Nonlinear Analysis: Modelling and Control, Vol. 22, No. 6, 861–876 ISSN 1392-5113
https://doi.org/10.15388/NA.2017.6.9

Lie symmetry analysis, exact solutions and conservation laws for the time fractional modified Zakharov–Kuznetsov equation

Dumitru Baleanu[a,b], **Mustafa Inc**[c], **Abdullahi Yusuf**[c,d], **Aliyu Isa Aliyu**[c,d]

[a]Department of Mathematics, Cankaya University,
Öğretmenler Cad. 1406530, Ankara, Turkey
dumitru@cankaya.edu.tr

[b]Institute of Space Sciences,
Magurele, Bucharest, Romania

[c]Science Faculty, Firat University,
23119, Elaziğ, Turkey
minc@firat.edu.tr

[d]Science Faculty, Federal University Dutse,
7156, Jigawa, Nigeria
yusufabdullahi@fud.edu.ng; aliyu.isa@fud.edu.ng

Received: July 20, 2017 / **Revised:** September 8, 2017 / **Published online:** October 27, 2017

Abstract. In this work, Lie symmetry analysis (LSA) for the time fractional modified Zakharov–Kuznetsov (mZK) equation with Riemann–Liouville (RL) derivative is analyzed. We transform the time fractional mZK equation to nonlinear ordinary differential equation (ODE) of fractional order using its point symmetries with a new dependent variable. In the reduced equation, the derivative is in Erdelyi–Kober (EK) sense. We obtained exact traveling wave solutions by using fractional $D_\xi^\alpha G/G$-expansion method. Using Ibragimov's nonlocal conservation method to time fractional nonlinear partial differential equations (FNPDEs), we compute conservation laws (CLs) for the mZK equation.

Keywords: modified Zakharov–Kuznetsov equation, Lie symmetry, Riemann–Liouville fractional derivative, exact solutions, conservation laws.

1 Introduction

LSA is one of the most efficient method for investigating the exact solutions of nonlinear partial differential equations (NLPDEs) arising in mathematics, physics and many other fields of science and engineering. It is well known that there is no general method for solving NLPDEs, however, LSA is one of the more powerful method for reaching new exact and explicit solutions for NLPDEs [9, 11, 28, 31–33, 42, 44, 45, 55, 56, 60].

Moreover, fractional calculus has been applied to analyze many complex nonlinear physical phenomena and dynamic processes in physics, engineering, electromagnetics, viscoelasticity, electrochemistry [3, 12, 23, 26, 35, 36, 38, 43, 46, 47, 54, 57, 65]. As far as we know, the study of symmetry analysis for FNPDEs and their group properties are very few in the literature [10, 13, 24, 25].

Furthermore, CLs possess an important role in the analysis of NLPDEs from physical viewpoint [55]. If the considered system has CLs, then its integrability will be possible [29]. More detail about CLs and their construction can be found in [2, 30, 37, 53].

Time fractional NLPDEs comes from classical NLPDEs by replacing its time derivative with fractional derivative. In the present work, we study Lie symmetry analysis, exact traveling wave solutions using fractional $D_\xi^\alpha G/G$-expansion method and Ibragimov's nonlocal CLs [30] of the time fractional mZK equation given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u^2 u_x + u_{xxx} + u_{xyy} = 0, \tag{1}$$

where $0 < \alpha \leqslant 1$, and α is the order of the fractional time derivative, while $u(x, y, t)$ implies the electrostatic wave potential in plasmas, that is, a function of the spatial variables x, y and the temporal variable t.

The term $\partial^\alpha u/\partial t^\alpha$ in (1) is the fractional temporal evolution (FTE) term. FTE is very significant since it yields a history of the problem, and also it gets rid of the slowness of evolution. For instance, when the order of the fractional derivative parameter can be controlled externally, the evolution of the soliton can be manifested artificially [4, 16, 17, 21, 48–51]. This important feature is being globally applied in several areas of physics and engineering. One immediate example is the temporal evolution of solitons in optical fibers, which can be slowed to address Internet bottleneck that is a growing problem in Internet industry [6, 14, 19, 20, 22, 39].

If $\alpha = 1$, (1) reduces to the classical mZK equation. The ZK and mZK equations determine the behaviour of weakly nonlinear ion-acoustic waves in incorporated hot isothermal electrons and cold ions in the presence of the uniform magnetic field [1, 5, 7, 8, 15, 34, 40, 41, 52, 58, 59, 64]. The mZK equation interprets an anisotropic two-dimensional generalization of the KdV equation and can be analysed in magnetized plasma for a tiny amplitude Alfven wave at a critical angle to the uninterrupted magnetic field [1, 5, 7, 8, 15, 34, 40, 41, 52, 58, 59, 64].

2 Preliminaries

Basic definitions and theorems for the RL fractional derivative can be found in [61, 66]. Consider a FNPDEs of the form

$$\partial_t^\alpha u = F(t, x, y, u, u_x, u_{xx}, \ldots) \quad (0 < \alpha < 1). \tag{2}$$

Given a one-parameter Lie group of infinitesimal transformations are as follows:

$$\begin{aligned} &\tilde{t} = t + \epsilon \xi^2(x, t, y, u) + O(\epsilon^2), \qquad \tilde{x} = x + \epsilon \xi^1(x, t, y, u) + O(\epsilon^2), \\ &\tilde{y} = y + \epsilon \xi^3(x, t, y, u) + O(\epsilon^2), \qquad \tilde{u} = u + \epsilon \eta(x, t, y, u) + O(\epsilon^2), \end{aligned} \tag{3a}$$

$$\begin{aligned}\frac{\partial^\alpha \tilde{u}}{\partial \tilde{t}} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon \eta_\alpha^0(x,t,y,u) + O(\epsilon^2),\\ \frac{\partial \tilde{u}}{\partial \tilde{x}} &= \frac{\partial u}{\partial x} + \epsilon \eta^x(x,t,y,u) + O(\epsilon^2),\\ \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \epsilon \eta^{xx}(x,t,y,u) + O(\epsilon^2),\\ \frac{\partial^3 \tilde{u}}{\partial \tilde{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \epsilon \eta^{xxx}(x,t,y,u) + O(\epsilon^2),\end{aligned} \tag{3b}$$

where

$$\begin{aligned}\eta^x &= D_x(\eta) - u_x D_x(\xi^1) - u_t D_t(\xi^2),\\ \eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\xi^1) - u_{xx} D_t(\xi^2),\\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_t(\xi^2).\end{aligned} \tag{4}$$

Here $D_x = \partial/\partial x + u_x \partial/\partial u + u_{xx} \partial/\partial u_x + \cdots$.

The corresponding Lie algebra of symmetries consists of a set of vector fields defined by

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \xi^3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u}. \tag{5}$$

The vector field (5) is a Lie point symmetry of (2), provided

$$P^{\alpha,3} r X(\nabla)|_{\nabla=0} = 0.$$

Also, the invariance condition [27] reads

$$\xi^2(x,t,y,u)|_{t=0} = 0, \tag{6}$$

and the αth extended infinitesimal related to RL fractional time derivative with (6) is given by [27, 62]

$$\begin{aligned}\eta_\alpha^0 = {}& \frac{\partial^\alpha \eta}{\partial t^\alpha} + \left(\eta_u - \alpha D_t(\xi^2)\right) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi^1) D_t^{\alpha-n}(u_x)\\ &+ \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\xi^2)\right] D_t^{\alpha-n}(u),\end{aligned} \tag{7}$$

where

$$\mu = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} [-u]^r \frac{\partial^m}{\partial t^m} [u^{k-r}] \frac{\partial^{n-m+k}}{\partial t^{n-m} \partial u^k}.$$

3 Lie symmetries and reduction for (1)

Suppose that (1) is an invariant under (3), we have that

$$\bar{u}_t^\alpha + \bar{u}^2 \bar{u}_x + \bar{u}_{xxx} + \bar{u}_{xyy} = 0, \tag{8}$$

thus, $u = u(x,t,y)$ satisfies (1). Using (3) in (8), we obtain the invariant equation

$$\eta_\alpha^0 + 2uu_x\eta + u^2\eta^x + 2\eta^{xxx} = 0. \tag{9}$$

Substituting the values of η_α^0, η^x and η^{xxx} from (4) and (7) into (9) and isolating coefficients in partial derivatives with respect to x and power of u, the determining equations are obtained. Solving the obtained determining equation, we get

$$\xi^1 = c_1 + x\alpha c_6, \qquad \xi^2 = c_5 + 3tc_6, \qquad \xi^3 = 0, \qquad \eta = -\alpha u c_6,$$

where c_1, c_5 and c_6 are arbitrary constants. Thus, infinitesimal symmetry group for (1) is spanned by the three vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = x\alpha\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - u\alpha\frac{\partial}{\partial u}.$$

In particular, for the symmetry X_3, the similarity variables for the infinitesimal generator X_3 can be obtained from the following equation:

$$\frac{\mathrm{d}x}{\alpha x} = \frac{\mathrm{d}t}{3t} = -\frac{\mathrm{d}u}{\alpha u},$$

which yields the similarity transformation and similarity variable

$$u = t^{\alpha/3} f(\xi), \qquad \xi = xt^{-\alpha/3} \tag{10}$$

as required.

Theorem 1. *The similarity transformation* (10) *reduces* (1) *to the nonlinear ODE of fractional order as below*:

$$\bigl(P_{3/\alpha}^{1-2\alpha/3,\,\alpha} f\bigr)(\xi) + f^2 f_\xi + 2f_{\xi\xi\xi} = 0$$

with the EK fractional differential operator [38]

$$\bigl(P_\beta^{\xi^2,\alpha} f\bigr) = \prod_{j=0}^{n-1}\left(\xi^2 + j - \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\bigl(K_\beta^{\xi^2+\alpha,\,n-\alpha} f\bigr)(\xi), \tag{11}$$

$$n = \begin{cases} [\alpha] + 1, & \alpha \neq \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases}$$

where

$$\bigl(K_\beta^{\xi^2,\alpha} f\bigr)(\xi) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1} u^{-(\xi^2+\alpha)} f(\xi u^{1/\beta})\,\mathrm{d}u, & \alpha > 0, \\ f(\xi), & \alpha = 0, \end{cases} \tag{12}$$

is the EK fractional integral operator [18, 63].

Proof. Let $n-1 < \alpha < 1$, $n = 1, 2, 3, \dots$. Based on the RL fractional derivative in (10), we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_1^t (t-s)^{n-\alpha-1}s^{\alpha/3}f\bigl(xs^{-(\alpha/3)}\bigr)\,\mathrm{d}s\right]. \tag{13}$$

Let $v = t/s$, $\mathrm{d}s = -t/v^2\,\mathrm{d}v$. Thus, (13) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\alpha/3}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1}v^{-(n+1-2\alpha/3)}f\bigl(\xi v^{\alpha/3}\bigr)\,\mathrm{d}v\right]. \tag{14}$$

Applying EK fractional integral operator (12) in (14), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\bigl[t^{n-2\alpha/3}\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\bigr]. \tag{15}$$

We simplify the right-hand side of (15). Considering $\xi = xt^{-\alpha/3}$, $\phi \in (0,\infty)$, we acquire

$$t\frac{\partial}{\partial t}\phi(\xi) = tx\left(-\frac{\alpha}{3}\right)t^{-\alpha/3-1}\phi'(\xi) = -\frac{\alpha}{3}\xi\frac{\partial}{\partial \xi}\phi(\xi).$$

Hence,

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\bigl[t^{n-2\alpha/3}\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\bigr]\\
&\quad= \frac{\partial^{n-1}}{\partial t^{n-1}}\bigl[t^{n-2\alpha/3}\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\bigr]\\
&\quad= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-2\alpha/3-1}\left(n+\frac{\alpha}{3}-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\right].
\end{aligned}$$

Repeating $n-1$ times, we have

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\bigl[t^{n-2\alpha/3}\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\bigr]\\
&\quad= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\bigl(t^{n-2\alpha/3}\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\bigr)\right]\\
&\quad= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha/3-1}\left(n+\frac{\alpha}{3}-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\right]\\
&\quad\;\vdots\\
&\quad= t^{-\alpha/3}\prod_{j=0}^{n-1}\left[\left(1-\frac{2\alpha}{3}+j-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\right].
\end{aligned} \tag{16}$$

Applying EK fractional differential operator (11) in (16), we get

$$\frac{\partial^n u}{\partial t^n}\bigl[\bigl(t^{n-\alpha/3}\bigl(K_{3/\alpha}^{1-\alpha/3,\,n-\alpha}f\bigr)(\xi)\bigr)\bigr] = t^{-\alpha/3}\bigl(P_{3/\alpha}^{1-2\alpha/3,\,\alpha}f\bigr)(\xi). \tag{17}$$

Substituting (17) into (15), we obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-2\alpha/3}\left(P_{3/\alpha}^{1-2\alpha/3,\,\alpha} f\right)(\xi).$$

Thus, (1) can be reduced into a fractional order ODE

$$\left(P_{3/\alpha}^{1-2\alpha/3,\,\alpha} f\right)(\xi) + f^2 f_\xi + 2f_{\xi\xi\xi} = 0.$$

The proof of the theorem is complete. □

4 Exact traveling wave solutions

In this section, the exact traveling wave solutions for (1) will be presented. For this aim, we consider the fractional derivative appears in (1) in the sense of modified RL derivative [35, 36].

4.1 Description of the $D_\xi^\alpha G/G$-expansion method

The description of the $D_\xi^\alpha G/G$-expansion method is presented in detail in [67]. Using the same steps as in [67], we present the application of $D_\xi^\alpha G/G$-expansion method to (1) in the following subsections.

4.2 Application

In this subsection, we apply $D_\xi^\alpha G/G$-expansion method to (1). Using the travelling wave transformation $\xi = x + y - ct$, we reduce (1) to the nonlinear fractional ODE below:

$$c^\alpha D_\xi^\alpha u + u^2 D_\xi^\alpha u + 2D_\xi^{3\alpha} u = 0. \tag{18}$$

Balancing the highest order derivative term $D_\xi^{3\alpha} u$ with nonlinear term $u^2 D_\xi^\alpha u$ in (18), we get $n = 1$ [13] and therefore

$$u(\xi) = a_0 + a_{-1}\left(\frac{G(\xi)}{D_\xi^\alpha G(\xi)}\right) + a_1\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right). \tag{19}$$

Substitute (19) into (18) collecting the coefficients of $(D_\xi^\alpha G(\xi)/G(\xi))^i$, $i = (0, \pm1, \pm2, \dots)$ and equating them to zero. A set of algebraic systems is obtained in a_0, a_{-1}, a_1. Solving the obtained algebraic systems with *Mathematica* 9, we obtain the following cases:

Case 1: $a_0 = 0$, $\lambda = 0$, $c^\alpha = -8\mu$, $a_1 = \pm 2\mathrm{i}\sqrt{3}$, $a_{-1} = (8\mu + 4\mu)\mathrm{i}\sqrt{3}/6$.
If $\mu < 0$, we have

$$\begin{aligned} u_1(x,t,y) = {}& \frac{1}{6}(8\mu + 4\mu)\mathrm{i}\sqrt{3}\left(\frac{\sqrt{-4\mu}}{2}\left[\frac{c_1\cosh(\frac{\sqrt{-4\mu}}{2}\eta) + c_2\sinh(\frac{\sqrt{-4\mu}}{2}\eta)}{c_1\sinh(\frac{\sqrt{-4\mu}}{2}\eta) + c_2\cosh(\frac{\sqrt{-4\mu}}{2}\eta)}\right]\right)^{-1} \\ & + 2\mathrm{i}\sqrt{3}\left(\frac{\sqrt{-4\mu}}{2}\left[\frac{c_1\cosh(\frac{\sqrt{-4\mu}}{2}\eta) + c_2\sinh(\frac{\sqrt{-4\mu}}{2}\eta)}{c_1\sinh(\frac{\sqrt{-4\mu}}{2}\eta) + c_2\cosh(\frac{\sqrt{-4\mu}}{2}\eta)}\right]\right). \end{aligned}$$

When $c_1 = 0$ and $c_2 \neq 0$, we get

$$u_2(x,t,y) = \frac{1}{6}(8\mu + 4\mu)\mathrm{i}\sqrt{3}\left[\frac{\sqrt{-4\mu}}{2}\tanh\left(\frac{\sqrt{-4\mu}}{2}\eta\right)\right]^{-1} + 2\mathrm{i}\sqrt{3}\left[\frac{\sqrt{-4\mu}}{2}\tanh\left(\frac{\sqrt{-4\mu}}{2}\eta\right)\right],$$

when $c_1 \neq 0$ and $c_2 = 0$, we obtain

$$u_3(x,t) = \frac{1}{6}(8\mu + 4\mu)\mathrm{i}\sqrt{3}\left[\frac{\sqrt{-4\mu}}{2}\coth\left(\frac{\sqrt{-4\mu}}{2}\eta\right)\right]^{-1} + 2\mathrm{i}\sqrt{3}\left[\frac{\sqrt{-4\mu}}{2}\coth\left(\frac{\sqrt{-4\mu}}{2}\eta\right)\right].$$

If $\mu > 0$, we get

$$u_4(x,t,y) = \frac{1}{6}(8\mu + 4\mu)\mathrm{i}\sqrt{3}\left(\frac{\sqrt{4\mu}}{2}\left[\frac{-c_1\cos(\frac{\sqrt{4\mu}}{2}\eta) + c_2\sin(\frac{\sqrt{4\mu}}{2}\eta)}{c_1\sin(\frac{\sqrt{4\mu}}{2}\eta) + c_2\cos(\frac{\sqrt{4\mu}}{2}\eta)}\right]\right)^{-1} + 2\mathrm{i}\sqrt{3}\left(\frac{\sqrt{4\mu}}{2}\left[\frac{c_1\cos(\frac{\sqrt{4\mu}}{2}\eta) + c_2\sin(\frac{\sqrt{4\mu}}{2}\eta)}{c_1\sinh(\frac{\sqrt{-4\mu}}{2}\eta) + c_2\cosh(\frac{\sqrt{4\mu}}{2}\eta)}\right]\right).$$

When $c_1 = 0$ and $c_2 \neq 0$, we acquire

$$u_5(x,t,y) = \frac{1}{6}(8\mu + 4\mu)\mathrm{i}\sqrt{3}\left[\frac{\sqrt{4\mu}}{2}\tan\left(\frac{\sqrt{4\mu}}{2}\eta\right)\right]^{-1} + 2\mathrm{i}\sqrt{3}\left[\frac{\sqrt{4\mu}}{2}\tan\left(\frac{\sqrt{4\mu}}{2}\eta\right)\right],$$

when $c_1 \neq 0$ and $c_2 = 0$, we have that

$$u_6(x,t,y) = \frac{1}{6}(8\mu + 4\mu)\mathrm{i}\sqrt{3}\left[\frac{\sqrt{4\mu}}{2}\cot\left(\frac{\sqrt{4\mu}}{2}\eta\right)\right]^{-1} + 2\mathrm{i}\sqrt{3}\left[\frac{\sqrt{4\mu}}{2}\cot\left(\frac{\sqrt{4\mu}}{2}\eta\right)\right],$$

where $\eta = \xi^\alpha/\Gamma(1+\alpha)$ and $\xi = x + y - ct$.

Case 2: $c^\alpha = (-24\mu + a_0^2)/3$, $\lambda \neq 0$, $a_1 = 2a_0/\lambda$, $a_{-1} = 2\mu a_0/\lambda$.

If $\mu < 0$, we obtain

$$\begin{aligned} u_7(x,t,y) &= a_0 + \frac{2\mu a_0}{\lambda}\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\left[\frac{c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}{c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}\right] - \frac{\lambda}{2}\right)^{-1} \\ &\quad + \frac{2a_0}{\lambda}\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\left[\frac{c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}{c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}\right] - \frac{\lambda}{2}\right). \end{aligned}$$

When $c_1 = 0$ and $c_2 \neq 0$, we acquire

$$u_8(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right]^{-1} + \frac{2a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right],$$

when $c_1 \neq 0$ and $c_2 = 0$, we get

$$u_9(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right]^{-1} + \frac{2a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right].$$

If $\mu > 0$, we have

$$u_{10}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{-c_1\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}{c_1\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}\right] - \frac{\lambda}{2}\right)^{-1} + \frac{2a_0}{\lambda}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}{c_1\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}\right] - \frac{\lambda}{2}\right).$$

When $c_1 = 0$ and $c_2 \neq 0$, we obtain

$$u_{11}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right) - \frac{\lambda}{2}\right]^{-1} + \frac{2a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right) - \frac{\lambda}{2}\right], \tag{20}$$

when $c_1 \neq 0$ and $c_2 = 0$, we get

$$u_{12}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right) - \frac{\lambda}{2}\right]^{-1} + \frac{2a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right) - \frac{\lambda}{2}\right],$$

where $\eta = \xi^\alpha/\Gamma(1+\alpha)$ and $\xi = x + y - ct$.

Case 3: $c^\alpha = (12\mu + a_0^2)/3$, $\lambda \neq 0$, $a_1 = 2a_0/\lambda$, $a_{-1} = 0$.
If $\mu < 0$, we get

$$u_{13}(x,t,y) = a_0 + \frac{2a_0}{\lambda}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}{c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}\right] - \frac{\lambda}{2}\right).$$

When $c_1 = 0$ and $c_2 \neq 0$, we obtain

$$u_{14}(x,t,y) = a_0 + \frac{2a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right], \tag{21}$$

when $c_1 \neq 0$ and $c_2 = 0$, we have

$$u_{15}(x,t,y) = a_0 + \frac{2a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right].$$

If $\mu > 0$, we get

$$\begin{aligned} &u_{16}(x,t,y) \\ &\quad = a_0 + \frac{2a_0}{\lambda}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-c_1\cosh(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta) + c_2\sinh(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta)}{c_1\sinh(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta) + c_2\cosh(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta)}\right] - \frac{\lambda}{2}\right). \end{aligned}$$

When $c_1 = 0$ and $c_2 \neq 0$, we obtain that

$$u_{17}(x,t,y) = a_0 + \frac{2a_0}{\lambda}\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right) - \frac{\lambda}{2}\right],$$

when $c_1 \neq 0$ and $c_2 = 0$, we acquire

$$u_{18}(x,t,y) = a_0 + \frac{2a_0}{\lambda}\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right) - \frac{\lambda}{2}\right],$$

where $\eta = \xi^\alpha/\Gamma(1+\alpha)$ and $\xi = x + ct$.

Case 4: $a_0 = 0$, $\lambda = 0$, $c^\alpha = 4\mu$, $a_1 = 0$, $a_{-1} = \pm 2\mathrm{i}\sqrt{3}\mu$, $\mu \neq 0$.

If $\mu < 0$, we get

$$u_{19}(x,t,y) = 2\mathrm{i}\sqrt{3}\mu\left(\frac{\sqrt{-4\mu}}{2}\left[\frac{c_1\cosh(\frac{\sqrt{-4\mu}}{2}\eta) + c_2\sinh(\frac{\sqrt{-4\mu}}{2}\eta)}{c_1\sinh(\frac{\sqrt{-4\mu}}{2}\eta) + c_2\cosh(\frac{\sqrt{-4\mu}}{2}\eta)}\right]\right)^{-1},$$

when $c_1 = 0$ and $c_2 \neq 0$, we get

$$u_{20}(x,t,y) = 2\mathrm{i}\sqrt{3}\mu\left[\frac{\sqrt{-4\mu}}{2}\tanh(\frac{\sqrt{-4\mu}}{2}\eta)\right]^{-1},$$

when $c_1 \neq 0$ and $c_2 = 0$, we have

$$u_{21}(x,t,y) = 2\mathrm{i}\sqrt{3}\mu\left[\frac{\sqrt{-4\mu}}{2}\coth(\frac{\sqrt{-4\mu}}{2}\eta)\right]^{-1}.$$

If $\mu > 0$, we get

$$u_{22}(x,t,y) = 2\mathrm{i}\sqrt{3}\mu\left(\frac{\sqrt{4\mu}}{2}\left[\frac{-c_1\cos(\frac{\sqrt{4\mu}}{2}\eta) + c_2\sin(\frac{\sqrt{4\mu}}{2}\eta)}{c_1\sin(\frac{\sqrt{4\mu}}{2}\eta) + c_2\cos(\frac{\sqrt{4\mu}}{2}\eta)}\right]\right)^{-1}.$$

When $c_1 = 0$ and $c_2 \neq 0$, we obtain

$$u_{23}(x,t,y) = 2\mathrm{i}\sqrt{3}\mu\left[\frac{\sqrt{4\mu}}{2}\tan\left(\frac{\sqrt{4\mu}}{2}\eta\right)\right]^{-1},$$

when $c_1 \neq 0$ and $c_2 = 0$, we get

$$u_{24}(x,t,y) = 2\mathrm{i}\sqrt{3}\mu\left[\frac{\sqrt{4\mu}}{2}\cot\left(\frac{\sqrt{4\mu}}{2}\eta\right)\right]^{-1},$$

where $\eta = \xi^\alpha/\Gamma(1+\alpha)$ and $\xi = x + y - ct$.

Case 5: $\mu \neq 0$, $c^\alpha = 2\lambda^2 + 4\mu + a_0^2$, $a_1 = 0$, $\lambda \neq 0$, $a_{-1} = 2\mu a_0/\lambda$.
If $\mu < 0$, we get

$$u_{25}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{-c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}{c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta) + c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta)}\right] - \frac{\lambda}{2}\right)^{-1}.$$

When $c_1 = 0$ and $c_2 \neq 0$, we obtain that

$$u_{26}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right]^{-1},$$

when $c_1 \neq 0$, and $c_2 = 0$, we obtain

$$u_{27}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right]^{-1}.$$

If $\mu > 0$, we get

$$u_{28}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-c_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta) + c_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta)}{c_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta) + c_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta)}\right] - \frac{\lambda}{2}\right)^{-1}.$$

When $c_1 = 0$ and $c_2 \neq 0$, we get

$$u_{29}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\tan\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right]^{-1},$$

when $c_1 \neq 0$ and $c_2 = 0$, we have

$$u_{30}(x,t,y) = a_0 + \frac{2\mu a_0}{\lambda}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\cot\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) - \frac{\lambda}{2}\right]^{-1},$$

where $\eta = \xi^\alpha/\Gamma(1+\alpha)$ and $\xi = x + y - ct$.

4.3 Graphical and physical explanation of the obtained solutions

Herein, we present some three-dimensional, two-dimensional and contour plots of some of the obtained results. The construction of the figures is carried out by taking suitable values of the parameters in order to see the mechanism of the original (1). One can see that the obtained solutions possess solutions like periodic wave, bell-shaped, kink-type and singular solitons. We take solutions (20) and (21) and illustrate them in Figs. 1, 2, respectively.

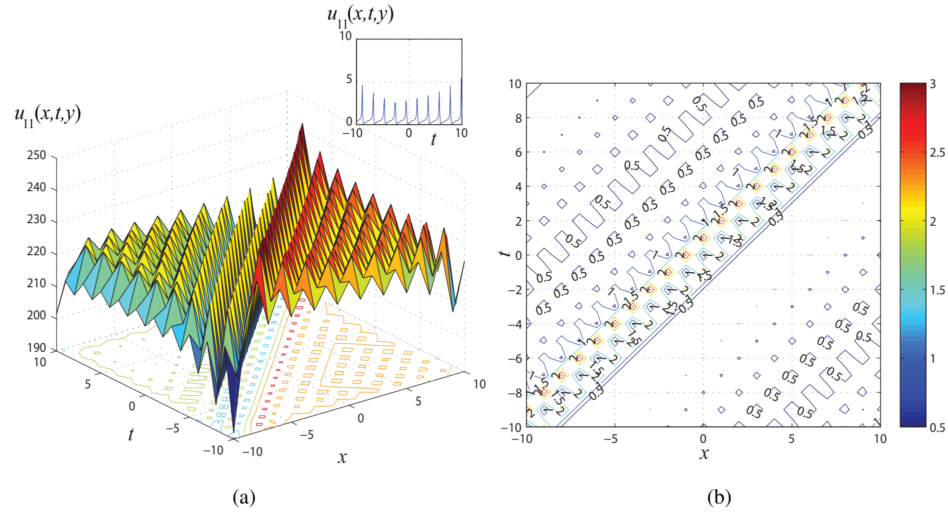

Figure 1. Solution (20) with $a_0 = 1$, $\mu = 0.75$, $\lambda = 1.5$, $\alpha = 0.8$: (a) 3D and 2D plot; (b) contour plot.

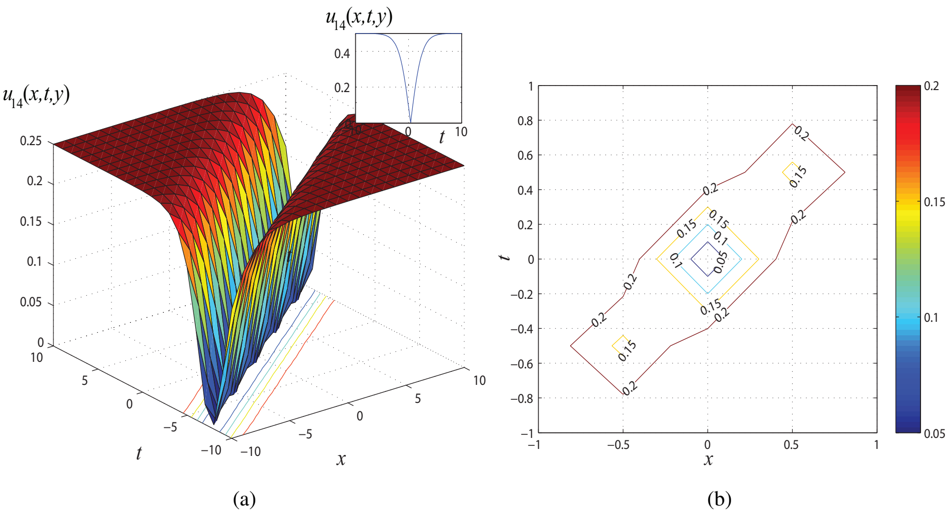

Figure 2. Solution (21) with $a_0 = 1$, $\mu = 0.5$, $\lambda = 0.1$, $\alpha = 0.5$: (a) 3D and 2D plot; (b) contour plot.

5 Conservation laws for (1)

For more details on CLs and their constructions, see [30, 37, 53]. Using the descriptions and basics definitions for CLs presented in [3], the CLs for (1) corresponding to the order of α are given as follows:

Case 1. When $\alpha \in (0,1)$, the components of the conserved vectors are

$$\begin{aligned} C_i^t &= \xi^2 L + (-1)^0 o D_t^{\alpha-1}(W_i) D_t^0 \frac{\partial L}{\partial(o D_t^\alpha u)} - (-1)^1 J\left(W_i, D_t^1 \frac{\partial L}{\partial(o D_t^\alpha u)}\right) \\ &= v_o D_t^{\alpha-1}(W_i) + J(W_i, v_t), \end{aligned}$$

$$\begin{aligned} C_i^x &= \xi^1 l + W_i\left(\frac{\partial}{\partial u_x} - D_x \frac{\partial}{\partial u_{xx}} + D_x^2 \frac{\partial}{\partial u_{xxx}}\right) \\ &\quad + D_x(W_i)\left(\frac{\partial}{\partial u_{xx}} - D_x \frac{\partial}{\partial u_{xxx}}\right) + D_x^2(W_i)\left(\frac{\partial}{\partial u_{xxx}}\right) \\ &= W_i\bigl(u^2 v + 2D_x^2 v\bigr) - D_x(W_i)(D_x v) + D_x^2(W_i) v, \end{aligned}$$

where $i = 1, 2, 3$, and the functions W_i are given by

$$W_1 = -u_x, \quad W_2 = -u_t, \quad W_3 = -u\alpha - 3tu_t - \alpha x u_x.$$

Case 2. When $\alpha \in (1,2)$, the components of the conserved vectors are

$$\begin{aligned} C_i^t &= \xi^2 L + (-1)^0 o D_t^{\alpha-1}(W_i) D_t^0 \frac{\partial L}{\partial(o D_t^\alpha u)} - (-1)^1 J\left(W_i, D_t^1 \frac{\partial L}{\partial(o D_t^\alpha u)}\right) \\ &\quad + (-1)^1 o D_t^{\alpha-2}(W_i) D_t^1 \frac{\partial L}{\partial(o D_t^\alpha u)} - (-1)^1 J\left(W_i, D_t^1 \frac{\partial L}{\partial(o D_t^\alpha u)}\right) \\ &= v_o D_t^{\alpha-1}(W_i) + J(W_i, v_t) - v_t o D_t^{\alpha-2}(W_i) - J(W_i, v_{tt}), \end{aligned}$$

$$\begin{aligned} C_i^x &= \xi^1 l + W_i\left(\frac{\partial}{\partial u_x} - D_x \frac{\partial}{\partial u_{xx}} + D_x^2 \frac{\partial}{\partial u_{xxx}}\right) \\ &\quad + D_x(W_i)\left(\frac{\partial}{\partial u_{xx}} - D_x \frac{\partial}{\partial u_{xxx}}\right) + D_x^2(W_i)\left(\frac{\partial}{\partial u_{xxx}}\right) \\ &= W_i\bigl(u^2 v + 2D_x^2 v\bigr) - D_x(W_i)(D_x v) + D_x^2(W_i) v, \end{aligned}$$

where $i = 1, 2, 3$, and the functions W_i are given by

$$W_1 = -u_x, \quad W_2 = -u_t, \quad W_3 = -u\alpha - 3tu_t - \alpha x u_x.$$

6 Concluding remarks

In this research, Lie symmetry analysis of (1) using RL derivative was investigated. We obtained point symmetries for the governing equation and reduced it to a fractional nonlinear ODE. An exact solutions for the reduced ODE are obtained using $D_\xi^\alpha G/G$-expansion method. The obtained solutions include hyperbolic, trigonometric and rational solutions. We computed the CLs for the governing equation by using new conservation theorem. The obtained solutions and CLs might be very vital for interpreting some physical phenomena in different fields of applied mathematics.

References

1. B.S. Ahmed, E. Zerrad, A. Biswas, Kinks and domain walls of the Zakharov–Kuznetsov equation in plasmas, *Proc. Rom. Acad., Ser. A, Math. Phys. Tech. Sci. Inf. Sci.*, **14**(4):281–286, 2013.
2. S.C. Anco, G.W. Bluman, Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications, *Eur. J. Appl. Math.*, **13**:545–566, 2002.
3. D. Baleanu, M. Inc, A. Yusuf, AI. Aliyu, Time fractional third-order evolution equation: Symmetry analysis, explicit solutions and conservation laws, *J. Comput. Nonlinear Dyn.*, 2010, `https://doi.org/10.1115/1.4037765`.
4. A. Bekir, O. Guner, A.H. Bhrawy, A. Biswas, Solving nonlinear fractional differential equations using exp-function and G'/G-expansion methods, *Rom. J. Phys.*, **60**(3-4):360–378, 2015.
5. A.H. Bhrawy, M.A. Abdelkawy, S. Kumar, S. Johnson, A. Biswas, Soliton and other solutions to quantum Zakharov–Kuznetsov equation in quantum magneto-plasmas, *Indian J. Phys.*, **87**(5):455–463, 2013.
6. A.H. Bhrawy, J.F. Alzaidy, M.A. Abdelkawy, A. Biswas, Jacobi spectral collocation approximation for multi-dimensional time fractional Schrödinger's equation, *Nonlinear Dyn.*, **84**(3):1553–1567, 2016.
7. A. Biswas, M. Song, Soliton solution and bifurcation analysis of the Zakharov–Kuznetsov–Benjamin–Bona–Mahoney equation with power law nonlinearity, *Commun. Nonlinear Sci. Numer. Simul.*, **18**(7):1676–1683, 2013.
8. A. Biswas, E. Zerrad, Solitary wave solution of the Zakharov–Kuznetsov equation in plasmas with power law nonlinearity, *Nonlinear Anal., Real World Appl.*, **11**(4):3272–3274, 2010.
9. G.W. Bluman, S. Kumei, *Symmetries and Differential Equations*, Springer, New York, 1989.
10. E. Buckwar, Y. Luchko, Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations, *J. Math. Anal. Appl.*, **227**:81–97, 1998.
11. M. Craddock, K. Lennox, Lie symmetry methods for multi-dimensional parabolic PDEs and diffusions, *J. Differ. Equations*, **252**:56–90, 2012.
12. K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.
13. VD. Djordjevic, TM. Atanackovic, Similarity solutions to nonlinear heat conduction and Burgers/Korteweg–de Vries fractional equations, *J. Comput. Appl. Math.*, **212**:701–714, 2008.
14. G. Ebadi, A. Biswas, The G'/G method and 1-soliton solution of Davey–Stewartson equation, *Math. Comput. Modelling*, **53**(5-6):694–698, 2011.
15. G. Ebadi, A. Mojaver, D. Milovic, S. Johnson, A. Biswas, Solitons and other solutions to the quantum Zakharov–Kuznetsov equation, *Astrophys. Space Sci.*, **341**(2):507–513, 2012.
16. M. Ekici, M. Mirzazadeh, M. Eslami, Q. Zhou, S.P. Moshokoa, A. Biswas, M. Belic, Optical soliton pertubation with fractional temporal evolution by first integral method with conformabal fractional derivatives, *Optik*, **127**(22):10659–10669, 2016.
17. M. Ekici, M. Mirzazadeh, Q. Zhou, S.P. Moshokoa, A. Biswas, M. Belic, Solitons in optical metamaterials with fractional temporal evolution, *Optik*, **127**(22):10879–10897, 2016.

18. E.H. El Kinani, Lie symmetry analysis of some time fractional partial differential equations, *Int. J. Mod. Phys. Conf. Ser.*, **38**:1560075, 2015.
19. M. Eslami, M. Mirzazadeh, A. Biswas, Soliton solutions of the resonant nonlinear Schrödinger's equation in optical fibers with time-dependent coefficients by simplest equation approach, *J. Mod. Opt.*, **60**(19):1627–1636, 2013.
20. M. Eslami, M. Mirzazadeh, B.F. Vajargah, A. Biswas, Optical solitons for the resonant nonlinear Schrödinger's equation with time-dependent coefficients by the first integral method, *Optik*, **125**(13):3107–3116, 2014.
21. M. Eslami, B.F. Vajargah, M. Mirzazadeh, A. Biswas, Application of the first integral method to fractional partial differential equations, *Indian J. Phys.*, **88**(2):177–184, 2014.
22. A.L. Fabian, R. Kohl, A. Biswas, Pertubation of topological solitons due to sine–Gordon equation and its type, *Commun. Nonlinear Sci. Numer. Simul.*, **14**(4):1227–1244, 2009.
23. R.K. Gazizov, A.A. Kasatkin, Construction of exact solutions for fractional order differential equations by the invariant subspace method, *Comput. Math. Appl.*, **66**:576–584, 2013.
24. R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk, Symmetry properties of fractional diffusion equations, *Phys. Scr.*, **136**:014016, 2009.
25. R.K. Gazizov, A.A. Kasatkin, Y.S. Lukashchuk, Continuous transformation groups of fractional differential equations, *Vestn. USATU*, **9**:125–135, 2007 (in Russian).
26. S. Guo, L. Mei, Y. Li, Y. Sun, The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics, *Phys. Lett. A*, **376**:407–411, 2012.
27. Q. Huanga, R. Zhdanov, Symmetries and exact solutions of the time fractional Harry–Dym equation with Riemann–Liouville derivative, *Physica A*, **409**:110–118, 2014.
28. N.H. Ibragimov, *Handbook of Lie Group Analysis of Differential Equations*, CRC Press, Boca Raton, FL, 1996.
29. N.H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, John Wiley & Sons, Chichester, 1999.
30. N.H. Ibragimov, A new conservation theorem, *J. Math. Anal. Appl.*, **28**:333–311, 2007.
31. M. Inc, A.I. Aliyu, A. Yusuf, Dark optical, singular solitons and conservation laws to the nonlinear Schrödinger's equation with spatio-temporal dispersion, *Mod. Phys. Lett. B*, **31**(14):1750163, 2017.
32. M. Inc, A.I. Aliyu, A. Yusuf, Traveling wave solutions and conservation laws of some fifth-order nonlinear equations, *Eur. Phys. J. Plus*, **132**:224, 2017.
33. M. Inc, A.I. Aliyu, A. Yusuf, D. Baleanu, New solitary wave solutions and conservation laws to the Kudryashov–Sinelshchikov equation, *Optik*, **142**:665–673, 2017.
34. A.G. Johnpillai, A.H. Kara, A. Biswas, Symmetry solutions and reductions of a class of generalized $(2+1)$ dimensional Zakharov–Kuznetsov equation, *Int. J. Nonlinear Sci. Numer. Simul.*, **12**(1-8):35–43, 2011.
35. G. Jumarie, Modified Riemann–Liouville derivative and fractional Taylor series of non differentiable functions further results, *Comput. Math. Appl.*, **51**:1367–1376, 2006.
36. G. Jumarie, Cauchy's integral formula via the modified Riemann–Liouville derivative for analytic functions of fractional order, *Appl. Math. Lett.*, **23**:1444–1450, 2010.

37. A.H. Kara, F.M. Mahomed, Noether-type symmetries and conservation laws via partial Lagragians, *Nonlinear Dyn.*, **45**(3–4):367–383, 2006.
38. V. Kiryakova, *Generalised Fractional Calculus and Applications*, Pitman Res. Notes Math. Ser., Vol. 301, Longman Scientific & Technical, Harlow, 1994.
39. R. Kohl, D. Milovic, E. Zerrad, A. Biswas, Optical solitons by He's variational principle in a non-Kerr law media, *J. Infrared Millimeter Terahertz Waves*, **30**(5):526–537, 2009.
40. E.V. Krishnan, A. Biswas, Solutions of the Zakharov–Kuznetsov equation with higher order nonlinearity by mapping and ansatz methods, *Phys. Wave Phenom.*, **18**(4):256–261, 2010.
41. E.V. Krishnan, Q. Zhou, A. Biswas, Solitons and shock waves to Zakharov–Kuznetsov equation with dual-power law nonlinearity in plasmas, *Proc. Rom. Acad., Ser. A, Math. Phys. Tech. Sci. Inf. Sci.*, **17**(2):137–143, 2016.
42. S. Kumar, K. Singh, R.K. Gupta, Painlev analysis, Lie symmetries and exact solutions for $(2 + 1)$-dimensional variable coefficients Broer–Kaup equations, *Commun. Nonlinear Sci. Numer. Simul.*, **17**:1529–1541, 2012.
43. X. Li, W. Chen, Analytical study on the fractional anomalous diffusion in a half-plane, *J. Phys. A, Math. Theor.*, **43**(49):495206, 2010.
44. S. Lie, On integration of a class of linear partial differential equations by means of definite integrals, *Arch. Math. Logic*, **6**(3):328–368, 1881.
45. H.Z. Liu, Y.X. Geng, Symmetry reductions and exact solutions to the systems of carbon nanotubes conveying fluid, *J. Differ. Equations*, **254**:2289–2303, 2013.
46. B. Lu, Backlund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations, *Phys. Lett. A*, **376**:2045–2048, 2012.
47. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
48. M. Mirzazadeh, M. Ekici, A. Sonomezoglu, M. Eslami, Q. Zhou, E. Zerrad, A. Biswas, M. Belic, Optical solitons in nano-fibers with fractional temporal evolution, *J. Comput. Theor. Nanosci.*, **13**(8):5361–5374, 2016.
49. M. Mirzazadeh, M. Ekici, A. Sonomezoglu, S. Ortakaya, M. Eslami, A. Biswas, Solitons solutions to a few fractional nonlinear evolution equations in shallow water wave dynamics, *Eur. Phys. J. Plus*, **131**(6):166–177, 2016.
50. M. Mirzazadeh, M. Eslami, B.S. Ahmed, A. Biswas, Dynamics of population growth model with fractional temporal evolution, *Life Sci. J.*, **11**(3):224–227, 2014.
51. M. Mirzazadeh, B.F. Vajargah M. Eslami, A. Biswas, Solitons and periodic solutions to a couple of fractional nonlinear evolution equations, *Pramana*, **82**(3):465–476, 2014.
52. R. Morris, A.H. Kara, A. Biswas, Soliton solution and conservation laws of the Zakharov–Kuznetsov equation in plasmas with power law nonlinearity, *Nonlinear Anal. Model. Control*, **18**(2):153–159, 2013.
53. E. Noether, Invariant variation problems, *Transp. Theory Stat. Phys.*, **1**(3):186–207, 1971.
54. K.B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, San Diego, CA, 1974.
55. P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1993.
56. L.V. Ovsiannikovr, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

57. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

58. M. Song, B. Ahmed, E. Zerrad, A. Biswas, Domain wall and bifurcation analysis of the Klein–Gordon–Zakharov equation in $(1+2)$-dimensions with power law nonlinearity, *Chaos*, **23**(3):033115, 2013.

59. P. Suarez, A. Biswas, Exact 1-soliton solution of the Zakharov–Kuznetsov equation in plasmas with power law nonlinearity, *Appl. Math. Comput.*, **217**(17):7372–7375, 2011.

60. F. Tchier, A. Yusuf, A.I. Aliyu, M. Inc, Soliton solutions and conservation laws for lossy nonlinear transmission line equation, *Superlattices Microstruct.*, **107**:320–336, 2017.

61. G. Wang, A.H. Kara, K. Fakhar, Symmetry analysis and conservation laws for the class of time-fractional nonlinear dispersive equation, *Nonlinear Dyn.*, **82**(1–2):281–287, 2015.

62. G. Wang, X. Tianzhou, Symmetry properties and explicit solutions of the nonlinear time fractional KdV equation, *Bound. Value Probl.*, **2013**:232, 2013.

63. G.W. Wang, T.Z. Xu, Invariant analysis and exact solutions of nonlinear time fractional Sharma–Tasso–Olver equation by Lie group analysis, *Nonlinear Dyn.*, **76**:571–580, 2014.

64. G.W. Wang, T.Z. Xu, S. Johnson, A. Biswas, Solitons and Lie group analysis to an extended quantum Zakharov–Kuznetsov equation, *Astrophys. Space Sci.*, **349**(1):317–327, 2014.

65. G. Wu, E.W.M. Lee, Fractional variational iteration method and its application, *Phys. Lett. A*, **374**:2506–2509, 2010.

66. E. Yasar, Y. Yıldı rım, C.M. Khalique, Lie symmetry analysis, conservation laws and exact solutions of the seventh-order time fractional Sawada–Kotera–Ito equation, *Results Phys.*, **6**:322–328, 2016.

67. E.M.E. Zayed, Y.A. Amer, R.M.A. Shohib, The fractional $(D^{\alpha}G/G)$-expansion method and its applications for solving four nonlinear space-time fractional PDEs in mathematical physics, *Ital. J. Pure Appl. Math.*, **34**:463–482, 2015.