Nonlinear Analysis: Modelling and Control, Vol. 22, No. 6, 821–840 ISSN 1392-5113
https://doi.org/10.15388/NA.2017.6.7

Eigenvalue problem for nonlinear elastic beam equation of fractional order*

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Received: April 21, 2017 / **Revised:** August 22, 2017 / **Published online:** October 27, 2017

Abstract. In this study, under some suitable assumptions, we determine an explicit eigenvalue interval for the existence of positive solution of singular fractional-order nonlinear elastic beam equation with bending term. Our analysis rely on cone theoretic techniques. Moreover, we consider some special cases and an example to affirm the applicability of the main result.

Keywords: eigenvalue problem, positive solution, boundary value problem, beam equation, fractional differential equation.

1 Introduction

Recently, fractional differential calculus has attracted a lot of attention by many researchers of different fields, such as: physics, chemistry, biology, economics, control theory, biophysics, etc. [11, 15, 16]. Since the fractional integrals and derivatives have more abilities to describe phenomena, it means that they can decrease errors occurring in modeling of real-life events, thus, studying of fractional systems solutions becomes one of the most significant challenging part of applied mathematics.

Also, fourth-order differential equations often used to describe the deformation of elastic beams and so are important in mechanics and engineering problems. Many authors have investigated fourth-order differential equations with different boundary conditions (see, for example, [1, 2, 5–7, 9, 17, 19, 25]).

So, due to the importance of fractional differential equations and fourth-order differential equations, the existence of positive solutions of fractional-order beam equations has been studied by many authors.

Xu et al. [24], considered the existence of positive solutions for fractional-order beam equation

$$D^\alpha u(t) = f\big(t, u(t)\big), \quad 0 < t < 1,\ 3 < \alpha \leqslant 4, \tag{1}$$

*This research was supported by Iran National Science Foundation (INSF) grant No. 93043645.

with the boundary condition

$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where f has singularity at $x = 0$, and D is the Riemann–Liouville fractional derivative.

The authors in [23] investigated the existence of positive solutions of singular superlinear (or sub-linear) integral boundary value problems (1) with Caputo derivative and the integral boundary condition

$$\begin{aligned}
&au(0) - bu'(0) = 0, \qquad cu(1) + du'(1) = 0,\\
&u''(0) + u'''(0) = \int_0^1 u''(\tau)\,\mathrm{d}p(\tau),\\
&u''(1) + u'''(1) + \int_0^1 u''(\tau)\,\mathrm{d}q(\tau) = 0.
\end{aligned}$$

Chen and Liu [4] studied the singular Riemann–Liouville fractional-order elastic beam equation (1) with the following conditions:

$$\begin{aligned}
&\lim_{t\to 0} t^{4-\alpha}u(t) = a, \qquad \lim_{t\to 0} D^{\alpha-3}u(t) = b,\\
&u(1) = D^{\alpha-3}u(1) = 0.
\end{aligned}$$

Their analysis relies on the well-known Schauder's fixed-point theorem.

In [14], by using fixed-point theorems on cones in a Banach space, there is discussed $D^\alpha u(t) = f(t, u(t), u'(t))$ with the conditions

$$\begin{aligned}
&\lim_{t\to 0} t^{4-\alpha}u(t) = \lim_{t\to 0} t^{4-\alpha}u'(t) = 0, \quad 0 < t < 1,\ 3 < \alpha \leqslant 4,\\
&u(1) = u'(1) = 0,
\end{aligned}$$

where D is the Riemann–Liouville fractional derivative.

Liu [13] considered the following boundary value problem for nonlinear singular Riemann–Liouville fractional-order elastic beam equation:

$$\begin{aligned}
&D^\alpha u(t) = f\bigl(t, u(t), u'(t), u''(t)\bigr), \quad 0 < t < 1,\ 3 < \alpha \leqslant 4,\\
&\lim_{t\to 0} t^{4-\alpha}u(t) = u(1) = 0,\\
&D^{\alpha-3}u(0) = u'(1) = 0.
\end{aligned}$$

Furthermore, the eigenvalue problems are one of the most noteworthy theories such that they have been concerned by some authors; see [3, 10, 18, 20–22, 27]. For example, Bai [3] considered the Caputo fractional ordinary differential equation boundary value problem

$$\begin{aligned}
&D^\alpha u(t) + \lambda h(t) f\bigl(u(t)\bigr) = 0, \quad 0 < t < 1,\ 2 < \alpha \leqslant 3,\\
&u(0) = u'(1) = u''(0) = 0.
\end{aligned}$$

Jiang [10], by using the fixed-point index theory and Krein–Rutman theorem, studied the eigenvalue interval of the multi-point boundary value problem

$$\begin{aligned} &D^\alpha u(t) - Mu = \lambda f\bigl(t, u(t)\bigr), \quad 0 < t < 1,\ 0 < \alpha < 1, \\ &u(0) = \sum_{i=1}^{n} \beta_i u(\xi_i), \end{aligned}$$

where, $M \geqslant 0$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_n \leqslant 1$ and D is the Caputo fractional derivative.

Wang et al. [20] gave the eigenvalue interval for the following nonlinear Caputo fractional differential equation with integral boundary condition:

$$\begin{aligned} &D^\alpha u(t) + \lambda f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1,\ n < \alpha \leqslant n+1,\ n \geqslant 2, \\ &u(0) = u''(0) = \cdots = u^{(n)}(0) = 0, \\ &u(1) = \xi \int_0^1 u(s)\,\mathrm{d}s, \quad 0 < \xi < 2. \end{aligned}$$

Wang and Guo [22], by using the fixed-point theory, investigated the following eigenvalue problem:

$$\begin{aligned} &D^\alpha u(t) + \lambda f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1,\ 1 < \alpha \leqslant 2, \\ &au(0) - bu'(0) = 0, \\ &u(1) = \int_0^1 k(s) g\bigl(u(s)\bigr)\,\mathrm{d}s + \mu, \quad \mu > 0, \end{aligned}$$

where D is the Caputo fractional derivative.

Therefore, it seems that few papers consider the eigenvalue problem of fractional differential equations, especially with integral boundary conditions.

Our aim in this paper is to give an eigenvalue interval for the existence of positive solution of the following nonlinear elastic beam equation with the bending term:

$$\begin{aligned} &D^\alpha u(t) + \lambda f\bigl(t, u(t), u''(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = \int_0^1 k(\tau) u(\tau)\,\mathrm{d}\tau, \\ &u''(0) + u'''(0) = \int_0^1 g(\tau) u''(\tau)\,\mathrm{d}\tau, \\ &u''(1) + u'''(1) = \int_0^1 g(\tau) u''(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{2}$$

where $3 < \alpha \leqslant 4$, $\lambda > 0$, $f \in C([0,1] \times [0,\infty) \times (-\infty,0],[0,\infty))$ and D is the Caputo fractional derivative. Also $f(t,x,y)$ may also have singularity at $x = 0$ and/or $y = 0$.

The innovation of this study is that the nonlinear term f involves the second-order derivative of the unknown function u as bending term, which represents bending effect. Besides, we consider the eigenvalue problem of fractional-order beam equation, which only a few results exist, so, from this point of view, we generalize some recent works.

Motivated by [12, 23], we first construct Green's function for problem (2) with the help of some lemmas and then obtain the property of Green's function in Section 2. In Section 3, we specify the range of the eigenvalue λ such that, in this interval, problem (2) has at least one positive solution. The paper concludes with an illustrative example.

2 Preliminaries and lemmas

In this section, we present some definitions and lemmas, which will be needed later.

Let $AC[0,1]$ be the space of functions, which are absolutely continuous on $[0,1]$,

$$AC^n[0,1] = \left\{u\colon\ [0,1] \to \mathbb{R} \text{ and } \left(D^{n-1}u\right)(t) \in AC[0,1],\ D = \frac{\mathrm{d}}{\mathrm{d}t}\right\}.$$

Definition 1. (See [11].) The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined by

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,\mathrm{d}s, \quad t > 0,$$

provided that the right-hand side is pointwise defined.

Definition 2. (See [11].) If $u(t) \in AC^n[0,1]$, then the Caputo fractional derivative of order $\alpha > 0$ exists almost everywhere on $[0,1]$ and is defined as

$$D^\alpha u(t) = \left(I^{n-\alpha} D^n u\right)(t).$$

Remark 1. The following property is well known:

$$D^\beta I^\alpha u(t) = I^{\alpha-\beta} u(t), \quad \alpha > \beta > 0,\ u(t) \in L^1(0,\infty).$$

Lemma 1. (*See* [11].) *Let* $\alpha > 0$. *If* $u(t) \in AC^n[0,1]$ *or* $u(t) \in C^n[0,1]$, *then*

$$I^\alpha D^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k.$$

Now, we mention the following fixed-point theorem on cones, which will be used to establish our main theorems.

Let K be a cone in a Banach space E, and let $K_r = \{x \in K\colon\ \|x\| < r\}$, $\partial K_r = \{x \in K\colon\ \|x\| = r\}$, and $\overline{K}_{r,R} = \{x \in K\colon\ r \leqslant \|x\| \leqslant R\}$, where $0 < r < R < \infty$.

Theorem 1. (*See* [8].) *Let K be a positive cone in a real Banach space E, $0 < r < R < \infty$, and let $T : \overline{K}_{r,R} \to K$ be a completely continuous operator such that*

(i) $\|Tx\| \leqslant \|x\|$ *for* $x \in \partial K_R$;
(ii) *There exists* $e \in \partial K_1$ *such that* $x \neq Tx + me$ *for any* $x \in \partial K_r$ *and* $m > 0$.

Then T has a fixed point in $\overline{K}_{r,R}$.

Remark 2. If (i) and (ii) are satisfied for $x \in \partial K_r$ and $x \in \partial K_R$, respectively, then Theorem 1 is still true.

The following hypotheses will be used in the sequel:

(H1) Let $h(t) \in C(0,1)$ be a given nonnegative function such that $\int_0^1 (1-s)^{\alpha-4} h(s)\,\mathrm{d}s$ exists.
(H2) The nonnegative functions g, k in (2) are in $L^1[0,1]$ such that

$$0 \leqslant \nu := \int_0^1 g(\tau)\,\mathrm{d}\tau < 1 \quad \text{and} \quad 0 \leqslant \int_0^1 k(\tau)\,\mathrm{d}\tau < 1.$$

Lemma 2. (*See* [23].) *Assume that* (H1) *holds. Then the boundary value problem*

$$\begin{aligned} &D^{\alpha-2} y(t) = h(t), \quad 0 < t < 1,\ 3 < \alpha \leqslant 4, \\ &y(0) + y'(0) = 0, \qquad y(1) + y'(1) = 0 \end{aligned} \tag{3}$$

has a unique solution

$$y(t) = \int_0^1 G_\alpha(t,s) h(s)\,\mathrm{d}s,$$

where

$$G_\alpha(t,s) = \begin{cases} \frac{(1-s)^{\alpha-3}(1-t)+(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{(1-s)^{\alpha-4}(1-t)}{\Gamma(\alpha-3)}, & 0 < s \leqslant t < 1, \\ \frac{(1-s)^{\alpha-3}(1-t)}{\Gamma(\alpha-2)} + \frac{(1-s)^{\alpha-4}(1-t)}{\Gamma(\alpha-3)}, & 0 < t \leqslant s < 1. \end{cases} \tag{4}$$

Lemma 3. *Assume that* (H1) *and* (H2) *hold, then the integral boundary-value problem*

$$\begin{aligned} &D^{\alpha-2} y(t) = h(t), \quad 0 < t < 1,\ 3 < \alpha \leqslant 4, \\ &y(0) + y'(0) = \int_0^1 g(\tau) y(\tau)\,\mathrm{d}\tau, \qquad y(1) + y'(1) = \int_0^1 g(\tau) y(\tau)\,\mathrm{d}\tau \end{aligned} \tag{5}$$

has a unique solution

$$y(t) = \int_0^1 G_2(t,s) h(s)\,\mathrm{d}s,$$

where

$$G_2(t,s) = G_\alpha(t,s) + H_\alpha(t,s) \tag{6}$$

such that

$$H_\alpha(t,s) = \frac{1}{1-\nu}\int_0^1 g(\tau)G_\alpha(\tau,s)\,\mathrm{d}\tau, \tag{7}$$

ν *defined in* (H2)*, and* $G_\alpha(t,s)$ *is given by* (4)*.*

Proof. Let $w(t) = \int_0^1 G_\alpha(t,s)h(s)\,\mathrm{d}s$ and satisfies

$$D^{\alpha-2}w(t) = h(t), \qquad w(0) + w'(0) = 0, \qquad w(1) + w'(1) = 0.$$

Let $y(t)$ be a solution of (5), $z(t) = y(t) - w(t)$,

$$\begin{aligned}
&D^{\alpha-2}z(t) = D^{\alpha-2}y(t) - D^{\alpha-2}w(t) = h(t) - h(t) = 0,\\
&z(0) + z'(0) = \int_0^1 g(\tau)y(\tau)\,\mathrm{d}\tau = \int_0^1 g(\tau)z(\tau)\,\mathrm{d}\tau + \int_0^1 g(\tau)w(\tau)\,\mathrm{d}\tau,\\
&z(1) + z'(1) = \int_0^1 g(\tau)y(\tau)\,\mathrm{d}\tau = \int_0^1 g(\tau)z(\tau)\,\mathrm{d}\tau + \int_0^1 g(\tau)w(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{8}$$

Also, $D^{\alpha-2}z(t) = 0$ implies $z(t) = c_0 + c_1 t$. Substituting $z(t)$ in (8), we obtain that

$$\begin{aligned}
c_0 + c_1 &= \int_0^1 g(\tau)(c_0 + c_1\tau)\,\mathrm{d}\tau + \int_0^1 g(\tau)w(\tau)\,\mathrm{d}\tau,\\
c_0 + 2c_1 &= \int_0^1 g(\tau)(c_0 + c_1\tau)\,\mathrm{d}\tau + \int_0^1 g(\tau)w(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{9}$$

By solving the above system, we obtain $c_1 = 0$,

$$z(t) = c_0 = \frac{\int_0^1 g(\tau)w(\tau)\,\mathrm{d}\tau}{1-\nu}.$$

Now,

$$\begin{aligned}
y(t) = z(t) + w(t) &= \frac{1}{1-\nu}\int_0^1 g(\tau)w(\tau)\,\mathrm{d}\tau + \int_0^1 G_\alpha(t,s)h(s)\,\mathrm{d}s\\
&= \frac{1}{1-\nu}\int_0^1 g(\tau)\left(\int_0^1 G_\alpha(\tau,s)h(s)\,\mathrm{d}s\right)\mathrm{d}\tau + \int_0^1 G_\alpha(t,s)h(s)\,\mathrm{d}s.
\end{aligned}$$

Then $y(t) = \int_0^1 G_2(t,s)h(s)\,\mathrm{d}s$ such that $G_2(t,s) = G_\alpha(t,s) + H_\alpha(t,s)$ and

$$H_\alpha(t,s) = \frac{1}{1-\nu}\int_0^1 g(\tau)G_\alpha(\tau,s)\,\mathrm{d}\tau. \qquad \square$$

Lemma 4. (*See* [26].) *Assume that* (H2) *holds. Then the boundary value problem*

$$\begin{aligned} -u''(t) &= y(t), \quad 0<t<1,\\ u(0) = u(1) &= \int_0^1 k(\tau)u(\tau)\,\mathrm{d}\tau \end{aligned} \tag{10}$$

has a unique solution u *given by*

$$u(t) = \int_0^1 H_1(t,\tau)y(\tau)\,\mathrm{d}\tau, \tag{11}$$

where

$$H_1(t,\tau) = G_1(t,\tau) + \frac{1}{1-\int_0^1 k(\tau)\,\mathrm{d}\tau}\int_0^1 G_1(\tau,\xi)k(\xi)d\xi, \tag{12}$$

$$G_1(t,\tau) = \begin{cases} \tau(1-t), & 0<\tau\leqslant t<1,\\ t(1-\tau), & 0<t\leqslant\tau<1. \end{cases} \tag{13}$$

Lemma 5. *Suppose that* (H1) *and* (H2) *hold. Then the integral boundary-value problem*

$$\begin{aligned} &D^\alpha u(t) = -h(t), \quad 0<t<1,\ 3<\alpha\leqslant 4,\\ &u(0) = u(1) = \int_0^1 k(\tau)u(\tau)\,\mathrm{d}\tau,\\ &u''(0) + u'''(0) = \int_0^1 g(\tau)u''(\tau)\,\mathrm{d}\tau,\\ &u''(1) + u'''(1) = \int_0^1 g(\tau)u''(\tau)\,\mathrm{d}\tau \end{aligned} \tag{14}$$

has a unique solution

$$u(t) = \int_0^1 H_1(t,\tau)\left(\int_0^1 G_2(\tau,s)h(s)\,\mathrm{d}s\right)\mathrm{d}\tau,$$

where $G_2(\tau,s), H_1(t,\tau)$ *defined in* (6), (12)*, respectively.*

Proof. Let $y(t) = -u''(t)$. Then $y(t)$ satisfies that

$$D^{\alpha-2}y(t) = h(t), \quad 0 \leqslant t \leqslant 1,$$
$$y(0) + y'(0) = \int_0^1 g(\tau)y(\tau)\,\mathrm{d}\tau, \qquad y(1) + y'(1) = \int_0^1 g(\tau)y(\tau)\,\mathrm{d}\tau.$$

Then, by Lemma 4, the boundary value problem

$$-u''(t) = y(t), \quad 0 < t < 1,$$
$$u(0) = u(1) = \int_0^1 k(\tau)u(\tau)\,\mathrm{d}\tau \tag{15}$$

has a unique solution

$$u(t) = \int_0^1 H_1(t,\tau)y(\tau)\,\mathrm{d}\tau, \tag{16}$$

and, by Lemma 3, we have

$$y(t) = \int_0^1 G_2(t,s)h(s)\,\mathrm{d}s,$$

so,

$$u(t) = \int_0^1 H_1(t,\tau)\left(\int_0^1 G_2(\tau,s)h(s)\,\mathrm{d}s\right)\mathrm{d}\tau. \qquad \square$$

Now, we obtain the properties of the Green's functions.

Lemma 6. (*See* [26].) *Let* (H2) *holds. Then*

$$\rho G_1(\tau,\tau) \leqslant H_1(t,\tau) \leqslant \gamma G_1(\tau,\tau) \leqslant \frac{\gamma}{4} \quad \forall t,\tau \in (0,1),$$

where

$$\rho = \frac{\int_0^1 G_1(s,s)h(s)\,\mathrm{d}s}{1 - \int_0^1 k(\tau)\,\mathrm{d}\tau}, \qquad \gamma = \frac{1}{1 - \int_0^1 k(\tau)\,\mathrm{d}\tau}.$$

Lemma 7. (*See* [23].) *Let* $3 < \alpha \leqslant 4$. *Then the Green's functions* $G_\alpha(t,s)$, $H_\alpha(t,s)$ *defined by* (4), (7) *have the following properties*:

(P1) $G_\alpha(t,s) \in C([0,1]\times[0,1])$, $G_\alpha(t,s) > 0$ *for* $t,s \in (0,1)$.
(P2) $G_\alpha(t,s) \geqslant ((1-t)/2)M_\alpha(s)$, $\max_{0\leqslant t\leqslant 1} G_\alpha(t,s) \leqslant M_\alpha(s)$, *where* $M_\alpha(s) = 2(1-s)^{\alpha-3}/\Gamma(\alpha-2) + (1-s)^{\alpha-4}/\Gamma(\alpha-3)$, $s \in [0,1)$.
(P3) *By* (H2), *we have* $H_\alpha(t,s) > 0$ *for all* $s \in [0,1]$.

Lemma 8. *Let* $3 < \alpha \leqslant 4$. *Then the Green's function* $G_2(t,s)$, *defined by* (6), *has the following properties*:

(R1) $G_2(t,s) \in C([0,1]\times[0,1))$, $G_2(t,s) > 0$ *for* $t,s \in (0,1)$.
(R2) $G_2(t,s) \geqslant ((1-t)/2)M_\alpha(s)$, $\max_{0\leqslant t\leqslant 1} G_2(t,s) \leqslant (1/(1-\nu))M_\alpha(s)$,

Proof. Clearly, property (R1) holds. For (R2), from property (P2) for $t \in [0,1]$, $s \in [0,1)$ we have

$$\begin{aligned} H_\alpha(t,s) &= \frac{1}{1-\nu}\int_0^1 g(\tau)G_\alpha(\tau,s)\,\mathrm{d}\tau \leqslant \frac{1}{1-\nu}\int_0^1 g(\tau)M_\alpha(s)\,\mathrm{d}\tau \\ &= \frac{\nu}{1-\nu}M_\alpha(s). \end{aligned}$$

So,

$$\begin{aligned} G_2(t,s) &= G_\alpha(t,s) + H_\alpha(t,s) \leqslant M_\alpha(s) + \frac{\nu}{1-\nu}M_\alpha(s) \\ &= \frac{1}{1-\nu}M_\alpha(s). \end{aligned}$$

Also,

$$\begin{aligned} G_2(t,s) &= G_\alpha(t,s) + H_\alpha(t,s) \\ &\geqslant \frac{1-t}{2}M_\alpha(s) + \frac{1}{1-\nu}\int_0^1 \frac{1-\tau}{2}g(\tau)M_\alpha(s)\,\mathrm{d}\tau \\ &\geqslant \frac{1-t}{2}M_\alpha(s). \end{aligned}$$

□

Taking into account (15) and (11), problem (2) reduces to the following problems:

$$\begin{gathered} D^{\alpha-2}y(t) = \lambda f\left(t, \int_0^1 H_1(t,\tau)y(\tau)\,\mathrm{d}\tau, -y(t)\right), \\ 0 \leqslant t \leqslant 1,\ 3 < \alpha \leqslant 4, \\ y(0) + y'(0) = y(1) + y'(1) = \int_0^1 g(\tau)y(\tau)\,\mathrm{d}\tau. \end{gathered} \tag{17}$$

If y^* is the solution of (17), then

$$u(t) = \int_0^1 H_1(t,\tau)y^*(\tau)\,\mathrm{d}\tau$$

is the solution of (2). So, the existence of a solution of problem (2) follows from the existence of a solution of problem (17).

Now, we define an integral operator $S : C[0,1] \to C[0,1]$ by

$$Sy(t) = \int_0^1 H_1(t,\tau) y(\tau)\,\mathrm{d}\tau.$$

Lemma 9. *The boundary value problem* (2) *has a positive solution if and only if the following integral-differential boundary value problem has a positive solution*:

$$\begin{aligned} &D^{\alpha-2} y(t) = \lambda f\bigl(t, Sy(t), -y(t)\bigr), \quad 0 < t < 1,\ 3 < \alpha \leqslant 4, \\ &y(0) + y'(0) = y(1) + y'(1) = \int_0^1 g(\tau) y(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{18}$$

Proof. If u is a positive solution (2), let $u = Sy$, then $y = -u''$. This implies $u'' = -y$ is a solution of (18). Conversely, if y is a positive solution of (18), let $u = Sy$, $u'' = -y$. Thus, $u = Sy$ is a positive solution of (2). □

Assume the following assumptions hold:

(H3) $f \in C((0,1) \times (0,\infty) \times (-\infty,0), [0,\infty))$ and $0 < \int_0^1 M_\alpha(s)\,\mathrm{d}s < \infty$.

(H4) For any $0 < r < R < \infty$,

$$\lim_{n\to\infty} \sup_{u,-v \in \overline{K}_{r,R}} \frac{1}{1-\nu} \int_{D_n} M_\alpha(s) f\bigl(s, u(s), -v(s)\bigr)\,\mathrm{d}s = 0,$$

where $D_n = [0, 1/n] \cup [(n-1)/n, 1]$, and put

$$Q = \frac{1}{1-\nu} \int_0^1 M_\alpha(s)\,\mathrm{d}s, \qquad l = \min_{t\in[a,b]\subset(0,1)} \int_0^1 G_2(t,s)\,\mathrm{d}s.$$

Note that, by (H3), there exist $a, b \in (0,1)$ such that

$$0 < \int_a^b M_\alpha(s)\,\mathrm{d}s < \infty,$$

then

$$\min_{t\in[a,b]\subset(0,1)} \int_0^1 G_2(t,s)\,\mathrm{d}s \leqslant \frac{1}{1-\nu} \int_0^1 M_\alpha(s)\,\mathrm{d}s < \infty.$$

Now, we concentrate our study on (18). Let

$$C^+[0,1] = \bigl\{x \in C[0,1]\colon\ x \geqslant 0\bigr\},$$

$$K = \Bigl\{x \in C^+[0,1]\colon\ x(t) \text{ is a concave function},\ \min_{t\in[a,b]\subset(0,1)} x(t) \geqslant \omega(1-\nu)\|x\|\Bigr\},$$

where $\omega = \min_{t\in[a,b]\subset(0,1)}(1-t)/2$, $\|x\| = \sup_{t\in[0,1]}|x(t)|$ for $x(t) \in C[0,1]$. It is easy to see that K is a cone in $C[0,1]$ and $\overline{K}_{r,R} \subset K \subset C^+[0,1]$. Now we define an operator $T : K \setminus \{0\} \to C^+[0,1]$ by

$$(Ty)(t) = \lambda \int_0^1 G_2(t,s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1].$$

Clearly, y is a solution of problem (18) if and only if y is a fixed point of the operator T.

Lemma 10. *Suppose that* (H2)–(H4) *hold. Then* $T : \overline{K}_{r,R} \to C^+[0,1]$ *is a completely continuous operator. Moreover* $T(\overline{K}_{r,R}) \subset K$.

Proof. We prove that, for any $r > 0$,

$$\sup_{y\in\partial K_r} \frac{\lambda}{1-\nu} \int_0^1 M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s < \infty. \tag{19}$$

Also, it shows that $T : K \setminus \{0\} \to C^+[0,1]$ is well defined.

By (H4), for any $r > 0$, there exists a $m \in \mathbb{N}$ such that

$$\sup_{y\in\partial K_r} \frac{\lambda}{1-\nu} \int_{D_m} M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s < 1.$$

For any $y \in \partial K_r$, let $y(t_0) = \max_{t\in[0,1]}|y(t)| = r$. By the concavity of $y(t)$ on $[0,1]$,

$$y(t) \geqslant \begin{cases} \frac{rt}{t_0}, & 0 \leqslant t \leqslant t_0, \\ \frac{r(1-t)}{1-t_0}, & t_0 \leqslant t \leqslant 1. \end{cases} \tag{20}$$

Then we get

$$y(t) \geqslant \begin{cases} rt, & 0 \leqslant t \leqslant t_0, \\ r(1-t), & t_0 \leqslant t \leqslant 1. \end{cases} \tag{21}$$

So, from (21) and Lemma 6, for any $t \in [1/m, (m-1)/m]$, we have $r/m \leqslant y(t) \leqslant r$ and

$$\begin{aligned} \frac{l_m r}{m} &= \frac{r}{m} \min_{t\in[1/m,(m-1)/m]} \int_0^1 H_1(t,s)\,\mathrm{d}s \leqslant Sy(t) \\ &\leqslant r \max_{t\in[1/m,(m-1)/m]} \int_0^1 H_1(t,s)\,\mathrm{d}s \leqslant \frac{\gamma}{4} r, \end{aligned}$$

where $l_m = \min_{t\in[1/m,(m-1)/m]} \int_0^1 H_1(t,s)\,\mathrm{d}s$.

Let

$$M_1 = \max\left\{f(t,x,y)\colon\ (t,x,y) \in \left[\frac{1}{m}, \frac{m-1}{m}\right] \times \left[\frac{l_m r}{m}, \frac{\gamma r}{4}\right] \times \left[-r, \frac{-r}{m}\right]\right\}.$$

By (H2)–(H4), we have

$$\begin{aligned}
&\sup_{y\in\partial K_r} \frac{\lambda}{1-\nu}\int_0^1 M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\quad\leqslant \sup_{y\in\partial K_r} \frac{\lambda}{1-\nu}\int_{D_m} M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\qquad + \sup_{y\in\partial K_r} \frac{\lambda}{1-\nu}\int_{1/m}^{(m-1)/m} M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\quad\leqslant 1 + M_1\frac{\lambda}{1-\nu}\int_0^1 M_\alpha(s)\,\mathrm{d}s < \infty.
\end{aligned}$$

This also implies $T(B)$ is uniformly bounded for any bounded set $B \subset \overline{K}_{r,R}$.

By standard proof, T is equicontinuous, and then, by the Arzela–Ascoli theorem, $T : \overline{K}_{r,R} \to C^+[0,1]$ is compact. Also, this is easily verified that T is continuous, so, we omit the proof. Thus, T is completely continuous. Now, we show that $T(\overline{K}_{r,R}) \subset K$. For any $y \in \overline{K}_{r,R}$, $t \in [0,1]$, we have

$$\begin{aligned}
(Ty)(t) &= \lambda\int_0^1 G_2(t,s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\leqslant \frac{\lambda}{1-\nu}\int_0^1 M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s.
\end{aligned}$$

Thus,

$$\|Ty\| \leqslant \frac{\lambda}{1-\nu}\int_0^1 M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s.$$

On the other hand, we have

$$\begin{aligned}
\min_{t\in[a,b]}(Ty)(t) &= \min_{t\in[a,b]} \lambda\int_0^1 G_2(t,s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\geqslant \lambda\omega\int_0^1 M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s.
\end{aligned}$$

This implies that $\min_{t\in[a,b]}(Ty)(t) \geqslant \omega(1-\nu)\|Ty\|$. Besides, it is clear that Ty is concave on $[0,1]$. Thus, $Ty \in K$ and, consequently, $T(\overline{K}_{r,R}) \subset K$. $\square$

3 Main result

Theorem 2. *Suppose that* (H1)–(H4) *and the following condition holds*:

(H5) $$0 \leqslant f^0 = \limsup_{|x|+|y|\to 0\,(x>0,\,y<0)} \max_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|} < L^{-1},$$
$$0 < l^{-1} < f_\infty = \liminf_{|x|+|y|\to \infty\,(x>0,\,y<0)} \min_{t\in[a,b]} \frac{f(t,x,y)}{|x|+|y|} \leqslant \infty.$$

Then problem (2) *has at least one positive solution for*

$$\lambda \in \left(\frac{1}{lf_\infty}, \frac{1}{Lf^0}\right), \tag{22}$$

where Q, l are defined by (H4), *and* $L := (\gamma/4+1)Q$.

Proof. We establish assumptions (i), (ii) of Theorem 1. First, we show that $\|Ty\| \leqslant \|y\|$ for all $y \in \partial K_r$. Let λ satisfy (22), and $\epsilon > 0$ be chosen such that

$$f_\infty - \epsilon > 0, \qquad \frac{1}{(f_\infty - \epsilon)l} \leqslant \lambda \leqslant \frac{1}{(f^0+\epsilon)L}. \tag{23}$$

Next, by (H5) there exists $r_0 > 0$ such that

$$f(t,x,y) \leqslant \big(f^0+\epsilon\big)\big(|x|+|y|\big) \quad \forall t \in [0,1],\ 0<|x|+|y|<r_0,\ x>0,\ y<0, \tag{24}$$
$$\big|Sy(t)\big| \leqslant \int_0^1 \big|H_1(t,\tau)y(\tau)\big|\,\mathrm{d}\tau \leqslant \frac{\gamma}{4}\|y\|.$$

Then, by taking $r = r_0/(\gamma/4+1)$,

$$0 < \big|Sy(t)\big| + \big|y(t)\big| \leqslant \left(\frac{\gamma}{4}+1\right)\|y\| = \left(\frac{\gamma}{4}+1\right)r = r_0, \quad 0 \leqslant t \leqslant 1. \tag{25}$$

It follows from (24) and (25) that, for any $y \in \partial K_r$,

$$\begin{aligned}\|Ty\| &= \max_{t\in[0,1]} \lambda \int_0^1 G_2(t,s) f\big(s, Sy(s), -y(s)\big)\,\mathrm{d}s \\ &\leqslant \lambda \int_0^1 \frac{1}{1-\nu} M_\alpha(s) f\big(s, Sy(s), -y(s)\big)\,\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{\lambda}{1-\nu} \int_0^1 M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\leqslant \frac{\lambda}{1-\nu} \bigl(f^0+\epsilon\bigr) \int_0^1 M_\alpha(s)\bigl(\bigl|Sy(s)\bigr| + \bigl|y(s)\bigr|\bigr)\,\mathrm{d}s \\
&= \lambda\bigl(f^0+\epsilon\bigr)\frac{1}{1-\nu}\left(\frac{\gamma}{4}+1\right) r \int_0^1 M_\alpha(s)\,\mathrm{d}s \\
&= \lambda\bigl(f^0+\epsilon\bigr)\left(\frac{\gamma}{4}+1\right) rQ \leqslant r = \|y\|.
\end{aligned}$$

Thus, $\|Ty\| \leqslant \|y\|$ for all $y \in \partial K_r$.

Now, we establish the second condition of Theorem 1. For the above ϵ, by (H5), there exists $R_0 > 0$ such that

$$f(t,x,y) > (f_\infty - \epsilon)\bigl(|x|+|y|\bigr), \quad t \in [a,b],\ |x|+|y| \geqslant R_0,\ x>0,\ y<0.$$

Let $R = \max\{2r, \omega^{-1}R_0\}$ and $\phi(t) = 1$, $t \in [0,1]$. Then $R > r$ and $\phi(t) \in \partial K_1$. By contradiction, suppose that there exists $y_0 \in \partial K_R$ and $m_0 > 0$ such that $y_0 = Ty_0 + m_0\phi$. Let $\zeta = \min\{y_0(t)\colon t \in [a,b]\}$ for any $s \in [a,b]$,

$$\begin{aligned}
\bigl|Sy_0(s)\bigr| + \bigl|y_0(s)\bigr| &\geqslant \min_{s\in[a,b]} \bigl[\bigl|Sy_0(s)\bigr| + \bigl|y_0(s)\bigr|\bigr] \geqslant \min_{s\in[a,b]} \bigl|y_0(s)\bigr| \\
&\geqslant \omega\|y_0\| \geqslant \omega R \geqslant R_0.
\end{aligned}$$

Consequently, for any $t \in [a,b]$, we have

$$\begin{aligned}
y_0(t) &= \lambda \int_0^1 G_2(t,s) f\bigl(s, Sy_0(s), -y_0(s)\bigr)\,\mathrm{d}s + m_0\phi(t) \\
&\geqslant \lambda \int_a^b G_2(t,s) f\bigl(s, Sy_0(s), -y_0(s)\bigr)\,\mathrm{d}s + m_0 \\
&\geqslant \lambda(f_\infty - \epsilon) \int_a^b G_2(t,s)\bigl(\bigl|Sy_0(s)\bigr| + \bigl|y_0(s)\bigr|\bigr)\,\mathrm{d}s + m_0 \\
&\geqslant \lambda(f_\infty - \epsilon) \int_a^b G_2(t,s)\bigl|y_0(s)\bigr|\,\mathrm{d}s + m_0 \\
&\geqslant \lambda(f_\infty - \epsilon)\zeta \min_{t\in[a,b]} \int_a^b G_2(t,s)\,\mathrm{d}s + m_0 \\
&\geqslant \zeta + m_0 > \zeta.
\end{aligned}$$

This implies that $\zeta > \zeta$, which is a contradiction. By Theorem 1, it follows that T has a fixed point y^* with $r < |y^*| < R$. Thus, y^* is a positive solution of problem (18). So, the boundary value problem (2) has a positive solution. $\square$

Remark 3. Since $0 < l < L < \infty$, we obtain $0 < 1/(lf_\infty) < 1, 1/(Lf^0) > 1$. Thus, $1 \in (1/(lf_\infty), 1/(Lf^0))$, so, when $\lambda = 1$, Theorem 2 holds.

Remark 4.

(i) If $f_\infty = \infty$, $f^0 > 0$, then Theorem 2 holds for each $\lambda \in (0, 1/(Lf^0))$.
(ii) If $f_\infty = \infty$, $f^0 = 0$, then Theorem 2 holds for each $\lambda \in (0, \infty)$.
(iii) If $f_\infty > l^{-1} > 0$, $f^0 = 0$, then Theorem 2 holds for each $\lambda \in (1/(lf_\infty), \infty)$.

Theorem 3. *Suppose that* (H1)–(H4) *and the following condition holds*:

(H6)
$$0 \leqslant f^\infty = \limsup_{|x|+|y|\to\infty\,(x>0,\,y<0)} \max_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|} < L^{-1},$$
$$0 < l^{-1} < f_0 = \liminf_{|x|+|y|\to 0\,(x>0,\,y<0)} \min_{t\in[a,b]} \frac{f(t,x,y)}{|x|+|y|} \leqslant \infty.$$

Then problem (2) *has at least one positive solution for any*

$$\lambda \in \left(\frac{1}{lf_0}, \frac{1}{Lf^\infty}\right), \tag{26}$$

where Q, l are defined by (H4)*, and* $L := (\gamma/4 + 1)Q$.

Proof. By similar argument as in proof of Theorem 2 and [12], we establish conditions (i) and (ii) of Theorem 1. Let λ satisfy (26), and $\epsilon_1 > 0$ be chosen such that $L^{-1} - \epsilon_1 > 0$ and $\lambda F^\infty < L^{-1} - \epsilon_1$. By (H6), there exists $(\gamma/4)R'_0$ such that

$$f(t,x,y) \leqslant \frac{1}{\lambda}\big(L^{-1} - \epsilon_1\big)\big(|x| + |y|\big) \quad \forall t \in [0,1],\ |x| + |y| > \frac{\gamma}{4}R'_0,\ x > 0,\ y < 0.$$

Let

$$M_0 = \sup_{y\in\partial K_{R'_0}} \frac{\lambda}{1-\nu} \int_0^1 M_\alpha(s) f\big(s, Sy(s), -y(s)\big)\,\mathrm{d}s.$$

Then $M_0 < \infty$ by (19). Take $R_1 > \max\{R'_0, M_0/(L\epsilon_1)\}$, then $M_0 < LR_1\epsilon_1$.

For $u \in \partial K_{R'_0}$,

$$u(t) \leqslant \|u\| = R'_0, \qquad Su(t) \leqslant \max_{t\in[0,1]} \int_0^1 H_1(t,\tau)\,\mathrm{d}\tau \|u\| = \frac{\gamma}{4}R'_0.$$

So, for any $y \in \partial K_{R_1}$, let

$$D(Sy, -y) = \left\{t \in [0,1]\colon (Sy, -y) \in \left[\frac{\gamma}{4}R'_0, \infty\right) \times (-\infty, -R'_0]\right\},$$

then, for any $t \in D(Sy, -y)$,

$$\frac{\gamma}{4}R_0' \leqslant |Sy| + |y| \leqslant \left(\frac{\gamma}{4} + 1\right)\|y\| = \left(\frac{\gamma}{4} + 1\right)R_1.$$

In addition, for any $y \in \partial K_{R_1}$, let $y_1(t) = \min\{y(t), R_0'\}$, then $y_1 \in \partial K_{R_0'}$. Thus, for any $y \in \partial K_{R_1}$, we have

$$\begin{aligned}
\|Ty\| &= \max_{t\in[0,1]} \frac{\lambda}{1-\nu} \int_0^1 G_2(t,s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\leqslant \frac{\lambda}{1-\nu} \int_{D(Sy,-y)} G_2(t,s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\quad + \frac{\lambda}{1-\nu} \int_{[0,1]\setminus D(Sy,-y)} M_\alpha(s) f\bigl(s, Sy(s), -y(s)\bigr)\,\mathrm{d}s \\
&\leqslant \frac{1}{\lambda}\bigl(L^{-1} - \epsilon_1\bigr)\frac{\lambda}{1-\nu} \int_0^1 M_\alpha(s)\bigl(\bigl|Sy(s)\bigr| + \bigl|y(s)\bigr|\bigr)\,\mathrm{d}s \\
&\quad + \frac{\lambda}{1-\nu} \int_0^1 M_\alpha(s) f\bigl(s, Sy_1(s), -y_1(s)\bigr)\,\mathrm{d}s \\
&\leqslant \bigl(L^{-1} - \epsilon_1\bigr)\left(\frac{\gamma}{4} + 1\right)R_1 \int_0^1 M_\alpha(s)\,\mathrm{d}s + M_0 \\
&\leqslant \bigl(L^{-1} - \epsilon_1\bigr)\left(\frac{\gamma}{4} + 1\right)R_1 Q + M_0 \\
&< R_1 = \|y\|.
\end{aligned}$$

Thus, $\|Ty\| \leqslant \|y\|$ for all $y \in \partial K_{R_1}$.

Next, let λ satisfy (26). Choose $\epsilon_2 > 0$ such that $l^{-1} + \epsilon_2 < \lambda f_0$. Then from (H6) there exists $0 < \delta < (\gamma/4 + 1)R_1$ such that

$$f(t,x,y) > \frac{1}{\lambda}\bigl(l^{-1} + \epsilon_2\bigr)\bigl(|x| + |y|\bigr), \quad t \in [a,b],\ 0 < |x| + |y| \leqslant \delta,\ x > 0,\ y < 0.$$

Let $r_1 = \delta/(\gamma/4 + 1)$ and $\phi(t) = 1$, $t \in [0,1]$. Then $R_1 > r_1$ and $\phi(t) \in \partial K_1$. By contradiction, suppose that there exists $y_0 \in \partial K_R$ and $m_0 > 0$ such that $y_0 = Ty_0 + m_0\phi$. Let $\eta = \min\{y_0(t)\colon t \in [a,b]\}$ and notice that, for any $s \in [a,b]$,

$$\bigl|Sy_0(s)\bigr| + \bigl|y_0(s)\bigr| < \left(\frac{\gamma}{4} + 1\right)r_1 = \delta.$$

Consequently, for any $t \in [a, b]$, we have

$$
\begin{aligned}
y_0(t) &= \lambda \int_0^1 G_2(t,s) f\bigl(s, Sy_0(s), -y_0(s)\bigr)\,\mathrm{d}s + m_0\phi(t) \\
&\geqslant \lambda \int_a^b G_2(t,s) f\bigl(s, Sy_0(s), -y_0(s)\bigr)\,\mathrm{d}s + m_0 \\
&\geqslant \frac{1}{\lambda}\bigl(l^{-1} + \epsilon_2\bigr)\lambda \int_a^b G_2(t,s)\bigl(\bigl|Sy_0(s)\bigr| + \bigl|y_0(s)\bigr|\bigr)\,\mathrm{d}s + m_0 \\
&\geqslant \bigl(l^{-1} + \epsilon_2\bigr) \int_a^b G_2(t,s)\bigl|y_0(s)\bigr|\,\mathrm{d}s + m_0 \\
&\geqslant \bigl(l^{-1} + \epsilon_2\bigr)\eta \min_{t\in[a,b]} \int_a^b G_2(t,s)\,\mathrm{d}s + m_0 \\
&\geqslant \bigl(1 + l\epsilon_2\bigr)\eta + m_0 > \eta.
\end{aligned}
$$

This implies that $\eta > \eta$, which is a contradiction. By Theorem 1, it follows that T has a fixed point y^{**} with $r < |y^{**}| < R$. Thus, y^{**} is a positive solution of the problem (18). So, the boundary value problem (2) has a positive solution. $\square$

Remark 5.

(i) If $f^\infty < L^{-1}$, $f^0 = \infty$, then Theorem 3 holds for each $\lambda \in (0, 1/(Lf^\infty))$.
(ii) If $f^\infty = 0$, $f_0 = \infty$, then Theorem 3 holds for each $\lambda \in (0, \infty)$.
(iii) If $f^\infty = 0$, $f_0 > l^{-1} > 0$, then Theorem 3 holds for each $\lambda \in (1/(lf_0), \infty)$.

Example 1. Consider the following boundary value problem:

$$
\begin{aligned}
&D^{3.5}u(t) + \lambda\biggl[t\tan\biggl[\frac{1}{10}\bigl(u(t) - u''(t)\bigr)\biggr]^2 \\
&\qquad + \frac{(t+3)(u(t)-u''(t))^{5/3}}{u(t)-u''(t)+10}\biggr]^{3/2} = 0, \quad t \in [0,1], \\
&u(0) = u(1) = \int_0^1 \tau u(\tau)\,\mathrm{d}\tau, \\
&u''(0) + u'''(0) = \int_0^1 \frac{1}{2}u''(\tau)\,\mathrm{d}\tau, \\
&u''(1) + u'''(1) = \int_0^1 \frac{1}{2}u''(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{27}
$$

Here $k(\tau) = \tau$, $g(\tau) = 1/2$, and

$$f(t,x,y) = \left[t \tan\left[\frac{1}{10}(x-y)\right]^2 + \frac{(t+3)(x-y)^{5/3}}{x-y+10}\right]^{3/2}.$$

Also,

$$M_\alpha(s) = \frac{2(1-s)^{1/2}}{\Gamma(\frac{3}{2})} + \frac{(1-s)^{-1/2}}{\Gamma(\frac{1}{2})}.$$

Then

$$Q = \frac{1}{1 - \int_0^1 g(\tau)\,\mathrm{d}\tau} \int_0^1 M_\alpha(s)\,\mathrm{d}s = 5.26$$

and

$$L = \left(\frac{\gamma}{4} + 1\right)Q = 7.89.$$

On the other hand,

$$l = \min_{t\in[0,1]} \int_0^1 G_2(t,s)\,\mathrm{d}s = 1.27.$$

By easy calculation, we have

$$0 \leqslant f^0 = 0.1 < 0.126 = \frac{1}{L}, \qquad 0 < \frac{1}{l} = 0.78 < f_\infty = 8 \leqslant \infty.$$

Thus, according to Theorem 2, if

$$0.098 \approx= \frac{1}{1.27\cdot 8} < \lambda < \frac{1}{7.89\cdot 0.1} \approx 1.26,$$

then the boundary value problem (27) has at least positive solution.

Acknowledgment. The authors would like to thank the referee for giving useful suggestions and comments for the improvement of this paper.

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