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Iterative unique positive solutions for singular p-Laplacian fractional differential equation system with several parameters*

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Abstract. By using the method of reducing the order of a derivative, the higher-order fractional differential equation is transformed into the lower-order fractional differential equation and combined with the mixed monotone operator, a unique positive solution is obtained in this paper for a singular p-Laplacian boundary value system with the Riemann–Stieltjes integral boundary conditions. This equation system is very wide because there are many parameters, which can be changeable in the equation system in this paper, and the nonlinearity is allowed to be singular in regard to not only the time variable but also the space variable. Moreover, the unique positive solution that we obtained in this paper is dependent on λ, and an iterative sequence and convergence rate are given, which are important for practical application. An example is given to demonstrate the application of our main results.

Keywords: fractional differential equation system, singular p-Laplacian, integral boundary condition; iterative positive solution, mixed monotone operator.

1 Introduction

During the last decades, boundary value problems for nonlinear fractional differential equations have gained its popularity and significance due to its distinguished applications

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as valuable tools in different areas of applied different areas such as physics, chemistry, electrical networks, economics, rheology, biology chemical, image processing, and so on. Fractional calculus have been shown to be more accurate and realistic than integer-order models, and it also provides an excellent tool to describe the hereditary properties of material and processes, particularly in viscoelasticity, electrochemistry, porous media, and so on. There has been a significant development in the study of fractional differential equations in recent years. For an extensive collection of such literature, readers can refer to [4,6–8,10–13,15,17,18,21–25,27–32], and there are a lot of methods to study differential equations such as degree theory (see [14]), mixed monotone operator (see [8, 13, 15, 29]), bifurcation method (see [14,20]), spectral analysis (see [2,19,27,31]), and so on. For some differential equation in which fractional derivatives are involved in the nonlinear terms, reader can refer to [6–8,28,29,31]. In order to meet the needs, the p-Laplacian equation is introduced in some boundary value problems. Fractional differential equation system of p-Laplacian with the Riemann–Stieltjes integral boundary conditions is a type of equation system that is very wide, and the general equation systems are special cases of p-Laplacian equation system. We refer the reader to [3, 9, 12, 16, 20, 26, 27] for some relevant work. In [27], the authors considered the following fractional differential equation:

$$
\begin{gathered}
-D_t^\beta\bigl(\varphi_p\bigl(D_t^\alpha x\bigr)\bigr)(t) = \lambda f\bigl(t, x(t)\bigr), \quad 0 < t < 1, \\
x(0) = 0, \quad D_{0+}^\alpha x(0) = 0, \qquad x(1) = \int\limits_0^1 x(s)\,\mathrm{d}A(s),
\end{gathered}
$$

where $\alpha, \beta \in \mathbb{R}_+ = [0, +\infty)$, $0 < \beta \leqslant 1$, $1 < \alpha \leqslant 2$, $\lambda > 0$ is a parameter, A is a function of bounded variation, $\int_0^1 x(s)\,\mathrm{d}A(s)$ denotes the Riemann–Stieltjes integral of x with respect to A, $f(t,x) : (0,1) \times (0,+\infty) \to (0,+\infty)$ is continuous and may be singular at $t = 0, 1$ and $x = 0$. $D_{t+}^\alpha, D_{t+}^\beta$ are the Riemann–Liouville differential fractional derivatives of order α, β, and the p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. The authors obtained the existence of positive solution by the upper and lower solutions and Schauder fixed-point theorems. In [12], the authors discussed a Hadamard fractional differential equation boundary value problem with p-Laplacian operator

$$
\begin{aligned}
&-D_t^\beta\bigl(\varphi_p\bigl(D_t^\alpha x\bigr)\bigr)(t) = f\bigl(t, x(t)\bigr), \quad 1 < t < \mathrm{e}, \\
&x(1) = x'(1) = x'(\mathrm{e}) = 0, \qquad D_{0+}^\alpha x(1) = D_{0+}^\alpha x(\mathrm{e}) = 0,
\end{aligned}
$$

where $\alpha, \beta \in \mathbb{R}_+$, $2 < \alpha \leqslant 3$, $1 < \beta \leqslant 2$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, and $f\colon [1,\mathrm{e}] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a positive continuous function, $D_{t+}^\alpha, D_{t+}^\beta$ are the Riemann–Liouville differential fractional derivative of order α, β. The authors obtained the existence and the uniqueness of positive solutions by using the Leray–Schauder-type alternative and the Guo–Krasnoselskii fixed-point theorem. In [23], the authors investigated the singular problem

$$
\begin{aligned}
&-D_t^\alpha u(t) + \lambda f\bigl(t, u(t), D_t^\beta u(t), v(t)\bigr) = 0, \quad 0 < t < 1, \\
&-D_t^\gamma v(t) + \lambda g\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1,
\end{aligned}
$$

$$
\begin{aligned}
&D_t^{\beta} u(0) = D_t^{\beta+1} u(0) = 0, \qquad D_t^{\beta} u(1) = \int\limits_0^1 D_t^{\beta} u(s)\,\mathrm{d}A(s),\\
&v(0) = v'(0) = 0, \qquad v(1) = \int\limits_0^1 v(s)\,\mathrm{d}B(s),
\end{aligned}
$$

where $\alpha, \beta, \gamma \in R_+^1$, $2 < \alpha, \gamma \leqslant 3$, $0 < \beta < 1$, u denotes the number of uninfected $CD4^+T$ cells and v denotes the number of infected cells, $\lambda > 0$ is a parameter, $\alpha - \beta > 2$, $\int_0^1 D_t^{\beta} u(s)\,\mathrm{d}A(s)$ and $\int_0^1 v(s)\,\mathrm{d}B(s)$ denote the Riemann–Stieltjes integrals of u, v with respect to A and B, respectively, A, B are bounded variations, $f : (0,1) \times \mathbb{R}_+^3 \to \mathbb{R}$, $g : (0,1) \times \mathbb{R}_+ \to \mathbb{R}$ are two continuous functions and may be singular at $t = 0, 1$, D_t^{α}, D_t^{β}, D_t^{γ} are the standard Riemann–Liouville derivatives. The authors obtained the existence of positive solution by the fixed-point theorem.

Motivated by the excellent results above, in this paper, we will devote to considering the following singular p-Laplacian fractional differential equation (PFDE) with the Riemann–Stieltjes integral boundary conditions:

$$
\begin{aligned}
&D_{0^+}^{\alpha}\big(\varphi_p\big(D_{0^+}^{\gamma} u\big)\big)(t) + \lambda^{1/(q-1)} f\big(t, u(t), D_{0^+}^{\mu_1} u(t), D_{0^+}^{\mu_2} u(t), \dots,\\
&\quad D_{0^+}^{\mu_{n-1}} u(t), v(t)\big) = 0, \quad 0 < t < 1,\\
&D_{0^+}^{\beta}\big(\varphi_p\big(D_{0^+}^{\delta} v\big)\big)(t) + \mu^{1/(q-1)} g\big(t, u(t), D_{0^+}^{\eta_1} u(t), D_{0^+}^{\eta_2} u(t), \dots,\\
&\quad D_{0^+}^{\eta_{m-1}} u(t)\big) = 0, \quad 0 < t < 1,\\
&u(0) = D_{0^+}^{\mu_i} u(0) = 0, \quad D_{0^+}^{\gamma} u(0) = D_{0^+}^{\gamma+\mu_i} u(0) = 0, \quad i = 1, 2, \dots, n-2,\\
&D_{0^+}^{\mu_{n-1}}(1) = \chi \int\limits_0^{\eta} h(t) D_{0^+}^{\mu_{n-1}} u(t)\,\mathrm{d}A(t),\\
&v(0) = D_{0^+}^{\eta_i} v(0) = 0, \quad D_{0^+}^{\delta} v(0) = D_{0^+}^{\delta+\eta_i} v(0) = 0, \quad i = 1, 2, \dots, m-2,\\
&D_{0^+}^{\eta_{m-1}} v(1) = \iota \int\limits_0^{\vartheta} a(t) D_{0^+}^{\eta_{m-1}} v(t)\,\mathrm{d}B(t),
\end{aligned}
\tag{1}
$$

where α, β, γ, δ, μ_κ, $\eta_\varrho \in \mathbb{R}_+$ ($\kappa = 1, 2, \dots, n-1$; $\varrho = 1, 2, \dots, m-1$), $n, m \in \mathbb{N}$ (natural number set), $n, m \geqslant 2$, and $1/2 < \alpha, \beta \leqslant 1$, $n - 1 < \gamma \leqslant n$, $m - 1 < \delta \leqslant m$, p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p, q > 1$, $1/p + 1/q = 1$, $n - \kappa < \gamma - \mu_\kappa \leqslant n + 1 - \kappa$, $m - \varrho < \delta - \eta_\varrho \leqslant m + 1 - \varrho$ ($\kappa = 1, 2, \dots, n-1$; $\varrho = 1, 2, \dots, m-1$), $\eta_i \leqslant \mu_{n-1}$ ($i = 1, 2, \dots, n-1$) and $0 < \eta$, $\vartheta \leqslant 1$, $\lambda, \mu, \chi, \iota > 0$ are parameters, $f \in C((0,1) \times (0, +\infty)^{n+1}, \mathbb{R}_+)$, and $f(t, x_1, x_2, \dots, x_{n+1})$ has singularity at $x_i = 0$ ($i = 1, 2, \dots, n+1$). $t = 0, 1$, $g \in C((0,1) \times \mathbb{R}_+^m, \mathbb{R}_+)$, $h, a \in C(0,1)$, A, B are functions of bounded variation, $\int_0^{\eta} h(t) D_{0^+}^{\mu_{n-1}} u(t)\,\mathrm{d}A(t)$, $\int_0^{\vartheta} a(t) D_{0^+}^{\eta_{m-1}} v(t)\,\mathrm{d}B(t)$ denote the Riemann–Stieltjes integral with respect to A and B,

$D^{\alpha}_{0+}u$, $D^{\beta}_{0+}v$, $D^{\gamma}_{0+}u$, $D^{\delta}_{0+}v$, $D^{\mu_\kappa}_{0+}u$, $D^{\eta_\varrho}_{0+}u$ are the standard Riemann–Liouville derivative. In this paper, the existence of positive solutions is obtained by means of mixed monotone operator in cones.

In this paper, we investigate the existence of positive solutions for a singular p-Laplacian boundary value system with the Riemann–Stieltjes integral boundary conditions. A vector $(u, v) \in C[0,1] \times C[0,1]$ is said to be a positive solution of system (1) if and only if (u, v) satisfies (1) and $u(t) > 0$, $v(t) > 0$ for any $t \in (0,1]$. Compared with the above results, our work presented in this paper has the following several new features. Firstly, fractional derivatives are involved in the nonlinear terms of fractional differential equation (1). Secondly, the method we used in this paper is reducing the order of derivative, that is, higher-order fractional differential equation is transformed into lower-order fractional differential equation. Thirdly, the uniqueness positive solution of Eq. (1) is dependent on λ.

For convenience in presentation, here we list some conditions to be used throughout the paper.

(S1) $f(t, x_1, x_2, \dots, x_{n+1}) = \phi(t, x_1, x_2, \dots, x_{n+1}) + \psi(t, x_1, x_2, \dots, x_{n+1})$, where $\phi : (0,1) \times (0, +\infty)^{n+1} \to \mathbb{R}_+$ is continuous, $\phi(t, x_1, x_2, \dots, x_{n+1})$ may be singular at $t = 0, 1$ and is nondecreasing on $x_i > 0$ $(i = 1, 2, \dots, n+1)$. $\psi : (0,1) \times (0, +\infty)^{n+1} \to \mathbb{R}_+$ is continuous, $\psi(t, x_1, x_2, \dots, x_{n+1})$ may be singular at $t = 0, 1$, $x_i = 0$ and is nonincreasing on $x_i > 0$ $(i = 1, 2, \dots, n+1)$.

(S2) There exists $0 < \sigma < 1$ such that, for all $x_i > 0$ $(i = 1, 2, \dots, n+1)$ and $t, l \in (0,1)$,

$$\phi(t, lx_1, lx_2, \dots, lx_{n+1}) \geqslant l^{\sigma^{1/(q-1)}} \phi(t, x_1, x_2, \dots, x_{n+1}),$$
$$\psi\bigl(t, l^{-1}x_1, l^{-1}x_2, \dots, l^{-1}x_{n+1}\bigr) \geqslant l^{\sigma^{1/(q-1)}} \psi(t, x_1, x_2, \dots, x_{n+1}).$$

(S3) $g \in C((0,1) \times (0, +\infty)^m, \mathbb{R}_+)$, $g(t, x_1, x_2, \dots, x_m)$ is nondecreasing on $x_i > 0$ $(i = 1, 2, \dots, m)$, and $g(t, 1, \dots, 1) \neq 0$, $t \in (0,1)$. Moreover, there exists $\varsigma \in (0,1)$ such that, for all $x_i > 0$ $(i = 1, 2, \dots, m)$ and $t, l \in (0,1)$,

$$g(t, lx_1, lx_2, \dots, lx_m) \geqslant l^{\varsigma^{1/(q-1)}} g(t, x_1, x_2, \dots, x_m).$$

(S4) $0 < \int_0^1 \phi^2(\tau, 1, 1, \dots, 1)\,\mathrm{d}\tau < +\infty$,

$$0 < \int_0^1 \tau^{-2(\gamma-1)\sigma^{1/(q-1)}} \psi^2(\tau, 1, 1, \dots, 1)\,\mathrm{d}\tau < +\infty,$$
$$0 < \int_0^1 g^2\bigl(\tau, 1, 1, \dots, 1\bigr)\,\mathrm{d}\tau < +\infty.$$

Here q is defined by (1).

Remark 1. According to (S2) and (S3), for all $x_i > 0$ $(i = 1, 2, \dots, n+1)$, $\sigma, \varsigma, t \in (0,1)$, $l \geqslant 1$, we have

$$\phi(t, lx_1, lx_2, \dots, lx_{n+1}) \leqslant l^{\sigma^{1/(q-1)}} \phi(t, x_1, x_2, \dots, x_{n+1}),$$
$$\psi\big(t, l^{-1}x_1, l^{-1}x_2, \dots, l^{-1}x_{n+1}\big) \leqslant l^{\sigma^{1/(q-1)}} \psi(t, x_1, x_2, \dots, x_{n+1}),$$
$$g(t, lx_1, lx_2, \dots, lx_m) \leqslant l^{\varsigma^{1/(q-1)}} g(t, x_1, x_2, \dots, x_m).$$

2 Preliminaries and lemmas

For some basic definitions and lemmas about the theory of fractional calculus, the reader can refer to the recent literature such as [17, 18].

Lemma 1. (*See* [17, 18].) *Assume that* $u \in C^n(0,1) \cap L(0,1)$*, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n},$$

where $n = [\alpha] + 1$, $C_i \in \mathbb{R}$ $(i = 1, 2, \dots, n)$.

Lemma 2. (*See* [10].)

(i) *If* $x \in L(0,1)$, $v > \sigma > 0$, *then*

$$I_{0+}^{v} I_{0+}^{\sigma} x(t) = I_{0+}^{v+\sigma} x(t),$$
$$D_{0+}^{\sigma} I_{0+}^{v} x(t) = I_{0+}^{v-\sigma} x(t), \qquad D_{0+}^{\sigma} I_{0+}^{\sigma} x(t) = x(t).$$

(ii) *If* $v > 0$, $\sigma > 0$, *then* $D_{0+}^{v} t^{\sigma-1} = (\Gamma(\sigma)/\Gamma(\sigma - v)) t^{\sigma-v-1}$.

Lemma 3. *Let* $\rho \in L^1(0,1) \cap C(0,1)$*, then the equation of the BVPs*

$$-D_{0+}^{\gamma-\mu_{n-1}} x(t) = \rho(t), \quad 0 < t < 1,$$
$$x(0) = 0, \qquad x(1) = \chi \int_0^{\eta} h(t) x(t) \, \mathrm{d}A(t),$$

$$-D_{0+}^{\sigma-\eta_{m-1}} y(t) = \rho(t), \quad 0 < t < 1,$$
$$y(0) = 0, \qquad y(1) = \iota \int_0^{\vartheta} a(t) y(t) \, \mathrm{d}B(t)$$

have integral representation

$$x(t) = \int_0^1 G(t,s)\rho(s) \, \mathrm{d}s = \int_0^1 \big(G_1(t,s) + G_2(t,s)\big)\rho(s) \, \mathrm{d}s,$$
$$y(t) = \int_0^1 H(t,s)\rho(s) \, \mathrm{d}s = \int_0^1 \big(H_1(t,s) + H_2(t,s)\big)\rho(s) \, \mathrm{d}s, \tag{2}$$

respectively, where

$$G_1(t,s) = \frac{1}{\Gamma(\gamma-\mu_{n-1})} \times \begin{cases} t^{\gamma-\mu_{n-1}-1}(1-s)^{\gamma-\mu_{n-1}-1} - (t-s)^{\gamma-\mu_{n-1}-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\gamma-\mu_{n-1}-1}(1-s)^{\gamma-\mu_{n-1}-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$G_2(t,s) = \frac{\chi t^{\gamma-\mu_{n-1}-1}}{1-F} j_A(s), \quad F = \chi \int_0^\eta t^{\gamma-\mu_{n-1}-1} h(t)\,\mathrm{d}A(t),$$

$$H_1(t,s) = \frac{1}{\Gamma(\delta-\eta_{m-1})} \times \begin{cases} t^{\delta-\eta_{m-1}-1}(1-s)^{\delta-\eta_{m-1}-1} - (t-s)^{\delta-\eta_{m-1}-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\delta-\eta_{m-1}-1}(1-s)^{\delta-\eta_{m-1}-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$H_2(t,s) = \frac{\iota t^{\delta-\eta_{m-1}-1}}{1-C} j_B(s), \quad C = \iota \int_0^\vartheta t^{\delta-\eta_{m-1}-1} a(t)\,\mathrm{d}B(t),$$

$$j_A(s) = \int_0^\eta h(t) G_1(t,s)\,\mathrm{d}A(t), \qquad j_B(s) = \int_0^\vartheta a(t) H_1(t,s)\,\mathrm{d}B(t).$$

Proof. The proof is similar to that for Lemma 2.3 in [25], we omit it here. □

Lemma 4. *Let* $0 \leqslant C,\ F < 1$, *and* $j_A(s), j_B(s) \geqslant 0$ *for* $s \in [0,1]$, *then the Green functions defined by* (2) *satisfy*:

(i) $G, H : [0,1] \times [0,1] \to R_+$ *are continuous, and* $G(t,s), H(t,s) > 0$ *for all* $t, s \in (0,1)$;

(ii) *There exist four positive constants* a_*, a^*, b_*, b^* *such that, for all* $t, s \in [0,1]$,

$$a_* t^{\gamma-\mu_{n-1}-1} j_A(s) \leqslant G(t,s) \leqslant a^* t^{\gamma-\mu_{n-1}-1},$$
$$b_* t^{\delta-\eta_{m-1}-1} j_B(s) \leqslant H(t,s) \leqslant b^* t^{\delta-\eta_{m-1}-1},$$

where

$$a_* = \frac{\chi}{1-F}, \qquad a^* = \frac{\chi \|j_A(s)\|}{1-F} + \frac{1}{\Gamma(\gamma-\mu_{n-1}-1)},$$
$$b_* = \frac{\iota}{1-C}, \qquad b^* = \frac{\iota \|j_B(s)\|}{1-C} + \frac{1}{\Gamma(\delta-\eta_{m-1}-1)}.$$

Proof. The proof is similar to that for Lemma 2.2 in [27], we omit it here. □

To study the PFDE (1), in what follows, we consider the associated linear PFDE:

$$
\begin{aligned}
&D_{0^+}^{\alpha}\big(\varphi_p\big(D_{0^+}^{\gamma-\mu_{n-1}}x\big)\big)(t)+\rho(t)=0, \quad 0<t<1,\\
&x(0)=0, \quad D_{0+}^{\gamma}x(0)=0, \qquad x(1)=\chi\int_0^{\eta} h(t)x(t)\,\mathrm{d}A(t),
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
&D_{0^+}^{\beta}\big(\varphi_p\big(D_{0^+}^{\delta-\eta_{m-1}}y\big)\big)(t)+\rho(t)=0, \quad 0<t<1,\\
&y(0)=0, \quad D_{0+}^{\delta}y(0)=0, \qquad y(1)=\mu\int_0^{\iota} a(t)y(t)\,\mathrm{d}B(t).
\end{aligned} \tag{4}
$$

Lemma 5. *The PFDE* (3), (4) *have the unique positive solution*

$$
x(t)=\int_0^1 G(t,s)\left(\int_0^s \overline{a}(s-\tau)^{\alpha-1}\rho(\tau)\,\mathrm{d}\tau\right)^{q-1}\mathrm{d}s, \quad t\in[0,1], \tag{5}
$$

$$
y(t)=\int_0^1 H(t,s)\left(\int_0^s \overline{b}(s-\tau)^{\beta-1}\rho(\tau)\,\mathrm{d}\tau\right)^{q-1}\mathrm{d}s, \quad t\in[0,1], \tag{6}
$$

respectively, where $\overline{a}=1/\Gamma(\alpha)$, $\overline{b}=1/\Gamma(\beta)$.

Proof. Let $h=D_{0^+}^{\gamma-\mu_{n-1}}x$, $k=\varphi_p(h)$, then the solution of the initial value problem

$$
D_{0^+}^{\alpha}k(t)+\rho(t)=0, \quad 0<t<1, \qquad k(0)=0
$$

is given by $k(t)=C_1t^{\alpha-1}-I_{0^+}^{\alpha}\rho(t)$, $t\in[0,1]$. By the relations $k(0)=0$, we have $C_1=0$, and hence

$$
k(t)=-I_{0^+}^{\alpha}\rho(t), \quad t\in[0,1]. \tag{7}
$$

By $D_{0^+}^{\gamma}x=h$, $h=\varphi_p^{-1}(k)$, we have from (7) that the solution of (3) satisfies

$$
\begin{aligned}
&D_{0^+}^{\gamma-\mu_{n-1}}x(t)=\varphi_p^{-1}\big(-I_{0^+}^{\alpha}\rho(t)\big), \quad 0<t<1,\\
&x(0)=0, \qquad x(1)=\chi\int_0^{\eta} h(t)x(t)\,\mathrm{d}A(t).
\end{aligned} \tag{8}
$$

By (2), the solution of Eq. (8) can be written as

$$
x(t)=-\int_0^1 G(t,s)\varphi_p^{-1}\big(-I_{0^+}^{\alpha}\rho(s)\big)\,\mathrm{d}s, \quad t\in[0,1]. \tag{9}
$$

Since $\rho(s)\geqslant 0$, $s\in[0,1]$, we have $\varphi_p^{-1}(-I_{0^+}^{\alpha}\rho(s))=-(I_{0^+}^{\alpha}\rho(s))^{q-1}$, $s\in[0,1]$, which implies that the solution of Eq. (3) is (5). Similarly, the solution of Eq. (4) is (6). $\square$

Let $u(t) = I_{0^+}^{\mu_{n-1}} x(t)$, $v(t) = I_{0^+}^{\eta_{m-1}} y(t)$, problem (1) can turn into the following modified problem of the PFDE:

$$\begin{aligned}
&D_{0+}^{\alpha}\bigl(\varphi_p\bigl(D_{0+}^{\gamma-\mu_{n-1}}x\bigr)\bigr)(t) + \lambda^{1/(q-1)} f\bigl(t,\, I_{0^+}^{\mu_{n-1}}x(t),\, I_{0^+}^{\mu_{n-1}-\mu_1}x(t),\, \dots,\\
&\quad I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(t),\, x(t),\, I_{0^+}^{\eta_{m-1}}y(t)\bigr) = 0, \quad 0 < t < 1,\\
&D_{0+}^{\beta}\bigl(\varphi_p\bigl(D_{0+}^{\delta-\eta_{m-1}}y\bigr)\bigr)(t) + \mu^{1/(q-1)} g\bigl(t,\, I_{0^+}^{\mu_{n-1}}x(t),\, I_{0^+}^{\mu_{n-1}-\eta_1}x(t),\, \dots,\\
&\quad I_{0^+}^{\mu_{n-1}-\eta_{m-1}}x(t)\bigr) = 0, \quad 0 < t < 1,\\
&x(0) = 0, \quad D_{0+}^{\gamma}x(0) = 0, \qquad x(1) = \chi \int_0^{\eta} h(t)x(t)\,\mathrm{d}A(t),\\
&y(0) = 0, \quad D_{0+}^{\delta}y(0) = 0, \qquad y(1) = \iota \int_0^{\vartheta} a(t)y(t)\,\mathrm{d}B(t).
\end{aligned} \tag{10}$$

Lemma 6. *Let* $u(t) = I_{0^+}^{\mu_{n-1}}x(t)$, $v(t) = I_{0^+}^{\eta_{m-1}}y(t)$, $x(t), y(t) \in C[0,1]$. *Then* (1) *can be transformed into* (10). *Moreover, if* $(x,y) \in C[0,1] \times C[0,1]$ *is a positive solution of problem* (10)*, then* $(I_{0^+}^{\mu_{n-1}}x,\, I_{0^+}^{\eta_{m-1}}y)$ *is a positive solution of problem* (1)*.*

Proof. The proof is similar to that for Lemma 2.5 in [8], we omit it here. □

The vector (u,v) is a solution of system (1) if and only if $(x,y) \in C[0,1] \times C[0,1]$ is a solution of the following nonlinear integral equation system:

$$\begin{aligned}
x(t) &= \lambda \int_0^1 G(t,s)\Biggl(\int_0^s \overline{a}(s-\tau)^{\alpha-1} f\bigl(\tau,\, I_{0^+}^{\mu_{n-1}}x(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1}x(\tau),\, \dots,\\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(\tau),\, x(\tau),\, I_{0^+}^{\eta_{m-1}}y(\tau)\bigr)\,\mathrm{d}\tau\Biggr)^{q-1}\mathrm{d}s, \quad t \in [0,1],\\
y(\tau) &= \mu \int_0^1 H(\tau,s)\Biggl(\int_0^s \overline{b}(s-w)^{\beta-1} g\bigl(w,\, I_{0^+}^{\mu_{n-1}}x(w),\, I_{0^+}^{\mu_{n-1}-\eta_1}x(w),\, \dots,\\
&\qquad I_{0^+}^{\mu_{n-1}-\eta_{m-1}}x(w)\bigr)\,\mathrm{d}w\Biggr)^{q-1}\mathrm{d}s, \quad t \in [0,1].
\end{aligned} \tag{11}$$

Obviously, system (11) is equivalent to the following integral equation:

$$\begin{aligned}
x(t) = \lambda \int_0^1 G(t,s)\Biggl(\int_0^s \overline{a}(s-\tau)^{\alpha-1} f\Biggl(\tau,\, I_{0^+}^{\mu_{n-1}}x(\tau),\\
I_{0^+}^{\mu_{n-1}-\mu_1}x(\tau),\, \dots,\, I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(\tau),\, x(\tau),
\end{aligned}$$

$$I_{0^+}^{\eta_{m-1}}\Biggl[\mu\int\limits_0^1 H(\tau,s)\Biggl(\int\limits_0^s \bar{b}(s-w)^{\beta-1}g\bigl(w,\,I_{0^+}^{\mu_{n-1}}x(w),\,\ldots,$$
$$I_{0^+}^{\mu_{n-1}-\eta_{m-1}}x(w)\bigr)\,\mathrm{d}w\Biggr)^{q-1}\mathrm{d}s\Biggr]\,\mathrm{d}\tau\Biggr)^{q-1}\mathrm{d}s.$$

Let P be a normal cone of a Banach space E, and $e \in P$, $e > \theta$, where θ is a zero element of E. Define a component of P by $Q_e = \{u \in P \mid \text{there exists a constant } C \geqslant 1 \text{ such that } e/c \leqslant u \leqslant Ce\}$. $A : Q_e \times Q_e \to P$ is said to be mixed monotone if $A(u,y)$ is nondecreasing in u and nonincreasing in y, i.e., $u_1 \leqslant u_2$ $(u_1,u_2 \in Q_e)$ implies $A(u_1,y) \leqslant A(u_2,y)$ for any $y \in Q_e$, and $y_1 \leqslant y_2$ $(y_1,y_2 \in Q_e)$ implies $A(u,y_1) \geqslant A(u,y_2)$ for any $u \in Q_e$. The element $u^* \in Q_e$ is called a fixed point of A if $A(u^*,u^*) = u^*$.

Lemma 7. (*See* [1,5].) *Suppose that* $A : Q_e \times Q_e \to Q_e$ *is a mixed monotone operator and there exists a constant* $0 < \sigma < 1$ *such that*

$$A\left(lx,\frac{1}{l}y\right) \geqslant l^{\sigma}A(x,y), \quad x,y \in Q_e,\ 0 < l < 1, \tag{12}$$

then A *has a unique fixed point* $x^* \in Q_e$*, and for any* $x_0 \in Q_e$*, we have* $\lim_{k\to\infty} x_k = x^*$*, where* $x_k = A(x_{k-1},x_{k-1})$ $(k = 1,2,\ldots)$*, and the convergence rate is* $\|x_k - x^*\| = o(1 - r^{\sigma^k})$*, where* r *is a constant dependent on* x_0 *and* $0 < r < 1$.

Lemma 8. (*See* [1,5].) *Suppose that* $A : Q_e \times Q_e \to Q_e$ *is a mixed monotone operator and there exists a constant* $\sigma \in (0,1)$ *such that* (12) *holds. If* x_λ^* *is a unique solution of equation* $\lambda A(x,x) = x$*,* $\lambda > 0$ *in* Q_e*, then*:

(i) *For any* $\lambda_0 \in (0,+\infty)$*,* $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$*,* $\lambda \to \lambda_0$;
(ii) *If* $0 < \sigma < 1/2$*, then* $0 < \lambda_1 < \lambda_2$ *implies* $x_{\lambda_1}^* \leqslant x_{\lambda_2}^*$*,* $x_{\lambda_1}^* \neq x_{\lambda_2}^*$;
(iii) *If* $0 < \sigma < 1/2$*, then* $\lim_{\lambda\to+\infty}\|x_\lambda^*\| = +\infty$*,* $\lim_{\lambda\to 0^+}\|x_\lambda^*\| = 0$.

Let $e(t) = t^{\gamma-\mu_{n-1}-1}$ for $t \in [0,1]$, we define a normal cone of C[0,1] by $P = \{x \in C[0,1]\colon x(t) \geqslant 0,\ t \in [0,1]\}$, also define a component of P by

$$Q_e = \left\{x \in P\colon \text{there exists } D \geqslant 1,\ \frac{1}{D}e(t) \leqslant x(t) \leqslant De(t),\ t \in [0,1]\right\}.$$

Remark 2. Let $s = \tau t$, by simple calculation, we have

$$\begin{aligned} I_{0^+}^{\mu_{n-1}}e(t) = I_{0^+}^{\mu_{n-1}}t^{\gamma-\mu_{n-1}-1} &= \frac{1}{\Gamma(\mu_{n-1})}\int\limits_0^t (t-s)^{\mu_{n-1}-1}s^{\gamma-\mu_{n-1}-1}\,\mathrm{d}s \\ &= \frac{t^{\gamma-1}}{\Gamma(n-2)}\int\limits_0^1 (1-\tau)^{\mu_{n-1}-1}\tau^{\gamma-\mu_{n-1}-1}\,\mathrm{d}\tau = \frac{\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma)}t^{\gamma-1}. \end{aligned} \tag{13}$$

Similarly, we have

$$I_{0^+}^{\eta_{m-1}} t^{\delta-\eta_{m-1}} = \frac{\Gamma(\delta-\eta_{m-1})}{\Gamma(\delta)} t^{\delta-1},$$

$$I_{0^+}^{\mu_{n-1}-\mu_\kappa} e(t) = \frac{\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\mu_\kappa)} t^{\gamma-\mu_\kappa-1}, \quad \kappa = 1,2,\dots,n-2, \tag{14}$$

$$I_{0^+}^{\mu_{n-1}-\eta_\varrho} e(t) = \frac{\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_\varrho)} t^{\gamma-\eta_\varrho-1}, \quad \varrho = 1,2,\dots,m-2. \tag{15}$$

3 Main results

Theorem 1. *Suppose that* (S1)–(S4) *hold. Then for all* $t \in [0,1]$, *the PFDE* (1) *has a unique positive solution* $(u_\lambda^*, v_\lambda^*)$, *which satisfies*

$$\frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)} t^{\gamma-1} \leqslant u_\lambda^*(t) \leqslant \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma)} t^{\gamma-1},$$

$$\frac{\mu b_* \Gamma(\delta-\eta_{m-1}) t^{\delta-1} \overline{b}^{q-1}}{\Gamma(\beta)} K_1 \leqslant v_\lambda^*(t) \leqslant \frac{\Gamma(\delta-\eta_{m-1}) b^* \mu t^{\delta-1} \overline{b}^{q-1}}{\Gamma(\delta)(2\beta-1)^{(q-1)/2}} K_2, \tag{$*$}$$

where K_1, K_2 *are two positive constants, and at the same time,* u_λ^* *satisfies*:

(i) *For* $\lambda_0 \in (0,\infty)$, $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$, $\lambda \to \lambda_0$;

(ii) *If* $0 < \sigma < 1/2$, *then* $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1}^* \leqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$;

(iii) *If* $0 < \sigma < 1/2$, *then* $\lim_{\lambda\to 0} \|u_\lambda^*\| = 0$, $\lim_{\lambda\to+\infty} \|u_\lambda^*\| = +\infty$.

Moreover, for any $u_0 \in Q_e$, *constructing a successively sequence*:

$$\begin{aligned} u_{k+1}(t) = I_{0^+}^{\mu_{n-1}} \Biggl\{ \lambda \int_0^1 G(t,s) \Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1} \bigl(\phi\bigl(\tau,\, u_k(\tau),\, D_{0^+}^{\mu_1} u_k(\tau),\, \dots, \\ D_{0^+}^{\mu_{n-1}} u_k(\tau),\, A u_k^{(\mu_{n-1})}(\tau)\bigr) + \psi\bigl(\tau,\, u_k(\tau),\, D_{0^+}^{\mu_1} u_k(\tau),\, \dots, \\ D_{0^+}^{\mu_{n-1}} u_k(\tau),\, A u_k^{(\mu_{n-1})}(\tau)\bigr)\bigr)\, \mathrm{d}\tau \Biggr]^{q-1} \mathrm{d}s \Biggr\}, \quad k = 0,1,2,\dots,\ t \in [0,1], \end{aligned}$$

and we have $\|u_k - u_\lambda^*\| \to 0$ *as* $k \to \infty$, *the convergence rate is* $\|u_k - u_\lambda^*\| = o(1 - r^{\sigma^k})$, *where* r *is a constant,* $0 < r < 1$, *and dependent on* u_0.

Proof. We now consider the existence of a positive solution to problem (1). From the discussion in Section 2 we only need to consider the existence of a positive solution to PFDE (11). In order to realize this purpose, define the operator $A : Q_e \to P$ by

$$\begin{aligned} Ax(\tau) = I_{0^+}^{\eta_{m-1}} \Biggl\{ \mu \int_0^1 H(\tau,s) \Biggl[\int_0^s \overline{b}(s-w)^{\beta-1} g\bigl(w,\, I_{0^+}^{\mu_{n-1}} x(w),\, I_{0^+}^{\mu_{n-1}-\eta_1} x(w),\, \dots, \\ I_{0^+}^{\mu_{n-1}-\eta_{m-1}} x(w)\bigr)\, \mathrm{d}w \Biggr]^{q-1} \mathrm{d}s \Biggr\}, \quad \tau \in [0,1], \end{aligned} \tag{16}$$

and define the operator $T_\lambda : Q_e \times Q_e \to P$ by

$$T_\lambda(x,z)(t) = \lambda \int_0^1 G(t,s)\Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\bigl(\phi\bigl(\tau,\, I_{0^+}^{\mu_{n-1}}x(\tau),\, \ldots,\, I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(\tau),\, x(\tau),\, Ax(\tau)\bigr) + \psi\bigl(\tau, I_{0^+}^{\mu_{n-1}}z(\tau),\, \ldots,\, I_{0^+}^{\mu_{n-1}-\mu_{n-2}}z(\tau),\, z(\tau),\, Az(\tau)\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{q-1}\mathrm{d}s, \quad t \in [0,1].$$

Now we prove that $T_\lambda : Q_e \times Q_e \to P$ is well defined. For any $x, z \in Q_e$, by (16), (S3), (13), (15), and Remark 1, for all $\tau \in [0,1]$, we have

$$\begin{aligned}
&\mu \int_0^1 H(\tau,s)\Biggl[\int_0^s \overline{b}(s-w)^{\beta-1} g\bigl(w,\, I_{0^+}^{\mu_{n-1}}x(w),\, I_{0^+}^{\mu_{n-1}-\eta_1}x(w),\, \ldots,\, I_{0^+}^{\mu_{n-1}-\eta_{m-1}}x(w)\bigr)\,\mathrm{d}w\Biggr]^{q-1}\mathrm{d}s \\
&\leqslant \mu b^* \tau^{\delta-m+1} \int_0^1 \Biggl[\int_0^s \overline{b}(s-w)^{\beta-1} g\biggl(w,\, \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma)}w^{\gamma-1},\, \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_1)}w^{\gamma-\eta_1-1},\, \ldots,\, \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_{m-1})}w^{\gamma-\eta_{m-1}-1}\biggr)\,\mathrm{d}w\Biggr]^{q-1}\mathrm{d}s \\
&\leqslant \mu b^* \tau^{\delta-\eta_{m-1}-1} \int_0^1 \Biggl[\int_0^s \overline{b}(s-w)^{\beta-1} g\biggl(w,\, \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_{m-2})}+1,\, \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_{m-1})}+1,\, \ldots,\, \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_{m-1})}+1\biggr)\,\mathrm{d}w\Biggr]^{q-1}\mathrm{d}s \\
&\leqslant \mu b^* \tau^{\delta-\eta_{m-1}-1}\overline{b}^{q-1}\biggl(\frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_{m-1})}+1\biggr)^{\varsigma} \\
&\quad \times \int_0^1 \Biggl[\int_0^s (s-w)^{\beta-1} g(w,1,1,\ldots,1)\,\mathrm{d}w\Biggr]^{q-1}\mathrm{d}s,
\end{aligned} \tag{17}$$

$$\mu \int_0^1 H(\tau,s)\Biggl[\int_0^s \overline{b}(s-w)^{\beta-1} g\bigl(w,\, I_{0^+}^{\mu_{n-1}}x(w),\, I_{0^+}^{\mu_{n-1}-\eta_1}x(w),\, \ldots,\, I_{0^+}^{\mu_{n-1}-\eta_{m-1}}x(w)\bigr)\,\mathrm{d}w\Biggr]^{q-1}\mathrm{d}s$$

$$
\begin{aligned}
&\geqslant \mu b_* \tau^{\delta-\eta_{m-1}-1} \int_0^1 j_B(s)\Bigg[\int_0^s \bar{b}(s-w)^{\beta-1} g\bigg(w,\, \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)} w^{\gamma-1}, \\
&\quad \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma-\eta_1)} w^{\gamma-\eta_1-1},\, \dots,\, \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma-\eta_{m-1})} w^{\gamma-\eta_{n-2}-1}\bigg)\,\mathrm{d}w\Bigg]^{q-1}\,\mathrm{d}s \\
&\geqslant \mu b_* \tau^{\delta-\eta_{m-1}-1} \int_0^1 j_B(s)\Bigg[\int_0^s \bar{b}(s-w)^{\beta-1} w^{(\gamma-1)\varsigma^{1/(q-1)}} \\
&\quad \bigg(\frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)}\bigg)^{\varsigma^{1/(q-1)}} g(w,1,1,\dots,1)\,\mathrm{d}w\Bigg]^{q-1}\,\mathrm{d}s \\
&= \mu b_* \tau^{\delta-\eta_{m-1}-1} \bar{b}^{q-1} \bigg(\frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)}\bigg)^{\varsigma} \\
&\quad \times \int_0^1 j_B(s)\Bigg[\int_0^s w^{(\gamma-1)\varsigma^{1/(q-1)}}(s-w)^{\beta-1} g(w,1,1,\dots,1)\,\mathrm{d}w\Bigg]^{q-1}\,\mathrm{d}s. \qquad (18)
\end{aligned}
$$

Hence, by (16), (17), (18), and the Hölder inequality, for $\tau \in [0,1]$, we have

$$
\begin{aligned}
Ax(\tau) &= I_{0^+}^{\eta_{m-1}}\Bigg(\mu \int_0^1 H(\tau,s)\Bigg[\int_0^s \bar{b}(s-w)^{\beta-1} g\big(w,\, I_{0^+}^{\mu_{n-1}} x(w), \\
&\qquad I_{0^+}^{\mu_{n-1}-\eta_1} x(w),\, \dots,\, I_{0^+}^{\mu_{n-1}-\eta_{m-1}} x(w)\big)\,\mathrm{d}w\Bigg]^{q-1}\,\mathrm{d}s\Bigg) \\
&\leqslant \frac{\Gamma(\delta-\eta_{m-1}) b^* \mu \tau^{\delta-1} \bar{b}^{q-1}}{\Gamma(\delta)(2\beta-1)^{(q-1)/2}} \bigg(\frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_{m-1})}+1\bigg)^{\varsigma} \\
&\quad \times \int_0^1 \Bigg[s^{(2\beta-1)/2}\bigg(\int_0^s g^2(w,1,1,\dots,1)\,\mathrm{d}w\bigg)^{1/2}\Bigg]^{q-1}\,\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
Ax(\tau) &= I_{0^+}^{m-2}\Bigg(\mu \int_0^1 H(\tau,s)\Bigg[\int_0^s \bar{b}(s-w)^{\beta-1} g\big(w,\, I_{0^+}^{\mu_{n-1}} x(w), \\
&\qquad I_{0^+}^{\mu_{n-1}-\eta_1} x(w),\, \dots,\, I_{0^+}^{\mu_{n-1}-\eta_{m-1}} x(w)\big)\,\mathrm{d}w\Bigg]^{q-1}\,\mathrm{d}s\Bigg) \\
&\geqslant \frac{\mu b_* \Gamma(\delta-\eta_{m-1}) \tau^{\delta-1} \bar{b}^{q-1}}{\Gamma(\delta)} \bigg(\frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)}\bigg)^{\varsigma} \\
&\quad \times \int_0^1 j_B(s)\Bigg[\int_0^s w^{(\gamma-1)\varsigma^{1/(q-1)}}(s-w)^{\beta-1} g(w,1,1,\dots,1)\,\mathrm{d}w\Bigg]^{q-1}\,\mathrm{d}s. \qquad (19)
\end{aligned}
$$

By (S4), we get that $Ax(\tau)$ is well defined. From (13), (14), (19), (S1), and Remark 1 we have

$$\begin{aligned}
&\phi\big(\tau,\, I_{0^+}^{\mu_{n-1}}x(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1}x(\tau),\, \dots,\, I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(\tau),\, x(\tau),\, Ax(\tau)\big)\\
&\quad\leqslant \phi\Bigg(\tau,\, \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma)}\tau^{\gamma-1}+1,\, \dots,\, \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\mu_{n-2})}\tau^{\gamma-\mu_{n-2}-1}+1,\\
&\qquad D\tau^{\gamma-\mu_{n-1}-1}, \frac{\Gamma(\delta-\eta_{m-1})b^*\mu\tau^{\delta-1}\bar{b}^{q-1}}{\Gamma(\delta)(2\beta-1)^{(q-1)/2}}\left(\frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_{m-1})}+1\right)^{\varsigma}\\
&\qquad\times\int_0^1\left[s^{(2\beta-1)/2}\left(\int_0^s g^2(w,1,\dots,1)\,\mathrm{d}w\right)^{1/2}\right]^{q-1}\mathrm{d}s+1\Bigg)\\
&\quad\leqslant \phi(\tau,\, Db+1,\, Db+1,\, \dots,\, D^{\varsigma}b+1)\\
&\quad\leqslant 2^{\sigma^{1/(q-1)}}b^{\sigma^{1/(q-1)}}D^{\sigma^{1/(q-1)}}\phi(\tau,1,1,\dots,1),\quad \tau\in(0,1),
\end{aligned}\tag{20}$$

where $D\geqslant 1$, b are two positive constants. By (13), (14), (19), (S1), and (S2), we also have

$$\begin{aligned}
&\psi\big(\tau, I_{0^+}^{\mu_{n-1}}x(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1}x(\tau),\, \dots,\, I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(\tau),\, x(\tau),\, Ax(\tau)\big)\\
&\quad\leqslant \psi\Bigg(\tau,\, \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)}\tau^{\gamma-1},\, \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma-\mu_1)}\tau^{\gamma-\mu_1-1},\, \dots,\, \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma-\mu_{n-2})}\tau^{\gamma-\mu_{n-2}-1},\\
&\qquad \frac{\tau^{\gamma-\mu_{n-1}-1}}{D},\, \frac{\mu b_*\Gamma(\delta-\eta_{m-1})\tau^{\delta-1}\bar{b}^{q-1}}{\Gamma(\delta)}\left(\frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)}\right)^{\varsigma}\\
&\qquad\times\int_0^1 j_B(s)\left[\int_0^s w^{(\gamma-1)\varsigma^{1/(q-1)}}(s-w)^{\beta-1}g(w,1,1,\dots,1)\,\mathrm{d}w\right]^{q-1}\mathrm{d}s\Bigg)\\
&\quad\leqslant \psi\left(\tau,\, \frac{c}{D}\tau^{\gamma-1},\, \frac{c}{D}\tau^{\gamma-1},\, \dots,\, \frac{c}{D}\tau^{\gamma-1}\right)\\
&\quad= c^{-\sigma^{1/(q-1)}}D^{\sigma^{1/(q-1)}}\tau^{(\gamma-1)\sigma^{1/(q-1)}}\psi(\tau,1,1,\dots,1),\quad \tau\in(0,1),
\end{aligned}\tag{21}$$

where c is a positive constant. Noting $(c/D)\tau^{\gamma-1}<1$ and by (13), (14), (19), (S1), and (S2), we have

$$\begin{aligned}
&\phi\big(\tau,\, I_{0^+}^{\mu_{n-1}}x(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1}x(\tau),\, \dots,\, I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(\tau),\, x(\tau),\, Ax(\tau)\big)\\
&\quad\geqslant \phi\Bigg(\tau,\, \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)}\tau^{\gamma-1},\, \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma-\mu_1)}\tau^{\gamma-\mu_1-1},\, \dots,\\
&\qquad \frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma-\mu_{n-2})}\tau^{\gamma-\mu_{n-2}-1},\, \frac{\tau^{\gamma-\mu_{n-1}-1}}{D},
\end{aligned}$$

$$
\begin{aligned}
&\frac{\mu b_* \Gamma(\delta-\eta_{m-1})\tau^{\delta-1}\overline{b}^{q-1}}{\Gamma(\delta)}\left(\frac{\Gamma(\gamma-\mu_{n-1})}{D\Gamma(\gamma)}\right)^{\varsigma} \\
&\times \int_0^1 j_B(s)\left[\int_0^s w^{(\gamma-1)\varsigma^{1/(q-1)}}(s-w)^{\beta-1}g(w,\,1,\,1,\,\ldots,\,1)\,\mathrm{d}w\right]^{q-1}\mathrm{d}s\Bigg) \\
\geqslant{}& \phi\left(\tau,\,\frac{c}{D}\tau^{\gamma-1},\,\frac{c}{D}\tau^{\gamma-1},\,\ldots,\,\frac{c}{D}\tau^{\gamma-1}\right) \\
={}& c^{\sigma^{1/(q-1)}}D^{-\sigma^{1/(q-1)}}\tau^{(\gamma-1)\sigma^{1/(q-1)}}\phi(\tau,1,1,\ldots,1), \quad \tau\in(0,1).
\end{aligned}
\tag{22}
$$

By (13), (14), (19), (S1), and Remark 1, we also get

$$
\begin{aligned}
&\psi\big(\tau,\,I_{0^+}^{\mu_{n-1}}x(\tau),\,I_{0^+}^{\mu_{n-1}-\mu_1}x(\tau),\,\ldots,\,I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(\tau),\,x(\tau),\,Ax(\tau)\big) \\
&\geqslant \psi\Bigg(\tau,\,\frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma)}\tau^{\gamma-1}+1,\,\frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\mu_1)}\tau^{\gamma-\mu_1-1}+1,\,\ldots, \\
&\quad \frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\mu_{n-2})}\tau^{\gamma-\mu_{n-2}-1}+1,\,D\tau^{\gamma-\mu_{n-1}-1},\,\frac{\Gamma(\delta-\eta_{m-1})b^*\mu\tau^{\delta-1}\overline{b}^{q-1}}{\Gamma(\delta)(2\beta-1)^{(q-1)/2}} \\
&\quad \times\left(\frac{D\Gamma(\gamma-\mu_{n-1})}{\Gamma(\gamma-\eta_{m-1})}+1\right)^{\varsigma}\int_0^1\left[s^{(2\beta-1)/2}\left(\int_0^s g^2(w,1,1,\ldots,1)\,\mathrm{d}w\right)^{1/2}\right]^{q-1}\mathrm{d}s+1\Bigg) \\
&\geqslant \psi\big(\tau,\,Db\tau^{\gamma-1}+1,\,Db\tau^{\gamma-\mu_1-1}+1,\,\ldots,\,Db\tau^{\gamma-\mu_{n-2}-1}+1,\,Db\tau^{\delta-1}+1\big) \\
&\geqslant 2^{-\sigma^{1/(q-1)}}b^{-\sigma^{1/(q-1)}}D^{-\sigma^{1/(q-1)}}\psi(\tau,1,1,\ldots,1), \quad \tau\in(0,1).
\end{aligned}
\tag{23}
$$

For any $x, z\in Q_e$, it follows from (20), (21) that

$$
\begin{aligned}
&T_\lambda(x,z)(t) \\
&\quad=\lambda\int_0^1 G(t,s)\Bigg[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau,\,I_{0^+}^{\mu_{n-1}}x(\tau),\,I_{0^+}^{\mu_{n-1}-\mu_1}x(\tau),\,\ldots, \\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}}x(\tau),\,x(\tau),\,Ax(\tau)\big)+\psi\big(\tau,\,I_{0^+}^{\mu_{n-1}}z(\tau),\,I_{0^+}^{\mu_{n-1}-\mu_1}z(\tau),\,\ldots, \\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}}z(\tau),\,z(\tau),\,Az(\tau)\big)\big)\,\mathrm{d}\tau\Bigg]^{q-1}\mathrm{d}s \\
&\quad\leqslant \lambda a^* t^{\gamma-\mu_{n-1}-1}D^{\sigma}\overline{a}^{q-1}\int_0^1\Bigg[2^{\sigma^{1/(q-1)}}b^{\sigma^{1/(q-1)}}\frac{1}{(2\alpha-1)^{1/2}}s^{(2\alpha-1)/2} \\
&\qquad\times\left(\int_0^s\phi^2(\tau,1,1,\ldots,1)\,\mathrm{d}\tau\right)^{1/2}+c^{-\sigma^{1/(q-1)}}\frac{1}{(2\alpha-1)^{1/2}}s^{(2\alpha-1)/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\int_0^s \tau^{-2(\gamma-1)\sigma^{1/(q-1)}} \psi^2(\tau,1,1,\dots,1)\,\mathrm{d}\tau \right)^{1/2} \Bigg]^{q-1} \mathrm{d}s \\
&\leqslant \lambda a^* t^{\gamma-\mu_{n-1}-1} D^{\sigma} \overline{a}^{q-1} \frac{1}{(2\alpha-1)^{(q-1)/2}} \\
&\quad \times \int_0^1 \Bigg[2^{\sigma^{1/(q-1)}} b^{\sigma^{1/(q-1)}} \left(\int_0^s \phi^2(\tau,1,1,\dots,1)\,\mathrm{d}\tau \right)^{1/2} \\
&\quad + c^{-\sigma^{1/(q-1)}} \left(\int_0^s \tau^{-2(\gamma-1)\sigma^{1/(q-1)}} \psi^2(\tau,1,1,\dots,1)\,\mathrm{d}\tau \right)^{1/2} \Bigg]^{q-1} \mathrm{d}s \\
&< +\infty, \quad t \in [0,1].
\end{aligned} \tag{24}
$$

By (S4), (24), we have that $T_\lambda : Q_e \times Q_e \to P$ is well defined.

Next, we will prove $T_\lambda : Q_e \times Q_e \to Q_e$. Formula (24) imply that

$$
T_\lambda(x,z)(t) \leqslant D t^{\gamma-n+1} = De(t), \quad t \in [0,1].
$$

At the same time, by (22) and (23), for $t \in [0,1]$, we have

$$
\begin{aligned}
&T_\lambda(x,z)(t) \\
&\quad = \lambda \int_0^1 G(t,s) \Bigg[\int_0^s \overline{a}(s-\tau)^{\alpha-1} \big(\phi\big(\tau,\, I_{0^+}^{\mu_{n-1}} x(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1} x(\tau),\, \dots, \\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}} x(\tau),\, x(\tau),\, Ax(\tau)\big) + \psi\big(\tau, I_{0^+}^{\mu_{n-1}} z(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1} z(\tau),\, \dots, \\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}} z(\tau),\, z(\tau), Az(\tau)\big)\big)\,\mathrm{d}\tau \Bigg]^{q-1} \mathrm{d}s \\
&\quad \geqslant \lambda \int_0^1 G(t,s) \Bigg[\int_0^s \overline{a}(s-\tau)^{\alpha-1} \big(c^{\sigma^{1/(q-1)}} D^{-\sigma^{1/(q-1)}} \tau^{(\gamma-1)\sigma^{1/(q-1)}} \\
&\qquad \times \phi(\tau,1,1,\dots,1) + 2^{-\sigma^{1/(q-1)}} b^{-\sigma^{1/(q-1)}} D^{-\sigma^{1/(q-1)}} \psi(\tau,1,1,\dots,1)\big)\,\mathrm{d}\tau \Bigg]^{q-1} \mathrm{d}s \\
&\quad \geqslant \lambda a_* t^{\gamma-\mu_{n-1}-1} D^{-\sigma} \overline{a}^{q-1} \int_0^1 j_A(s) \Bigg[\int_0^s (s-\tau)^{\alpha-1} \big(c^{\sigma^{1/(q-1)}} \tau^{(\gamma-1)\sigma^{1/(q-1)}} \\
&\qquad \times \phi(\tau,1,1,\dots,1) + 2^{-\sigma^{1/(q-1)}} b^{-\sigma^{1/(q-1)}} \psi(\tau,1,1,\dots,1)\big)\,\mathrm{d}\tau \Bigg]^{q-1} \mathrm{d}s.
\end{aligned} \tag{25}
$$

Formula (25) imply that

$$
T_\lambda(x,z)(t) \geqslant \frac{1}{D} t^{\gamma-\mu_{n-1}-1} = \frac{1}{D} e(t), \quad t \in [0,1].
$$

Hence, $T_\lambda : Q_e \times Q_e \to Q_e$. It is easy to prove that $T_\lambda : Q_e \times Q_e \to Q_e$ is a mixed monotone operator.

Finally, we show that the operator T_λ satisfies (12). For any $x, z \in Q_e$ and $l \in (0,1)$, by (S2) and Remark 1, for all $t \in [0,1]$, we have

$$\begin{aligned}
&\lambda \int_0^1 G(t,s)\Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\biggl(\phi\bigl(\tau,\, I_{0^+}^{\mu_{n-1}} lx(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1} lx(\tau),\, \dots,\\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}} lx(\tau),\, lx(\tau),\, Alx(\tau)\bigr) + \psi\biggl(\tau,\, I_{0^+}^{\mu_{n-1}}\frac{1}{l}z(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1}\frac{1}{l}z(\tau),\, \dots,\\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}}\frac{1}{l}z(\tau),\, \frac{1}{l}z(\tau),\, A\frac{1}{l}z(\tau)\biggr)\biggr)\,\mathrm{d}\tau\Biggr]^{q-1}\mathrm{d}s\\
&\quad\geqslant \lambda \int_0^1 G(t,s)\Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1} l^{\sigma^{1/(q-1)}}\bigl(\phi\bigl(\tau,\, I_{0^+}^{\mu_{n-1}} x(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1} x(\tau),\, \dots,\\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}} x(\tau),\, x(\tau),\, Ax(\tau)\bigr) + \psi\bigl(\tau,\, I_{0^+}^{\mu_{n-1}} z(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1} z(\tau),\, \dots,\\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}} z(\tau),\, z(\tau),\, Az(\tau)\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{q-1}\mathrm{d}s\\
&\quad\geqslant l^\sigma \lambda \int_0^1 G(t,s)\Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\bigl(\phi\bigl(\tau, I_{0^+}^{\mu_{n-1}} x(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1} x(\tau),\, \dots,\\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}} x(\tau),\, x(\tau),\, Ax(\tau)\bigr) + \psi\bigl(\tau,\, I_{0^+}^{\mu_{n-1}} z(\tau),\, I_{0^+}^{\mu_{n-1}-\mu_1} z(\tau),\, \dots,\\
&\qquad I_{0^+}^{\mu_{n-1}-\mu_{n-2}} z(\tau),\, z(\tau),\, Az(\tau)\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{q-1}\mathrm{d}s.
\end{aligned}\tag{26}$$

Formula (26) imply that

$$T_\lambda\left(lx, \frac{1}{l}z\right) \geqslant l^\sigma T_\lambda(x,z), \quad x, z \in Q_e.$$

Hence, Lemma 7 assume that there exists a unique positive solution $x_\lambda^* \in Q_e$ such that $T_\lambda(x_\lambda^*, x_\lambda^*) = x_\lambda^*$. It is easy to check that x_λ^* is a unique positive solution of (10) for any given $\lambda > 0$. Moreover, by Lemma 8, we have:

(i) For any $\lambda_0 \in (0, +\infty)$, $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$, $\lambda \to \lambda_0$;
(ii) If $0 < \sigma < 1/2$, then $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1}^* \leqslant x_{\lambda_2}^*$, $x_{\lambda_1}^* \neq x_{\lambda_2}^*$;
(iii) If $0 < \sigma < 1/2$, then $\lim_{\lambda\to 0}\|x_\lambda^*\| = 0$, $\lim_{\lambda\to+\infty}\|x_\lambda^*\| = +\infty$.

By Lemma 6, for any $t \in [0,1]$, we have

$$u_\lambda^*(t) = I_{0^+}^{\mu_{n-1}} x_\lambda^*(t), \qquad v_\lambda^*(t) = I_{0^+}^{\eta_{m-1}} y_\lambda^*(t). \tag{27}$$

Hence, by (27) and the monotonicity and continuity of $I_{0^+}^{\mu_{n-1}}$, we get:

(i) For any $\lambda_0 \in (0, +\infty)$, $\|u^*_\lambda - u^*_{\lambda_0}\| \to 0$, $\lambda \to \lambda_0$;
(ii) If $0 < \sigma < 1/2$, then $0 < \lambda_1 < \lambda_2$ implies $u^*_{\lambda_1} \leqslant u^*_{\lambda_2}$, $u^*_{\lambda_1} \neq u^*_{\lambda_2}$;
(iii) If $0 < \sigma < 1/2$, then $\lim_{\lambda\to 0} \|u^*_\lambda\| = 0$, $\lim_{\lambda\to+\infty} \|u^*_\lambda\| = +\infty$.

Moreover, for any $u_0(t) = I^{n-2}_{0^+} x_0 \in Q_e$ and $t \in [0,1]$, by Lemma 7, constructing a successively sequence

$$\begin{aligned} &x_{k+1}(t) \\ &\quad= \lambda \int_0^1 G(t,s)\Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\bigl(\phi\bigl(\tau,\, I^{\mu_{n-1}}_{0^+}x_k(\tau),\, I^{\mu_{n-1}-\mu_1}_{0^+}x_k(\tau),\, \dots, \\ &\qquad I^{\mu_{n-1}-\mu_{n-2}}_{0^+}x_k(\tau),\, x_k(\tau),\, Ax_k(\tau)\bigr) + \psi\bigl(\tau,\, I^{\mu_{n-1}}_{0^+}x_k(\tau),\, I^{\mu_{n-1}-\mu_1}_{0^+}x_k(\tau),\, \dots, \\ &\qquad I^{\mu_{n-1}-\mu_{n-2}}_{0^+}x_k(\tau),\, x_k(\tau),\, Ax_k(\tau)\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{q-1}\mathrm{d}s, \quad k = 0,1,2,\dots, \end{aligned}$$

by $u_{k+1}(t) = I^{\mu_{n-1}}_{0^+} x_{k+1}(t)$,

$$\begin{aligned} &u_{k+1}(t) \\ &\quad= I^{\mu_{n-1}}_{0^+}\Biggl\{\lambda \int_0^1 G(t,s)\Biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\bigl(\phi\bigl(\tau,\, u_k(\tau),\, D^{\mu_1}_{0^+}u_k(\tau),\, \dots, \\ &\qquad D^{\mu_{n-1}}_{0^+}u_k(\tau),\, u_k(\tau),\, Au_k^{(\mu_{n-1})}(\tau)\bigr) + \psi\bigl(\tau, u_k(\tau),\, D^{\mu_1}_{0^+}u_k(\tau),\, \dots, \\ &\qquad D^{\mu_{n-1}}_{0^+}u_k(\tau),\, u_k(\tau),\, Au_k^{(\mu_{n-1})}(\tau)\bigr)\bigr)\,\mathrm{d}\tau\Biggr]^{q-1}\mathrm{d}s\Biggr\}, \quad k = 0,1,2,\dots,\ t \in [0,1], \end{aligned}$$

and we have $\|u_k - u^*_\lambda\| = \|I^{\mu_{n-1}}_{0^+}x_k - I^{\mu_{n-1}}_{0^+}x^*_\lambda\| \to 0$ as $k \to \infty$, the convergence rate is

$$\|u_k - u^*_\lambda\| = \bigl\|I^{\mu_{n-1}}_{0^+}x_k - I^{\mu_{n-1}}_{0^+}x^*_\lambda\bigr\| = o\bigl(1 - r^{\sigma^m}\bigr),$$

where r is a constant, $0 < r < 1$, and dependent on u_0. By (11), we easily get

$$\begin{aligned} y^*_\lambda(t) = \mu \int_0^1 H(t,s)\Biggl[\int_0^s \overline{b}(s-w)^{\beta-1} g\bigl(w,\, I^{\mu_{n-1}}_{0^+}x^*_\lambda(w),\, I^{\mu_{n-1}-\eta_1}_{0^+}x^*_\lambda(w),\, \dots, \\ I^{\mu_{n-1}-\eta_{m-1}}_{0^+}x^*_\lambda(w)\bigr)\,\mathrm{d}w\Biggr]^{q-1}\mathrm{d}s, \quad t \in [0,1]. \end{aligned} \tag{28}$$

By (19), (27), (28), and $x^*_\lambda \in Q_e$, we get $(u^*_\lambda, v^*_\lambda)$, which satisfies $(*)$. Therefore, the proof of Theorem 1 is completed. □

Remark 3. p-Laplacian boundary value system is an great extension from general fractional-order differential equation, general fractional-order differential equation is a special case of p-Laplacian fractional-order differential equation, p-Laplacian fractional-order differential equation is general fractional-order differential equation when $q = 2$.

4 An example

Example 1. Consider the following boundary value problem:

$$
\begin{aligned}
&D_{0^+}^{3/4}\big(\varphi_2\big(D_{0+}^{5/2}u\big)\big)(t)+\lambda^2 f\big(t,u(t),\ D_{0^+}^{1/2}u(t),v(t)\big)=0, \quad 0<t<1,\\
&D_{0^+}^{3/4}\big(\varphi_2\big(D_{0+}^{3/2}v\big)\big)(t)+\mu^2 g\big(t,u(t)\big)=0, \quad 0<t<1,\\
&u(0)=u'(0)=0,\ \ D_{0+}^{\gamma}u(0)=D_{0+}^{\gamma+1}u(0)=0, \quad u'(1)=\frac{1}{2}\int_0^{3/4}u'(t)\,\mathrm{d}A(t),\\
&v(0)=v'(0)=0,\ \ D_{0+}^{\delta}v(0)=D_{0+}^{\delta+1}v(0)=0, \quad v(1)=\frac{5}{6}\int_0^{3/4}v(t)\,\mathrm{d}B(t),
\end{aligned}
\tag{29}
$$

where $\gamma=5/2$, $\delta=3/2$, $\alpha=\beta=3/4$, $h(s)=a(s)=1$, $\eta=\vartheta=3/4$, $\chi=1/2$, $\iota=5/6$, $p=3$, $q=3/2$, and

$$
\begin{aligned}
\phi(t,x_1,x_2,x_3)&=\big(t^{-1/4}+\cos t\big)x_1^{1/9}+2tx_2^{1/8}+2x_3^{1/16},\\
\psi(t,x_1,x_2,x_3)&=t^{-1/16}x_1^{-1/8}+x_2^{-1/16}+(2-t)x_3^{-1/15},\\
g(t,u)&=\big(3t+t^2\big)u^{3/5}+(t\sin t+t)u^{2/3},
\end{aligned}
$$

$$
A(t)=\begin{cases}0, & t\in[0,1/2),\\ 6, & t\in[1/2,3/4),\\ 2, & t\in[3/4,1],\end{cases}\qquad
B(t)=\begin{cases}0, & t\in[0,1/2),\\ 4, & t\in[1/2,3/4),\\ 3, & t\in[3/4,1].\end{cases}
$$

Hence,

$$
\begin{aligned}
&\int_0^{\eta}\chi t^{\gamma-n+1}h(t)\,\mathrm{d}A(t)=\frac{1}{2}\int_0^{3/4}t^{1/2}\,\mathrm{d}A(t)=3\bigg(\frac{1}{2}\bigg)^{1/2}-2\bigg(\frac{3}{4}\bigg)^{1/2}<1,\\
&\int_0^{\vartheta}\iota t^{\delta-m+1}a(t)\,\mathrm{d}B(t)=\frac{5}{6}\int_0^{3/4}t^{1/2}\,\mathrm{d}B(t)\approx 0.1117<1.
\end{aligned}
$$

Moreover, for any $(t,x_1,x_2,x_3)\in(0,1)\times(0,\infty)^3$ and $0<l<1$, we have

$$
\begin{aligned}
\phi(t,lx_1,lx_2,lx_3)&=\big(t^{-1/4}+\cos t\big)(lx_1)^{1/9}+2t(lx_2)^{1/8}+2(lx_3)^{1/16}\\
&\geqslant l^{1/8}\big(\big(t^{-1/4}+\cos t\big)x_1^{1/9}+2tx_2^{1/8}+2x_3^{1/16}\big)\\
&=l^{1/8}\phi(t,x_1,x_2,x_3)=l^{\sigma^{1/(q-1)}}\phi(t,x_1,x_2,x_3),
\end{aligned}
$$

$$
\begin{aligned}
\psi\big(t,l^{-1}x_1,l^{-1}x_2,l^{-1}x_3\big)&=t^{-1/16}\big(l^{-1}x_1\big)^{-1/8}+\big(l^{-1}x_2\big)^{-1/16}\\
&\quad+(2-t)\big(l^{-1}x_3\big)^{-1/15}
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant l^{1/8}\big(t^{-1/16}x_1^{-1/8} + x_2^{-1/16} + (2-t)x_3^{-1/15}\big)\\
&= l^{1/8}\psi(t,x_1,x_2,x_3) = l^{\sigma^{1/(q-1)}}\psi(t,x_1,x_2,x_3),
\end{aligned}
$$

$$
\begin{aligned}
g(t,lu) &= \big(3t+t^2\big)(lu)^{3/5} + (t\sin t + t)(lu)^{2/3}\\
&\geqslant l^{2/3}\big(\big(3t+t^2\big)u^{3/5} + (t\sin t+t)u^{2/3}\big) = l^{2/3}g(t,u)\\
&= l^{\varsigma^{1/(q-1)}}g(t,u).
\end{aligned}
$$

Noting $\sigma = 1/(2\sqrt{2}) < 1$, $\varsigma = \sqrt{6}/3$, $\psi(\tau,1,1,1) = \tau^{-1/16} + 3 - \tau$, $\phi(\tau,1,1,1) = \tau^{-1/4} + \cos\tau + 2\tau + 2$, $g(\tau,1) = 3\tau + \tau^2 + \tau\sin\tau + \tau$, we have

$$
\begin{aligned}
&0 < \int_0^1 \phi^2(\tau,1,1,\dots,1)\,\mathrm{d}\tau = \int_0^1 \big(\tau^{-1/4} + \cos\tau + 2\tau + \tau\big)^2\,\mathrm{d}\tau \leqslant 27 + \frac{16}{7}\\
&\quad < +\infty,\\
&0 < \int_0^1 \tau^{-2(\gamma-1)\sigma^{1/(q-1)}}\psi^2(\tau,1,1,\dots,1)\,\mathrm{d}\tau \leqslant \int_0^1 \tau^{-3/8}\big(\tau^{-1/16}+3\big)^2\,\mathrm{d}\tau\\
&\quad = 11 + \frac{96}{9} < +\infty,\\
&0 < \int_0^1 g^2(\tau,1,1,\dots,1)\,\mathrm{d}\tau \leqslant \int_0^1 \big(3\tau + \tau^2 + \tau\sin\tau + \tau\big)^2\,\mathrm{d}\tau \leqslant 36 < +\infty.
\end{aligned}
$$

Thus, assumptions (S1)–(S4) of Theorem 1 hold. Then Theorem 1 implies that problem (29) has a unique solution. Furthermore, when $\lambda \to \lambda_0$, $\lambda_0 \in (0,+\infty)$, we have $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$. Since $\sigma = 1/(2\sqrt{2}) \in (0,1/2)$, $0 < \lambda_1 < \lambda_2$ implies

$$
x_{\lambda_1}^*(t) \leqslant x_{\lambda_2}^*(t), \qquad x_{\lambda_1}^*(t) \neq x_{\lambda_2}^*(t), \qquad \lim_{\lambda\to 0}\|x_\lambda^*\| = 0, \qquad \lim_{\lambda\to+\infty}\|x_\lambda^*\| = +\infty.
$$

By $u_\lambda^*(t) = I_{0^+}^1 x_\lambda^*(t)$, we can easily get that:

(i) $\lambda_0 \in (0,+\infty)$, $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$, $\lambda \to \lambda_0$;
(ii) $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$;
(iii) $\lim_{\lambda\to 0}\|u_\lambda^*\| = 0$, $\lim_{\lambda\to+\infty}\|u_\lambda^*\| = +\infty$.

In addition, for any initial $u_0 = I_{0^+}^1 x_0 \in Q_e$, we construct a successively sequence

$$
\begin{aligned}
&x_{k+1}(t)\\
&\quad = \lambda\int_0^1 G(t,s)\Bigg[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(t,\, I_{0^+}^1 x_k(t),\, I_{0^+}^{1/2}x_k(t),\, Ax_k(t)\big)\\
&\qquad + \psi\big(t,\, I_{0^+}^1 x_k(t),\, I_{0^+}^{1/2}x_k(t),\, Ax_k(t)\big)\big)\,\mathrm{d}\tau\Bigg]^{q-1}\mathrm{d}s, \quad t\in[0,1],\ k = 0,1,2,\dots,
\end{aligned}
$$

by $u_{k+1}(t) = I^1_{0^+} x_{k+1}(t)$, we have

$$
\begin{aligned}
& u_{k+1}(t) \\
&\quad = I^1_{0^+}\left\{ \int_0^1 \lambda G(t,s) \left[\int_0^s \overline{a}(s-\tau)^{\alpha-1} \big(\phi\big(t,\, u_k(t),\, D^{1/2}_{0^+} u_k(t),\, Au'_k(t)\big) \right.\right. \\
&\qquad \left.\left. + \psi\big(t, u_k(t),\, D^{1/2}_{0^+} u_k(t),\, Au'_k(t)\big)\big)\, \mathrm{d}\tau \right]^{q-1} \mathrm{d}s \right\}, \quad t \in [0,1],\ k = 0,1,2,\dots,
\end{aligned}
$$

and we have $\|u_k - u^*_\lambda\| = \|I^1_{0^+} x_k - I^1_{0^+} x^*_\lambda\| \to 0$ as $k \to \infty$, the convergence rate is

$$
\|u_k - u^*_\lambda\| = \big\|I^1_{0^+} x_k - I^1_{0^+} x^*_\lambda\big\| = o\big(1 - r^{\sigma^k}\big),
$$

where r is a constant dependent on u_0 and $0 < r < 1$.

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