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Hopf-pitchfork bifurcation of coupled van der Pol oscillator with delay*

Yuting Cai, Chunrui Zhang[1]

Department of Mathematics, Northeast Forestry University,
Harbin 150040, China
math@nefu.edu.cn

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Abstract. In this paper, the Hopf-pitchfork bifurcation of coupled van der Pol with delay is studied. The interaction coefficient and time delay are taken as two bifurcation parameters. Firstly, the normal form is gotten by performing a center manifold reduction and using the normal form theory developed by Faria and Magalhães. Secondly, bifurcation diagrams and phase portraits are given through analyzing the unfolding structure. Finally, numerical simulations are used to support theoretical analysis.

Keywords: delay, Hopf-pitchfork bifurcation, stability, coupled van der Pol model, normal form.

1 Introduction or the first section

The coupling of nonlinear systems comes naturally from physics and engineering, for example, in electronics, nonlinear systems have been long used as an efficient system to generate higher harmonics from a given signal [26]. Studying of coupled nonlinear systems is significant in a number of areas of fundamental and applied mathematics, such as bifurcation in the presence of symmetries, chaos theory, nonlinear electronics.

In the research of nonlinear dynamical system, van der Pol equation is one of the most intensely studied equation (see [14, 15]). This celebrated equation has a nonlinear damping

$$\ddot{x} + \varepsilon\bigl(x^2 - 1\bigr)\dot{x} + x = f(x),$$

which originally was a model for an electrical circuit with a triode valve and was extensively studied as a host of a rich class of dynamical behavior, including relaxation oscillations, quasi-periodicity, elementary bifurcations, and chaos [1]. Noting that most practical implementations of feedback have inherent delays, some researchers have considered the effect of time delay in van der Pol's oscillator [8, 18, 19, 25, 32, 35, 36]. It is shown that the presence of time delay can change the amplitude of limit cycle oscillations. Thus, time

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[1]Corresponding author.

delay is inevitable in coupled systems, and effects of time delay are also very popular in the study of dynamical systems with many delay factors that appear in state variables, and some of them appear in parameters [42]. When time delay becomes a parameter, structural properties of dynamical systems (such as the number of equilibrium, stability, ect.) will change, then the question belongs to the bifurcation question. Now, there are some articles on Hopf bifurcation in delay differential equations (see [2,29,34]). However, there are other articles on Hopf-pitchfork bifurcation in delay differential equations (see [3–6, 10, 17, 22, 24, 27, 30, 31, 38, 41]). Particularly, by our existing knowledge, there is no study in Hopf-pitchfork bifurcation of coupled van der Pol's equation with time delay.

In recent years, various aspects of the van der Pol have been studied [28, 40]. Wen et al. [33] investigate the dynamics of Mathieu equation with two kinds of van der Pol fractional-order terms. Euzebio and Llibre [9] discuss some aspects on the periodic solutions of the extended Duffing–van der Pol oscillator. They show that it can bifurcate one or three periodic solutions from a two-dimensional manifold filled by periodic solutions of the referred system. Kumar et al. [20] carry out investigations on the bifurcation characteristics of a Duffing–van der Pol oscillator subjected to white noise excitations. Fu et al. [13] discuss noise-induced and delay-induced bifurcations in a bistable Duffing–van der Pol oscillator under time delay and join noises theoretically and numerically. Dubkov and Litovsky [7] investigate that the exact Fokker–Planck equation for the joint probability distribution of amplitude and phase of a van der Pol oscillator perturbed by both additive and multiplicative noise sources with arbitrary nonlinear damping is first derived by the method of functional splitting of averages. Yonkeu et al. [39] propose to compute the effective activation energy, usually referred to a pseudopotential or quasipotential, of a birhythmic system – a van der Pol-like oscillator – in the presence of correlated noise. Ji and Zhang [16] use the method of multiple time scales to investigate the following system with both external force and feedback control:

$$\begin{aligned}
&\ddot{x} - \bigl(\mu - \beta x^2\bigr)\dot{x} + \omega x + \alpha x^3 \\
&\quad = e_0 \cos(\Omega_0 t) + px(t-\tau) + q\dot{x}(t-\tau) + k_1 x^3(t-\tau) \\
&\qquad + k_2 \dot{x}^3 + k_3 \dot{x}(t-\tau) x^2(t-\tau) + k_4 \dot{x}^2(t-\tau) x(t-\tau).
\end{aligned}$$

Njah [23] studied the synchronization and antisynchronization of the following van der Pol systems based on the theory of Lyapunov stability and Routh–Hurwitz criteria:

$$\ddot{x} - \mu\bigl(1 - x^2\bigr)\dot{x} + \alpha x + \beta x^3 = 2F\cos(\Omega_0 t).$$

Yamapi and Filatrella [37] studied the strange attractors of the following coupled van der Pol systems:

$$\begin{aligned}
&\ddot{x} - \mu\bigl(1 - x^2\bigr)\dot{x} + x + \beta x^3 = F\cos(\Omega_0 t), \\
&\ddot{y} - \mu\bigl(1 - y^2\bigr)\dot{y} + y + \beta y^3 = F\cos(\Omega_0 t) - K(y - x)H(t - T_0),
\end{aligned}$$

where $H(t)$ is the Heaviside function.

In [37], they have obtained the stability of the equilibrium and the existence of Hopf bifurcation. Using the center manifold reduction technique and normal form theory, they give the direction of the Hopf bifurcation. Therefore, I want to know if this model can produce Hopf-pitchfork bifurcation, and whether we can apply these theories to analyse of the Hopf-pitchfork bifurcation.

Because there are only some articles to study Hopf-pitchfork bifurcation of coupled van der Pol with delay, in order to get more dynamic behaviors, we have the reason to believe that investigating Hopf-pitchfork bifurcation of coupled van der Pol with delay is interesting and worthwhile. Consider the following coupled van der Pol systems:

$$\begin{aligned}&\ddot{x} - \left(\alpha - x^2\right)\dot{x} + x + \beta x^3 = k_1 g\left(y(t-\tau)\right),\\ &\ddot{y} - \left(\alpha - y^2\right)\dot{y} + y + \beta y^3 = k_2 g\left(x(t-\tau)\right).\end{aligned} \tag{1}$$

Let $x = x_1$, $\dot{x} = x_2$, $y = x_3$, $\dot{y} = x_4$, then equation (1) can be written as follows:

$$\begin{aligned}&\dot{x}_1 = x_2,\\ &\dot{x}_2 = -x_1 + \alpha x_2 - x_1^2 x_2 - \beta x_1^3 + k_1 g\left(x_3(t-\tau)\right),\\ &\dot{x}_3 = x_4,\\ &\dot{x}_4 = -x_3 + \alpha x_4 - x_3^2 x_4 - \beta x_3^3 + k_2 g\left(x_1(t-\tau)\right).\end{aligned} \tag{2}$$

Because the activate function $g(t)$ belongs to sigmoidal function (see [12, p. 356]), we assume $g(0) = g''(0) = 0$, $g'(0) = 1$, and $g'''(0) \neq 0$ throughout this paper. Clearly, we probe dynamical behaviors of system (1) equaling to investigate that of system (2).

The rest of the article is organized as follows. In Section 2, we will give the existence condition of the Hopf-pitchfork bifurcation by taking interaction coefficient and delay as two parameters. In Section 3, we use center manifold theory and normal form method [11, 30] to investigate Hopf-pitchfork bifurcation with original parameters. In Section 4, we given some numerical simulations to support the analytic results. Finally, we draw the conclution in Section 5.

2 The existence of Hopf-pitchfork bifurcation

In the following, if the characteristic equation (2) has a simple root 0 and a simple pair of purely imaginary roots $\pm \mathrm{i}\omega_0$, and all other roots of the characteristic equation have negative real parts, then the Hopf-zero bifurcation will occur. The linearization equation of system (2) at the origin is

$$\begin{aligned}&\dot{x}_1 = x_2,\\ &\dot{x}_2 = -x_1 + \alpha x_2 + k_1 x_3(t-\tau),\\ &\dot{x}_3 = x_4,\\ &\dot{x}_4 = -x_3 + \alpha x_4 + k_2 x_1(t-\tau).\end{aligned} \tag{3}$$

The characteristic equation of system (3) is

$$E(\lambda) = \left(\lambda^2 - \alpha\lambda + 1 - a\mathrm{e}^{-\lambda\tau}\right)\left(\lambda^2 - \alpha\lambda + 1 + a\mathrm{e}^{-\lambda\tau}\right) = 0, \tag{4}$$

where $a = \sqrt{k_1 k_2}$, $k_1 k_2 > 0$. If $\lambda = 0$ is one root of equation (4), we obtain $a = \pm 1$. By above analysis, we know $a > 0$, so we get $a = 1$. If $\tau = 0$, then

$$H(\lambda) = \left(\lambda^2 - \alpha\lambda + 2\right)\left(\lambda^2 - \alpha\lambda\right) = 0.$$

We obtain that if $\tau = 0$, then $\alpha < 0$. Except a single zero eigenvalue, all the roots of equation (4) have negative real parts.

Next, we consider the case of $\tau \neq 0$. Let $\mathrm{i}\omega$, $\omega > 0$, be such a root of $\lambda^2 - \alpha\lambda + 1 + \mathrm{e}^{-\lambda\tau} = 0$, then the following holds:

$$(\mathrm{i}\omega)^2 - \alpha(\mathrm{i}\omega) + 1 - \mathrm{e}^{-\mathrm{i}\omega\tau} = 0.$$

Separating the real and imaginary parts, we have

$$\omega^2 - 1 = \cos\omega\tau, \qquad -\alpha\omega = \sin\omega\tau. \tag{5}$$

It follows that ω satisfies

$$\omega^2\left(\omega^2 + \alpha^2 - 2\right) = 0.$$

If $\alpha^2 - 2 > 0$, then equation (5) has no positive solutions. If $\alpha^2 - 2 < 0$, then equation (5) has a positive solution ω_0 with

$$\omega_0 = \sqrt{2 - \alpha^2}.$$

If $-\sqrt{2} < \alpha < 0$, then $\sin\omega_0\tau > 0$, $\cos\omega_0\tau > 0$, we know the point in the first quadrant, then

$$\tau_k = \frac{1}{\omega_0}\left\{\arccos\left(\omega_0^2 - 1\right) + 2k\pi\right\}, \quad k = 0, 1, 2, \dots .$$

We can obtain the following lemma immediately.

Lemma 1. *If* $a = 1$ *that means* $\sqrt{k_1 k_2} = 1$ *and* $-\sqrt{2} < \alpha < 0$ *hold, when* $\tau = \tau_k$ ($k = 0, 1, 2, \dots$)*, system* (2) *undergoes a Hopf-zero bifurcation at equilibrium* $(0, 0, 0, 0)$.

3 Normal form for Hopf-zero bifurcation

In this section, center manifold theory and normal form method [11, 30] are used to study Hopf-pitchfork bifurcation. After scaling $t \to t/\tau$, system (2) can be written as

$$\begin{aligned}
\dot{x}_1 &= \tau x_2,\\
\dot{x}_2 &= \tau\left(-x_1 + \alpha x_2 - x_1^2 x_2 - \beta x_1^3\right) + \tau k_1 g\left(x_3(t-1)\right),\\
\dot{x}_3 &= \tau x_4,\\
\dot{x}_4 &= \tau\left(-x_3 + \alpha x_4 - x_3^2 x_4 - \beta x_3^3\right) + \tau k_2 g\left(x_1(t-1)\right).
\end{aligned} \tag{6}$$

Let the Taylor expansion of g be

$$g(s) = g(0) + g'(0)s + \frac{1}{2}g''(0)s^2 + \frac{1}{6}g'''(0)s^3 + O\left(|s|^4\right),$$

where $g(0) = g''(0) = 0$, $g'(0) = 1$.

Let $\tau = \tau_0 + \mu_1$ and $k_1 = 1/k_2 + \mu_2$, μ_1 and μ_2 are bifurcation parameters and expand the function g, equation (6) becomes

$$\begin{aligned}
\dot{x}_1 &= (\tau_0 + \mu_1)x_2, \\
\dot{x}_2 &= (\tau_0 + \mu_1)\bigl(-x_1 + \alpha x_2 - x_1^2 x_2 - \beta x_1^3\bigr) \\
&\quad + (\tau_0 + \mu_1)\left(\frac{1}{k_2} + \mu_2\right)\left[x_3(t-1) + \frac{g'''(0)}{6}x_3^3(t-1)\right], \\
\dot{x}_3 &= (\tau_0 + \mu_1)x_4, \\
\dot{x}_4 &= (\tau_0 + \mu_1)\bigl(-x_3 + \alpha x_4 - x_3^2 x_4 - \beta x_3^3\bigr) \\
&\quad + (\tau_0 + \mu_1)k_2\left[x_1(t-1) + \frac{g'''(0)}{6}x_1^3(t-1)\right].
\end{aligned} \tag{7}$$

Choosing the phase space $C = C([-1,0];\mathbb{R}^4)$ with supreme norm, $X_t \in C$ is defined by $X_t(\theta) = X(t+\theta)$, $-\tau \leqslant \theta \leqslant 0$, and $\|X_t\| = \sup|X_t(\theta)|$. Then system (7) becomes

$$\dot{X}(t) = L(\mu)X_t + F(X_t, \mu), \tag{8}$$

where

$$L(\mu)X_t = (\tau_0 + \mu_1)\begin{pmatrix} x_2(t) \\ -x_1(t) + \alpha x_2(t) + (\frac{1}{k_2} + \mu_2)x_3(t-1) \\ x_4(t) \\ -x_3(t) + \alpha x_4 + k_2 x_1(t-1) \end{pmatrix}$$

and

$$F(X_t, \mu) = \begin{pmatrix} 0 \\ (\tau_0 + \mu_1)(-x_1^2(t)x_2(t) - \beta x_1^3(t) + (\frac{1}{k_2} + \mu_2)\frac{g'''(0)}{6}x_3^3(t-1)) \\ 0 \\ (\tau_0 + \mu_1)(-x_3^2(t)x_4(t) - \beta x_3^3(t) + k_2\frac{g'''(0)}{6}x_1^3(t-1)) \end{pmatrix},$$

where $L(\mu)\varphi = \int_{-1}^{0} \mathrm{d}\eta(\theta,\mu)\varphi(\xi)\,\mathrm{d}\xi$ for $\varphi \in ([-1,0],\mathbb{R}^4)$,

$$\eta(\theta,\mu) = \begin{cases} 0, & \theta = 0, \\ -(\tau_0 + \mu_1)A, & \theta \in (-1,0), \\ -(\tau_0 + \mu_1)(A + B), & \theta = -1, \end{cases}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & \alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{k_2} & 0 \\ 0 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the following linear system:

$$\dot{X}(t) = L(0)X_t.$$

Define the bilinear form between C and $C' = C([0,\tau], C^{n*})$ by

$$\big(\psi(s), \varphi(\theta)\big) = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi-\theta)\,\mathrm{d}\eta(\theta,0)\varphi(\xi)\,\mathrm{d}\xi \quad \forall\psi\in C',\ \forall\varphi\in C,$$

where $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta)) \in C$, $\psi(s) = (\psi_1(s), \psi_2(s), \psi_3(s))^{\mathrm{T}} \in C^*$.

Because $L(0)$ has a simple 0 and a pair of purely imaginary eigenvalues $\pm\mathrm{i}\omega_0$, $\omega > 0$, all other eigenvalues have negative real parts. Let $\Lambda = \{0, \mathrm{i}\omega_0, -\mathrm{i}\omega_0\}$, P can be the generalized eigenspace associated with Λ, and P^* – the space adjoint with P. Then C can be decomposed as $C = P \oplus Q$, where $Q = \{\varphi \in C\colon (\psi,\varphi) = 0 \text{ for all } \psi \in P^*\}$. Choose the bases Φ and Ψ for P and P^* such that $(\Psi(s), \Phi(\theta)) = I$, $\dot{\Phi} = \Phi J$, and $-\Psi = J\Psi$, where $J = \operatorname{diag}(0, \mathrm{i}\omega_0, -\mathrm{i}\omega_0)$.

By calculating, we choose

$$\Phi(\theta) = \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}w_0\tau_0\theta} & \mathrm{e}^{-\mathrm{i}w_0\tau_0\theta} \\ 0 & \mathrm{i}w_0\mathrm{e}^{\mathrm{i}w_0\tau_0\theta} & -\mathrm{i}w_0\mathrm{e}^{-\mathrm{i}w_0\tau_0\theta} \\ k_2 & -k_2\mathrm{e}^{\mathrm{i}w_0\tau_0\theta} & -k_2\mathrm{e}^{-\mathrm{i}w_0\tau_0\theta} \\ 0 & -\mathrm{i}w_0k_2\mathrm{e}^{\mathrm{i}w_0\tau_0\theta} & \mathrm{i}w_0k_2\mathrm{e}^{-\mathrm{i}w_0\tau_0\theta} \end{pmatrix}$$

and

$$\Psi(s) = \begin{pmatrix} D_1(-\alpha k_2) & D_1k_2 & D_1(-\alpha) & D_1 \\ D_2(\alpha-\mathrm{i}w_0)k_2\mathrm{e}^{-\mathrm{i}w_0\tau_0 s} & -D_2k_2\mathrm{e}^{-\mathrm{i}w_0\tau_0 s} & D_2(\mathrm{i}w_0-\alpha)\mathrm{e}^{-\mathrm{i}w_0\tau_0 s} & D_2\mathrm{e}^{-\mathrm{i}w_0\tau_0 s} \\ \overline{D}_2(\alpha+\mathrm{i}w_0)k_2\mathrm{e}^{\mathrm{i}w_0\tau_0 s} & -\overline{D}_2k_2\mathrm{e}^{\mathrm{i}w_0\tau_0 s} & \overline{D}_2(-\mathrm{i}w_0-\alpha)\mathrm{e}^{\mathrm{i}w_0\tau_0 s} & \overline{D}_2\mathrm{e}^{\mathrm{i}w_0\tau_0 s} \end{pmatrix},$$

where

$$D_1 = \frac{1}{2\tau_0k_2 - 2\alpha k_2}, \qquad D_2 = \frac{1}{2\alpha k_2 - 4\mathrm{i}wk_2 + 2\tau_0k_2\mathrm{e}^{-\mathrm{i}w_0\tau_0}}.$$

To consider system (8), we need to enlarge the space C to the following:

$$BC = \Big\{\varphi \text{ is continuous functions on } [-1,0),\ \text{and } \lim_{\theta\to0^-}\varphi(\theta) \text{ exists}\Big\}.$$

Its elements can be written as $\phi = \varphi + Y_0c$ with $\varphi \in C$, $c \in \mathbb{R}^4$, and

$$Y_0(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ I, & \theta = 0. \end{cases}$$

In BC, equation (8) becomes an abstract ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}X_t = Au + Y_0\tilde{F}(u,\mu), \tag{9}$$

where $u \in C$, A is defined by

$$A : C^1 \to BC, \quad Au = \dot{u} + Y_0\big[L(0)u - \dot{u}(0)\big],$$

and

$$\tilde{F}(u,\mu) = \big[L(\mu) - L_0\big]u + F(u,\mu).$$

Then the enlarged phase space BC can be decomposed as $BC = P \oplus \operatorname{Ker}\pi$. Let $X_t = \Phi z(t) + \tilde{y}(\theta)$, where $z(t) = (z_1, z_2, z_3)^{\mathrm{T}}$, namely

$$\begin{aligned}
x_1(\theta) &= z_1 + \mathrm{e}^{\mathrm{i}w_0\tau_0\theta} z_2 + \mathrm{e}^{-\mathrm{i}w_0\tau_0\theta} z_3 + y_1(\theta),\\
x_2(\theta) &= \mathrm{i}w\mathrm{e}^{\mathrm{i}w_0\tau_0\theta} z_2 - \mathrm{i}w\mathrm{e}^{-\mathrm{i}w_0\tau_0\theta} z_3 + y_2(\theta),\\
x_3(\theta) &= k_2 z_1 - k_2\mathrm{e}^{\mathrm{i}w_0\tau_0\theta} z_2 - k_2\mathrm{e}^{-\mathrm{i}w_0\tau_0\theta} z_3 + y_3(\theta),\\
x_4(\theta) &= -\mathrm{i}wk_2\mathrm{e}^{\mathrm{i}w_0\tau_0\theta} z_2 +^{\mathrm{i}} wk_2\mathrm{e}^{-\mathrm{i}w_0\tau_0\theta} z_3 + y_4(\theta).
\end{aligned}$$

Let

$$\begin{aligned}
\Psi(0) &= \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14}\\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24}\\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \end{pmatrix}\\
&= \begin{pmatrix} D_1(-\alpha k_2) & D_1k_2 & D_1(-\alpha) & D_1\\ D_2(\alpha - \mathrm{i}w_0)k_2 & -D_2k_2 & D_2(\mathrm{i}w_0 - \alpha) & D_2\\ \overline{D}_2(\alpha + \mathrm{i}w_0)k_2 & -\overline{D}_2k_2 & \overline{D}_2(-\mathrm{i}w_0 - \alpha) & \overline{D}_2 \end{pmatrix}.
\end{aligned}$$

System (9) can be decomposed as

$$\begin{aligned}
\dot{z} &= Jz + \Psi(0)\tilde{F}\big(\Phi z + \tilde{y}(\theta), \mu\big),\\
\dot{\tilde{y}} &= A_{Q1}\tilde{y} + (I - \pi)Y_0\tilde{F}\big(\Phi z + \tilde{y}(0), \mu\big),
\end{aligned} \tag{10}$$

where $\tilde{y}(\theta) \in Q^1 := Q \cap C^1 \subset \operatorname{Ker}\pi$, A_{Q1} is the restriction of A as an operator from Q_1 to the Banach space $\operatorname{Ker}\pi$. Neglecting higher-order terms with respect to parameters μ_1 and μ_2, equation (11) can be written as

$$\begin{pmatrix} \dot{z}_1\\ \dot{z}_2\\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14}\\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24}\\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \end{pmatrix} \begin{pmatrix} F_2^1 + F_3^1 + O(\|x\|^4)\\ F_2^2 + F_3^2 + O(\|x\|^4)\\ F_2^3 + F_3^3 + O(\|x\|^4)\\ F_2^4 + F_3^4 + O(\|x\|^4) \end{pmatrix},$$

where

$$\begin{aligned}
F_2^1 &= \mu_1\big(\mathrm{i}\omega z_2 - \mathrm{i}\omega z_3 + y_2(0)\big),\\
F_2^2 &= -\mu_1\big(z_1 + z_2 + z_3 + y_1(0)\big) + \alpha\mu_1\big(\mathrm{i}\omega z_2 - \mathrm{i}\omega z_3 + y_2(0)\big)\\
&\quad + \left(\tau_0\mu_2 + \mu_1\mu_2 + \mu_1\left(\frac{1}{k_2}\right)\right)\big(k_2z_1 - k_2\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} z_2 - k_2\mathrm{e}^{\mathrm{i}\omega_0\tau_0} z_3 + y_3(-1)\big),\\
F_2^3 &= \mu_1\big(-\mathrm{i}\omega k_2 z_2 + \mathrm{i}\omega k_2 z_3 + y_4(0)\big),\\
F_2^4 &= -\mu_1\big(k_2z_1 - k_2z_2 - k_2z_3 + y_3(0)\big) + \alpha\mu_1\big(-\mathrm{i}\omega k_2z_2 + \mathrm{i}\omega k_2 z_3 + y_4(0)\big)\\
&\quad + \mu_1k_2\big(z_1 + \mathrm{e}^{-\mathrm{i}\tau_0\omega_0} z_2 + \mathrm{e}^{\mathrm{i}\tau_0\omega_0} z_3 + y_1(-1)\big),
\end{aligned}$$

$$
\begin{aligned}
F_3^1 &= 0,\\
F_3^2 &= (\tau_0+\mu_1)\bigl(-\bigl(z_1+z_2+z_3+y_1(0)\bigr)^2\bigl(\mathrm{i}\omega z_2-\mathrm{i}\omega z_3+y_2(0)\bigr)\\
&\quad -\beta\bigl(z_1+z_2+z_3+y_1(0)\bigr)^3,\\
&\quad +\left(\frac{1}{k_2}+\mu_2\right)\frac{g'''(0)}{6}\bigl(k_2z_1-k_2\mathrm{e}^{-\mathrm{i}\omega_0\tau_0}z_2-k_2\mathrm{e}^{\mathrm{i}\omega_0\tau_0}z_3+y_3(-1)^3\bigr),\\
F_3^3 &= 0,\\
F_3^4 &= (\tau_0+\mu_1)\bigl(-\bigl(k_2z_1-k_2z_2-k_2z_3+y_3(0)\bigr)^2\bigl(-\mathrm{i}\omega k_2z_2+\mathrm{i}\omega k_2z_3+y_4(0)\bigr)\\
&\quad -\beta\bigl(k_2z_1-k_2z_2-k_2z_3+y_3(0)\bigr)^3\\
&\quad +k_2\frac{g'''(0)}{6}\bigl(z_1+\mathrm{e}^{-\mathrm{i}\omega_0\tau_0}z_2+\mathrm{e}^{\mathrm{i}\omega_0\tau_0}z_3+y_1(-1)^3\bigr).
\end{aligned}
$$

According to [30], $(\mathrm{Im}(M_2^1))^c$ is spanned by

$$\bigl\{z_1^2e_1,\ z_2z_3e_1,\ z_1\mu_\mathrm{i}e_1,\ \mu_1\mu_2e_1,\ z_1z_2e_2,\ z_2\mu_\mathrm{i}e_2,\ z_1z_3e_3,\ z_3\mu_\mathrm{i}e_3\bigr\},\quad i=1,2,$$

with $e_1=(1,0,0)^\mathrm{T}$, $e_2=(0,1,0)^\mathrm{T}$, $e_3=(0,0,1)^\mathrm{T}$.

If g is an odd function, the Hopf-pitchfork will occur.

$(\mathrm{Im}(M_3^1))^c$ is spanned by

$$\bigl\{z_1^3e_1,\ z_1z_2z_3e_1,\ z_1^2z_2e_2,\ z_2^2z_3e_2,\ z_1^2z_3e_3,\ z_2z_3^2e_3\bigr\}.$$

Then we get

$$
\begin{aligned}
g_2^1(x,0,\mu) &= \mathrm{Proj}_{(\mathrm{Im}(M_2^1))^c}\, f_2^1(x,0,\mu)=\mathrm{Proj}_{S_1}\, f_2^1(x,0,\mu)+O\bigl(|\mu|^2\bigr),\\
g_3^1(x,0,\mu) &= \mathrm{Proj}_{(\mathrm{Im}(M_3^1))^c}\, \tilde f_3^1(x,0,\mu)=\mathrm{Proj}_{S_1}\, \tilde f_3^1(x,0,0)+O\bigl(|\mu|^2|x|+|\mu||x|^2\bigr),
\end{aligned}
$$

where S_1 and S_2 are spanned, respectively, by

$$\bigl\{z_1\mu_\mathrm{i}e_1,\ z_2\mu_\mathrm{i}e_2,\ z_3\mu_\mathrm{i}e_3\bigr\},\quad i=1,2,$$

and

$$\bigl\{z_1^3e_1,\ z_1z_2z_3e_1,\ z_1^2z_2e_2,\ z_2^2z_3e_2,\ z_1^2z_3e_3,\ z_2z_3^2e_3\bigr\}.$$

On the center manifold, (8) can be transform as the following normal form:

$$\dot z = Jz+\frac{1}{2!}g_2^1(z,0,\mu)+\frac{1}{3!}g_3^1(z,0,0)+\text{h.o.t.}$$

with $g_3^1(z,0,0)=\mathrm{Proj}_{(\mathrm{Im}(M_3^1))^c}\, f_3^1(z,0,0)$. According to [11, Thm. 2.1], we obtain the dynamical behavior of (8) near $X_t=0$, which is governed by the general normal form of the third order. Then the equation becomes

$$
\begin{aligned}
\dot z_1 &= b_{11}\mu_1z_1+b_{12}\mu_2z_1+c_{11}z_1^3+c_{12}z_1z_2z_3+\text{h.o.t.},\\
\dot z_2 &= \mathrm{i}\tau_0\omega_0z_2+b_{21}\mu_1z_2+b_{22}\mu_2z_2+c_{21}z_1^2z_2+c_{22}z_2^2z_3+\text{h.o.t.},\\
\dot z_3 &= -\mathrm{i}\tau_0\omega_0z_3+\overline{b}_{21}\mu_1z_3+\overline{b}_{22}\mu_2z_3+\overline{c}_{21}z_1^2z_3+\overline{c}_{22}z_2z_3^2+\text{h.o.t.},
\end{aligned}
\tag{11}
$$

where

$$\begin{aligned}
b_{11} &= 0, \qquad b_{12} = D_1\tau_0 k_2^2,\\
c_{11} &= D_1\tau_0 \frac{(g'''(0)-6\beta)(k_2^3+k_2)}{6},\\
c_{12} &= D_1\tau_0\big(g'''(0)-6\beta\big)\big(k_2^3+k_2\big),\\
b_{21} &= 2D_2k_2\big(\omega^2+\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}+1\big),\\
b_{22} &= D_2\tau_0k_2^2\mathrm{e}^{-\mathrm{i}\tau_0\omega_0},\\
c_{21} &= D_2\tau_0k_2\left[\big(1+k_2^2\big)\left(\mathrm{i}\omega+3\beta+\frac{g'''(0)}{2}\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}\right)\right],\\
c_{22} &= D_2\tau_0k_2\left[\big(1+k_2^2\big)\left(\mathrm{i}\omega+3\beta+\frac{g'''(0)}{2}\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}\right)\right].
\end{aligned}$$

Through the change of variables $z_1 = \omega_1$, $z_2 = \omega_2 + \mathrm{i}\omega_3$, $z_3 = \omega_2 - \mathrm{i}\omega_3$ and then a change to cylindrical coordinates according to $\omega_1 = \zeta$, $\omega_2 = r\cos\theta$, $\omega_3 = r\sin\theta$, $r > 0$, system (11) becomes

$$\begin{aligned}
\dot r &= \mathrm{Re}(b_{21})\mu_1 r + \mathrm{Re}(b_{22})\mu_2 r + \mathrm{Re}(c_{21}) r\zeta^2 + \mathrm{Re}(c_{22}) r^3,\\
\dot\zeta &= b_{12}\mu_2\zeta + c_{11}\zeta^3 + c_{12}\zeta r^2,\\
\dot\theta &= \tau_0\omega_0 + \mu_1\,\mathrm{Im}(b_{21}) + \mu_2\,\mathrm{Im}(b_{22}) + \mathrm{Im}(c_{21})\zeta^2 + \mathrm{Im}(c_{22}) r^2.
\end{aligned} \tag{12}$$

Let $\hat\zeta = \zeta\sqrt{|c_{11}|}$ and $\hat r = r\sqrt{|Re(c_{22})|}$, after dropping the hats, equation (12) can be written as

$$\begin{aligned}
\dot r &= r\left(c_1 + \frac{\mathrm{Re}(c_{22})}{|\,\mathrm{Re}(c_{22})|} r^2 + \frac{\mathrm{Re}(c_{21})}{|c_{21}|}\zeta^2\right),\\
\dot\zeta &= \zeta\left(c_2 + \frac{c_{12}}{|\,\mathrm{Re}(c_{22})|} r^2 + \frac{c_{11}}{|c_{11}|}\zeta^2\right),
\end{aligned} \tag{13}$$

where $c_1 = \mathrm{Re}(b_{21})\mu_1 + \mathrm{Re}(b_{22})\mu_2$, $c_2 = b_{12}\mu_2$.

If $c_{11} < 0$ and $\mathrm{Re}(c_{22}) < 0$, then (13) becomes

$$\dot r = r\big(c_1 - r^2 - \sigma\zeta^2\big), \qquad \dot\zeta = \zeta\big(c_2 - \delta r^2 - \zeta^2\big), \tag{14}$$

where $\sigma = \mathrm{Re}(c_{21})/c_{11}$, $\delta = c_{12}/\mathrm{Re}(c_{22})$.

In equation (14), $M_0 = (r,\zeta) = (0,0)$ is always an equilibrium, and the other equilibria are

$$M_1 = (\sqrt{c_1}, 0) \quad \text{for } c_1 > 0, \qquad M_2^{\pm} = (0, \pm\sqrt{c_2}) \quad \text{for } c_2 > 0,$$

$$M_3^{\pm} = \left(\sqrt{\frac{\sigma c_2 - c_1}{\sigma\delta - 1}}, \pm\sqrt{\frac{\delta c_1 - c_2}{\sigma\delta - 1}}\right) \quad \text{for } \frac{\sigma c_2 - c_1}{1-\sigma\delta} > 0,\ \frac{\delta c_1 - c_2}{1-\sigma\delta} > 0.$$

Table 1. The five unfoldings of system (14) as $\sigma \geqslant \delta$.

Case	I	II	III	IV	V
σ	$+$	$+$	$+$	$-$	$-$
δ	$+$	$+$	$-$	$-$	$-$
$\sigma\delta - 1$	$+$	$-$		$-$	$+$

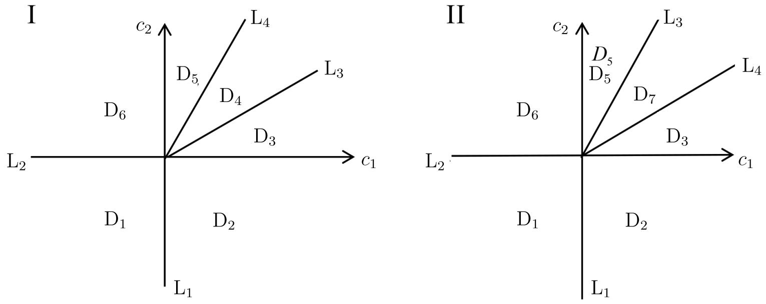

Figure 1. Bifurcation diagrams for system (14) with parameter (c_1, c_2) around $(0,0)$ (see [31].)

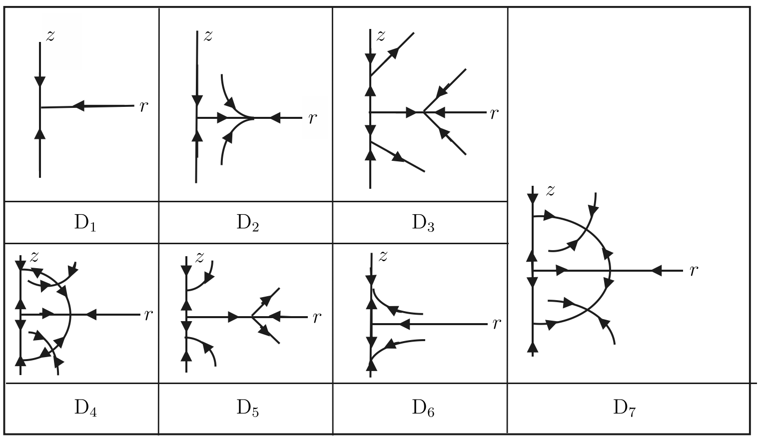

Figure 2. Phase portraits in D_1–D_7 (see [31]).

From [21] we obtain that if $c_{11} < 0$ and $\mathrm{Re}(c_{22}) < 0$, σ, δ, $\sigma - \delta$, and $\sigma\delta - 1$ are five distinct types of unfolding with respect to different signs in system (14), we demonstrate this in [21, Sect. 8.6.2] corresponding to Table 1.

From [31] we can obtain

Theorem 1. *If the assumptions of Lemma* 1 *are satisfied,* $\sigma \geqslant \delta$, $\sigma\delta > 1$, *and* $\mathrm{sign}(g'''(0))\,\mathrm{sign}(\beta) < 0$ *hold, then system* (2) *undergoes a Hopf-pitchfork bifurcation of case* I *at equilibrium* $(0,0,0,0)$, *which is shown in Fig.* 2, *where* σ, δ *are expressed as* (14).

Since system (12) rotates around the Z-axis, the correspondences between two-dimensional flows for (14) and three-dimensional flows (12) can be established. Thus, for (12), the equilibria on the Z-axis in (14) remain equilibria, while the equilibria outside the Z-axis in (14) become periodic orbits.

Therefore, if case I occurs, then the detailed dynamics of system (2) in D_1–D_6 near the original parameters $(1/k_2, \tau_0)$ are as follows:

- In D_1, (2) has only one trivial equilibrium M_0, which is a sink.
- In D_2, the trivial equilibrium (corresponding to M_0) becomes a saddle from a sink, and a stable periodic orbit (corresponding to M_1) appears.
- In D_3, the trivial equilibrium (corresponding to M_0) becomes a source from a saddle, a pair of unstable semitrivial equilibria (corresponding to $M_2^{\pm}$) appear, and the periodic orbit (corresponding to M_1) remains stable.
- In D_4, the semitrivial equilibria (corresponding to $M_2^{\pm}$) become stable from its unstable state, a pair of unstable periodic orbits (corresponding to $M_3^{\pm}$) appear, and the periodic orbit (corresponding to M_1) remains stable.
- In D_5, the unstable periodic orbit (corresponding to $M_3^{\pm}$) disappear, the periodic orbit (corresponding to M_1) becomes unstable, and the semitrivial equilibria (corresponding to $M_2^{\pm}$) remains stable.
- In D_6, the periodic orbit (corresponding to M_1) disappears, the trivial equilibrium (corresponding to M_0) becomes a saddle from a source, and the semitrivial equilibria (corresponding to $M_2^{\pm}$) remains stable.

Theorem 2. *If the assumptions of Lemma* 1 *are satisfied,* $\sigma \geqslant \delta$*,* $\sigma\delta < 1$*, and* $\operatorname{sign}(f'''(0))\operatorname{sign}(\beta) < 0$ *hold, then system* (2) *undergoes a Hopf-pitchfork bifurcation of case* II *at equilibrium* $(0,0,0,0)$*, which is shown in Fig.* 2*, where* σ, δ *are expressed as* (14).

Noticing that if case II arises, then the detailed dynamics of system (2) in D1, D2, D3, D5, and D6 are the same as that in case I, except in D7. In D7, system (2) has a pair of stable periodic orbits (corresponding to $M_3^{\pm}$), a pair of unstable semitrivial equilibria (corresponding to $M_2^{\pm}$), an unstable periodic orbit (corresponding to M_1), and an unstable trivial equilibrium (corresponding to M_0).

By analysing above, we can obtain the bifurcation critical lines as follows:

$$\mathrm{L}_1\colon\ \tau = \frac{\mathrm{Re}(b_{22})}{\mathrm{Re}(b_{21})}\left(k_1 - \frac{1}{k_2}\right) + \tau_0$$

corresponding to

$$\mu_1 = \frac{\mathrm{Re}(b_{22})}{\mathrm{Re}(b_{21})}\mu_2;$$

$$\mathrm{L}_2\colon\ k_1 = \frac{1}{k_2}$$

corresponding to

$$\mu_2 = 0;$$

$$\mathrm{L}_3\colon\ \tau = \left[\frac{\mathrm{Re}(c_{21})b_{12}}{c_{11}\,\mathrm{Re}(b_{21})} - \frac{\mathrm{Re}(b_{22})}{\mathrm{Re}(b_{21})}\right]\left(k_1 - \frac{1}{k_2}\right) + \tau_0$$

corresponding to

$$\mu_1 = \left[\frac{\sigma b_{12}}{\mathrm{Re}(b_{21})} - \frac{\mathrm{Re}(b_{22})}{\mathrm{Re}(b_{21})}\right]\mu_2;$$

$$\mathrm{L}_4\colon\ \tau = \frac{\mathrm{Re}(c_{22})b_{12} + c_{12}\,\mathrm{Re}(b_{22})}{c_{12}\,\mathrm{Re}(b_{21})}\left(k_1 - \frac{1}{k_2}\right) + \tau_0$$

corresponding to

$$\mu_1 = \frac{b_{12} + \delta\mathrm{Re}(b_{22})}{\delta\mathrm{Re}(b_{21})}\mu_2.$$

4 Numerical simulations

In this section, some examples are given to illustrate the theoretical results. We select $\alpha = -1.3$ and $g(t) = \tanh(t)$ into system (2). From Theorem 1, if $k_1 = 1/k_2 = 2$ and $\tau = \tau_0 = 4.1887$, then system (2) undergoes a Hopf-pitchfork bifurcation at $(0, 0)$. According to the calculation, we obtain $\omega = 0.5568$, $\sin\omega\tau = 0.5285$, $\cos\omega\tau = 0.8490$, $D_1 = 0.1822$, $D_2 = -0.1206 + 0.1193\mathrm{i}$, $g'''(0) = -2$, $b_{11} = 0$, $b_{12} = 0.1908$, $c_{11} = -0.6358$, $c_{12} = -3.8150$, $b_{21} = 0.0116 + 0.1613\mathrm{i}$, $b_{22} = 0.1776 + 0.0052\mathrm{i}$, $c_{21} = -1.5650 + 0.7481\mathrm{i}$, $c_{22} = -1.5650 + 0.7481\mathrm{i}$, $\mathrm{Re}(c_{22}) < 0$, $c_{11} < 0$, $c_1 = 0.0116\mu_1 + 0.1776\mu_2$, $c_2 = 0.1908\mu_2$, $\sigma = 2.4615 > 0$, $\delta = 2.4377 > 0$, $\sigma\delta = 6.0004 > 1$. Here, in Fig. 1, bifurcation critical lines are, respectively,

$$\mathrm{L}_1\colon\ \tau = 15.3101(k_1 - 2) + 4.1887, \quad \text{i.e. } \mu_1 = 15.310\mu_2;$$
$$\mathrm{L}_2\colon\ k_1 = 2, \quad \text{i.e. } \mu_2 = 0;$$
$$\mathrm{L}_3\colon\ \tau = 40.4869(k_1 - 2) + 4.1887, k_1 > 2, \quad \text{i.e. } \mu_1 = 40.4869\mu_2,\ \mu_2 > 0;$$
$$\mathrm{L}_4\colon\ \tau = 22.0578(k_1 - 2) + 4.1887, k_1 > 2, \quad \text{i.e. } \mu_1 = 22.0578\mu_2,\ \mu_2 > 0.$$

Through the above analysis, we can obtain Figs. 3–7.

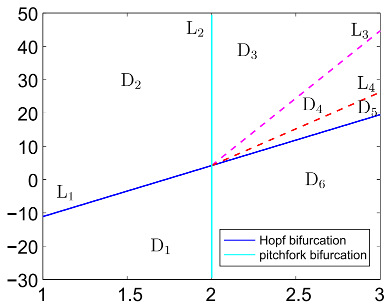

Figure 3. The bifurcation set.

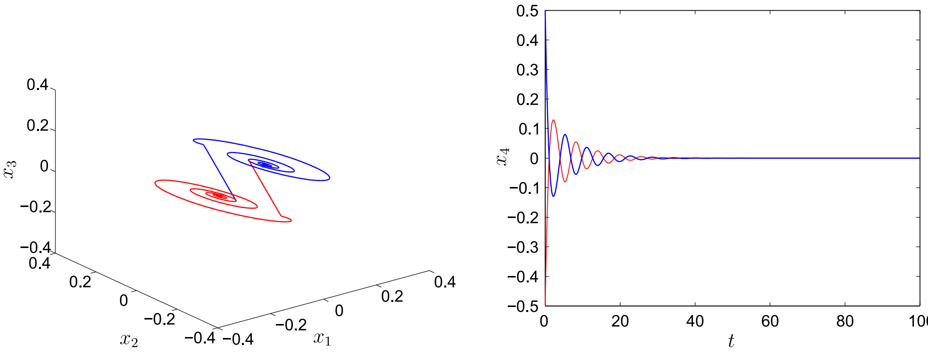

Figure 4. The stable trivial equilibrium in D_1: $(\mu_1, \mu_2) = (-1.87, -1.913)$, using the red line is $(0.1, 0.1, 0.02, -0.5)$ and the blue line is $(-0.1, -0.1, -0.02, 0.5)$. Phase diagram for variable (x_1, x_2, x_3) in left. Waveform diagram for variable of x_4 in right. (Online version in color.)

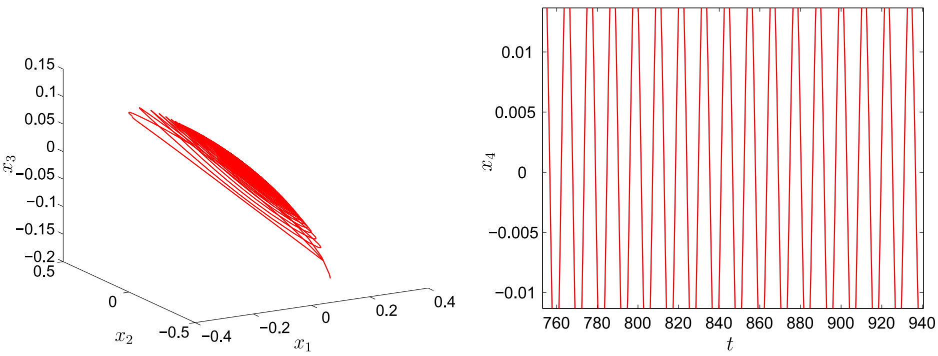

Figure 5. The stable periodic orbit in D_2: $(\mu_1, \mu_2) = (-0.01, 0.01)$, the initial value is $(0.2, -0.2, -0.2, 0.2)$. Phase diagram for variable (x_1, x_2, x_3) in left. Waveform diagram for variable of x_4 in right.

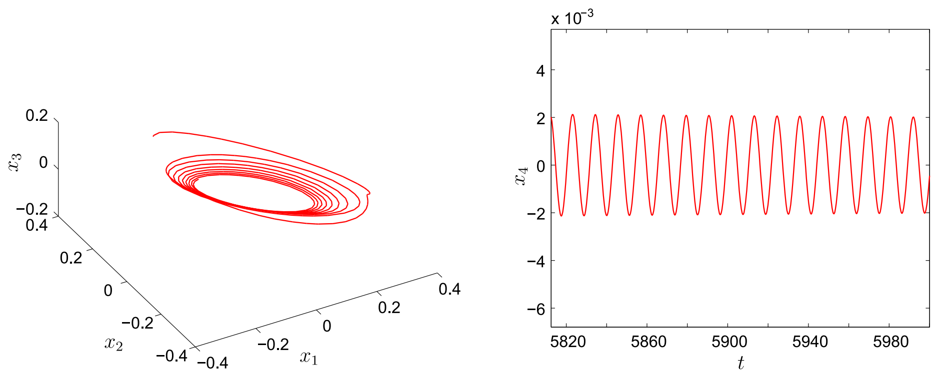

Figure 6. The stable periodic orbit in D3: $(\mu_1, \mu_2) = (-0.02, 0.01)$, the initial value is $(-0.2, 0.2, 0.2, -0.2)$. Phase diagram for variable (x_1, x_2, x_3) in left. Waveform diagram for variable of x_4 in right.

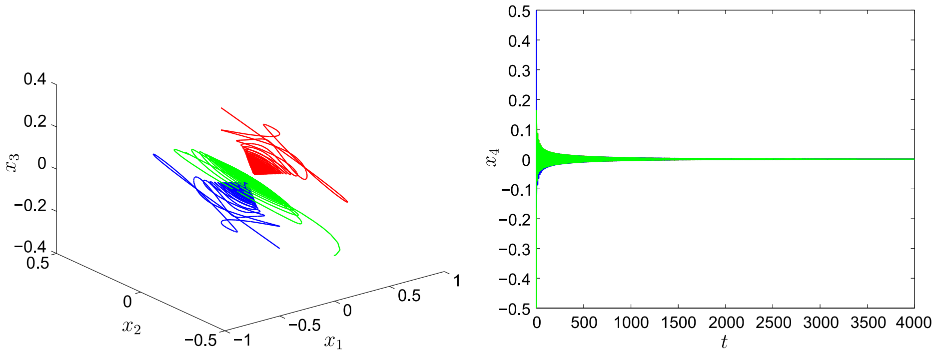

Figure 7. Two stable nontrivial equilibria and a stable periodic orbit coexist in D_4: $(\mu_1, \mu_2) = (0.01, -0.05)$. The red line expresses the initial values of $(0.5, 0.5, 0.08, -0.5)$, the blue line expresses the value of $(-0.5, -0.5, -0.08, 0.5)$ and $(0.3, -0.3, -0.3, 0.3)$ for magenta line. Phase diagram for variable (x_1, x_2, x_3) in left. Waveform diagram for variable of x_4 in right. (Online version in color.)

5 Conclusions

In this paper, we have investigated the Hopf-pitchfork bifurcation of coupled van der Pol oscillator with delay. Our contributions include the following:

1. By analyzing the distribution of the eigenvalues of the corresponding characteristic equation of its linearized equation, we find the conditions for the occurrence of Hopf-pitchfork bifurcation.
2. By using the normal form method and the center manifold theorem, we have derived the normal form of the reduced system on the center manifold, discussed the Hopf-pitchfork bifurcation with the parameter in system (2), and analyzed the stability. Furthermore, we can obtain the coexistence of periodic orbits.
3. By comparing system (2) in this paper with non-coupled van der Pol system in [20], we obtain the coexistence of a pair of stable nontrivial equilibria and the coexistence of a stable periodic orbit and a pair of stable nontrivial equilibria. We know that the periodic orbit is stable, corresponding to the periodic spiking behavior.

Our work is a further study of van der Pol oscillator, which is helpful in the study of the complex phenomenon caused by high co-dimensional bifurcation of delay differential equation.

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