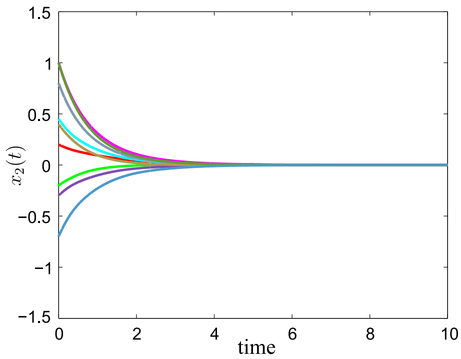

Figure 3. Time response of the desired trajectories $x_2(t)$ for $\alpha = 0.5$ in (24).

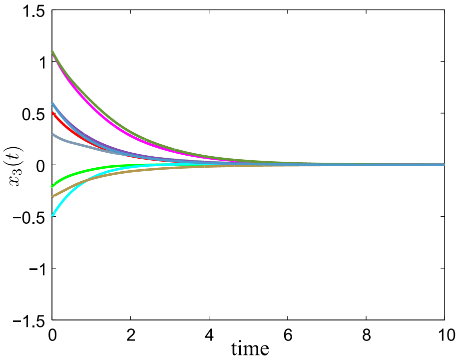

Figure 4. Time response of the desired trajectories $x_3(t)$ for $\alpha = 0.5$ in (24).

5 Conclusion

This paper provides some sufficient conditions for characterizing the stability properties of a class of fractional-order delayed neural networks. The main statements are divided into two steps. In the first step, the existence and uniqueness of the equilibrium point is verified. It should be pointed out that the active functions that possess both Lipschitz continuous and inverse Lipschitz restrictions are all considered, in which, by using the arguments of the topological degree theory and nonsmooth analysis approach, the criteria that ensure the uniqueness of the equilibrium point for the target model that possess inverse Lipschitz functions are derived. When the functions are restricted by Lipschitz conditions, the nonlinear measure method is employed. Subsequently, the stability analysis of the fractional-order system contains Lipschitz continuous are also derived. Thanks to these conclusions, one can simply check some analytical properties of the dynamic system, which can be automatically rewarded with a suitable selection of parameters. Finally, the simulation results are listed to confirm these facts.

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