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On the compactness of the set of L_2 trajectories of the control system

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Abstract. In this paper, the compactness of the set of L_2 trajectories of the control system described by the Urysohn-type integral equation is studied. The control functions are chosen from the closed ball of the space L_2 with radius r and centered at the origin. Existence of an optimal trajectory of the optimal control problem with lower semicontinuous payoff functional is discussed.

Keywords: control system, Urysohn integral equation, integral constraint, set of L_2 trajectories, compactness.

1 Introduction

The mathematical models of many processes in mechanics, physics, economy, biology are described via nonlinear integral equations (see, e.g., [2,6,7,18,23] and references therein). W. Heisenberg in his well-known *Physics and Philosophy* writes: "The final equation of motion for matter will probably be some quantized nonlinear wave equation... This wave equation will probably be equivalent to rather complicated sets of integral equations..." (see [10, p. 68]). It should be noted that the theory of integral equations is considered one of the origins of contemporary functional analysis (see, e.g., [11, p. 2, Chap. 1]). Some processes described by the integral equations quite often include a parameters, which characterize the control efforts or describe the model uncertainties. Many of control efforts and some of uncertainties have limited resources and as usual they are exhausted by consumption, say as fuel, energy, finance etc. These kinds of efforts in general are characterized by an integral constraint on the control functions (see, e.g., [8, 9, 17, 19]).

Control systems described by the integral equations with geometric constraints on the controls are discussed in [1, 3, 4, 22] (see also the references in these papers). The properties of the set of trajectories of the control systems with integral constraints on the control functions and described by different type integral equations are considered in [13, 14]. In [12, 15] and [16], the methods for approximate construction of the set of trajectories are discussed. Note that in aforementioned papers, it is accepted that the trajectories of the considered equations are continuous function. In this paper, the functions from the space L_2 are chosen as a trajectory of the Urysohn-type integral equation. Note that L_2 solution concept is very useful tool for investigation various problems arising in theory and applications (see [5,20,21] and references therein). In the presented paper, the compactness of the set of trajectories is established, which is applied to prove existence theorem for optimal control problem with semicontinuous payoff functional. The distance between the trajectories generated by admissible control functions is evaluated.

The paper is organized as follows: In Section 2, the basic conditions, which satisfy the system's equation, are given. In Section 3, it is proved that every admissible control function generates unique trajectory (Proposition 3). In Section 4, it is shown that the set of trajectories is a bounded subset of the space L_2 (Proposition 4). The distance evaluation between the trajectories is given in Section 5 (Proposition 5). In Section 6, it is proved that the set of trajectories is a compact subset of the space L_2 (Theorem 1), and the existence of optimal trajectories in given optimal control problem, where the payoff of the control is a lower semicontinuous functional, is discussed (Proposition 6).

2 Preliminaries

Consider control system described by the Urysohn-type integral equation

$$x(\xi) = f\bigl(\xi, x(\xi)\bigr) + \lambda \int\limits_{\Omega} \bigl[K_1\bigl(\xi, s, x(s)\bigr) + K_2\bigl(\xi, s, x(s)\bigr)u(s)\bigr]\,\mathrm{d}s, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, $\xi \in \Omega$, $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable set such that $\mu(\Omega) < \infty$, $\mu(\Omega)$ is the Lebesgue measure of the set Ω.

Let $r > 0$ be given number,

$$U_r = \bigl\{u(\cdot) \in L_2\bigl(\Omega; \mathbb{R}^m\bigr)\colon \bigl\|u(\cdot)\bigr\|_2 \leqslant r\bigr\},$$

where $L_2(\Omega; \mathbb{R}^m)$ is the space of Lebesgue measurable functions $u(\cdot)\colon\ \Omega \to \mathbb{R}^m$ such that $\|u(\cdot)\|_2 < +\infty$, $\|u(\cdot)\|_2 = (\int_{\Omega} \|u(s)\|^2\,\mathrm{d}s)^{1/2}$, $\|\cdot\|$ denotes the Euclidean norm.

U_r is called the set of admissible control functions, and every $u(\cdot) \in U_r$ is said to be an admissible control function.

It is assumed that the functions and a number $\lambda \in \mathbb{R}$ given in system (1) satisfy the following conditions:

(A) The function $f(\cdot, x) : \Omega \to \mathbb{R}^n$ is Lebesgue measurable for every fixed $x \in \mathbb{R}^n$, $f(\cdot, 0) \in L_2(\Omega; \mathbb{R}^n)$, and there exists $l_0(\cdot) \in L_\infty(\Omega; \mathbb{R})$ such that for almost all (a.a.) $\xi \in \Omega$, the inequality

$$\left\|f(\xi, x_1) - f(\xi, x_2)\right\| \leqslant l_0(\xi)\|x_1 - x_2\|$$

is satisfied for every $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$, where $L_\infty(\Omega; \mathbb{R}^m)$ is the space of Lebesgue measurable functions $u(\cdot) : \Omega \to \mathbb{R}^m$ such that $\|u(\cdot)\|_\infty < +\infty$, $\|u(\cdot)\|_\infty = \inf\{c > 0\colon \|u(s)\| \leqslant c$ for almost all $s \in \Omega\}$;

(B) The function $K_1(\cdot, \cdot, x) : \Omega \times \Omega \to \mathbb{R}^n$ is Lebesgue measurable for every fixed $x \in \mathbb{R}^n$, $K_1(\cdot, \cdot, 0) \in L_2(\Omega \times \Omega; \mathbb{R}^n)$, and there exists $l_1(\cdot, \cdot) \in L_2(\Omega \times \Omega; \mathbb{R})$ such that for a.a. $(\xi, s) \in \Omega \times \Omega$, the inequality

$$\left\|K_1(\xi, s, x_1) - K_1(\xi, s, x_2)\right\| \leqslant l_1(\xi, s)\|x_1 - x_2\|$$

is satisfied for every $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$;

(C) The function $K_2(\cdot, \cdot, x) : \Omega \times \Omega \to \mathbb{R}^{n \times m}$ is Lebesgue measurable for every fixed $x \in \mathbb{R}^n$, $K_2(\cdot, \cdot, 0) \in L_2(\Omega \times \Omega; \mathbb{R}^{n \times m})$, and there exists $l_2(\cdot, \cdot) \in L_\infty(\Omega \times \Omega; \mathbb{R})$ such that for a.a. $(\xi, s) \in \Omega \times \Omega$, the inequality

$$\left\|K_2(\xi, s, x_1) - K_2(\xi, s, x_2)\right\| \leqslant l_2(\xi, s)\|x_1 - x_2\|$$

is satisfied for every $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$;

(D) The inequality $6[h_0^2 + \lambda^2 h_1^2 + \lambda^2 r^2 h_2^2 \mu(\Omega)] < 1$ is satisfied, where $h_0 = \|l_0(\cdot)\|_\infty$, $h_1 = \|l_1(\cdot, \cdot)\|_2 = (\int_\Omega \int_\Omega l_1(\xi, s)^2 \,\mathrm{d}s \,\mathrm{d}\xi)^{1/2}$, $h_2 = \|l_2(\cdot, \cdot)\|_\infty$.

Let us define a trajectory of system (1) generated by an admissible control function $u(\cdot) \in U_r$. A function $x(\cdot) \in L_2(\Omega; \mathbb{R}^n)$ satisfying the integral equation (1) for a.a. $\xi \in \Omega$ is said to be a trajectory of system (1) generated by the admissible control function $u(\cdot) \in U_r$. The set of trajectories of system (1) generated by all admissible control functions $u(\cdot) \in U_r$ is denoted by $\mathbf{X}_r$ and is called briefly the set of trajectories of system (1).

Now we will formulate some auxiliary propositions, which will be used in following arguments.

Proposition 1. *The inequalities*

$$\begin{aligned}
\left\|f(\xi, x)\right\| &\leqslant h_0\|x\| + \left\|f(\xi, 0)\right\|, \\
\left\|K_1(\xi, s, x)\right\| &\leqslant l_1(\xi, s)\|x\| + \left\|K_1(\xi, s, 0)\right\|, \\
\left\|K_2(\xi, s, x)\right\| &\leqslant h_2\|x\| + \left\|K_2(\xi, s, 0)\right\|
\end{aligned}$$

are satisfied for every $x \in \mathbb{R}^n$ and a.a. $(\xi, s) \in \Omega \times \Omega$.

The proof of the proposition follows from conditions (A), (B) and (C).

Proposition 2. *For every $a_1, a_2, \ldots, a_n \in \mathbb{R}$, the following inequality is satisfied*:

$$(a_1 + a_2 + \ldots + a_n)^2 \leqslant n\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

3 Existence and uniqueness of trajectory

Denote

$$
\begin{aligned}
f_* &= \int\limits_{\Omega} \left\| f(\xi,0) \right\|^2 \mathrm{d}\xi, \\
k_i &= \int\limits_{\Omega} \int\limits_{\Omega} \left\| K_i(\xi,s,0) \right\|^2 \mathrm{d}s\, \mathrm{d}\xi, \quad i=1,2.
\end{aligned} \tag{2}
$$

Proposition 3. *Every admissible control function $u(\cdot) \in U_r$ generates unique trajectory of system* (1).

Proof. Let us choose an arbitrary $u_*(\cdot) \in U_r$. Define a map $x(\cdot) \to F(x(\cdot))|(\cdot)$, $x(\cdot) \in L_2(\Omega;\mathbb{R}^n)$, setting

$$
F\bigl(x(\cdot)\bigr)\big|(\xi) = f\bigl(\xi, x(\xi)\bigr) + \lambda \int\limits_{\Omega} \bigl[K_1\bigl(\xi,s,x(s)\bigr) + K_2\bigl(\xi,s,x(s)\bigr)u_*(s)\bigr]\, \mathrm{d}s. \tag{3}
$$

According to conditions (A), (B) and (C), the function $f(\cdot,x)$ is measurable for every fixed $x \in \mathbb{R}^n$, for a.a. $\xi \in \Omega$, the function $f(\xi,\cdot)$ is continuous, the functions $K_i(\cdot,\cdot,x)$ $(i=1,2)$ are measurable for every fixed $x \in \mathbb{R}^n$, for a.a. $(\xi,s) \in \Omega \times \Omega$, the functions $K_i(\xi,s,\cdot,)$ $(i=1,2)$ are continuous. Then we obtain that for every $x(\cdot) \in L_2(\Omega;\mathbb{R}^n)$, the function $\xi \to F(x(\cdot))|(\xi)$, $\xi \in \Omega$, is Lebesgue measurable (see [24, p. 71]).

Now we will prove that the inclusion $F(x(\cdot))|(\cdot) \in L_2(\Omega;\mathbb{R}^n)$ is satisfied for every $x(\cdot) \in L_2(\Omega;\mathbb{R}^n)$. Taking into consideration that $u_*(\cdot) \in U_r$, from Proposition 1, (3) and Hölder's inequality it follows that

$$
\begin{aligned}
&\bigl\|F\bigl(x(\cdot)\bigr)\big|(\xi)\bigr\| \\
&\quad \leqslant h_0\bigl\|x(\xi)\bigr\| + \bigl\|f(\xi,0)\bigr\| + \lambda \int\limits_{\Omega} \bigl[l_1(\xi,s)\bigl\|x(s)\bigr\| + \bigl\|K_1(\xi,s,0)\bigr\|\bigr]\, \mathrm{d}s \\
&\qquad + \lambda \int\limits_{\Omega} \bigl[h_2\bigl\|x(s)\bigr\| + \bigl\|K_2(\xi,s,0)\bigr\|\bigr]\bigl\|u_*(s)\bigr\|\, \mathrm{d}s \\
&\quad \leqslant h_0\bigl\|x(\xi)\bigr\| + \bigl\|f(\xi,0)\bigr\| + \lambda \biggl(\int\limits_{\Omega} l_1(\xi,s)^2\, \mathrm{d}s\biggr)^{1/2} \biggl(\int\limits_{\Omega} \bigl\|x(s)\bigr\|^2 \mathrm{d}s\biggr)^{1/2} \\
&\qquad + \lambda\mu(\Omega)^{1/2} \biggl(\int\limits_{\Omega} \bigl\|K_1(\xi,s,0)\bigr\|^2 \mathrm{d}s\biggr)^{1/2} + \lambda h_2 r \biggl(\int\limits_{\Omega} \bigl\|x(s)\bigr\|^2 \mathrm{d}s\biggr)^{1/2} \\
&\qquad + \lambda r \biggl(\int\limits_{\Omega} \bigl\|K_2(\xi,s,0)\bigr\|^2 \mathrm{d}s\biggr)^{1/2}
\end{aligned}
$$

$$= h_0\|x(\xi)\| + \|f(\xi,0)\| + \lambda\biggl(\int_\Omega l_1(\xi,s)^2\,\mathrm{d}s\biggr)^{1/2}\|x(\cdot)\|_2 + \lambda\mu(\Omega)^{1/2}\biggl(\int_\Omega \|K_1(\xi,s,0)\|^2\,\mathrm{d}s\biggr)^{1/2} + \lambda h_2 r\|x(\cdot)\|_2 + \lambda r\biggl(\int_\Omega \|K_2(\xi,s,0)\|^2\,\mathrm{d}s\biggr)^{1/2} \tag{4}$$

for a.a. $\xi \in \Omega$. (4) and Proposition 2 imply

$$\|F(x(\cdot))|(\xi)\|^2 \leqslant 6\biggl[h_0^2\|x(\xi)\|^2 + \|f(\xi,0)\|^2 + \lambda^2\int_\Omega l_1(\xi,s)^2\,\mathrm{d}s\cdot\|x(\cdot)\|_2^2 + \lambda^2\mu(\Omega)\int_\Omega \|K_1(\xi,s,0)\|^2\,\mathrm{d}s + \lambda^2h_2^2r^2\|x(\cdot)\|_2^2 + \lambda^2r^2\int_\Omega \|K_2(\xi,s,0)\|^2\,\mathrm{d}s\biggr]$$

for a.a. $\xi \in \Omega$, and hence

$$\int_\Omega \|F(x(\cdot))|(\xi)\|^2\,\mathrm{d}\xi \leqslant 6\biggl[h_0^2\int_\Omega \|x(\xi)\|^2\,\mathrm{d}\xi + \int_\Omega \|f(\xi,0)\|^2\,\mathrm{d}\xi + \lambda^2\int_\Omega\int_\Omega l_1(\xi,s)^2\,\mathrm{d}s\,\mathrm{d}\xi\cdot\|x(\cdot)\|_2^2 + \lambda^2\mu(\Omega)\int_\Omega\int_\Omega \|K_1(\xi,s,0)\|^2\,\mathrm{d}s\,\mathrm{d}\xi + \lambda^2h_2^2r^2\mu(\Omega)\|x(\cdot)\|_2^2 + \lambda^2r^2\int_\Omega\int_\Omega \|K_2(\xi,s,0)\|^2\,\mathrm{d}s\,\mathrm{d}\xi\biggr].$$

The last inequality and (2) yield

$$\|F(x(\cdot))|(\cdot)\|_2 \leqslant \sqrt{6}\bigl[h_0^2\|x(\cdot)\|_2^2 + f_* + \lambda^2h_1^2\|x(\cdot)\|_2^2 + \lambda^2h_2^2r^2\mu(\Omega)\|x(\cdot)\|_2^2 + \lambda^2\mu(\Omega)k_1 + \lambda^2r^2k_2\bigr]^{1/2}. \tag{5}$$

Since $x(\cdot) \in L_2(\Omega;\mathbb{R}^n)$, then we have from (5) that $\|F(x(\cdot))|(\cdot)\|_2 < +\infty$ and consequently $F(x(\cdot))|(\cdot) \in L_2(\Omega;\mathbb{R}^n)$.

Now let us prove that the map $F(x(\cdot))|(\cdot) : L_2(\Omega;\mathbb{R}^n) \to L_2(\Omega;\mathbb{R}^n)$ is contractive.

Let us choose arbitrary $x_1(\cdot) \in L_2(\Omega; \mathbb{R}^n)$ and $x_2(\cdot) \in L_2(\Omega; \mathbb{R}^n)$. Since $u_*(\cdot) \in U_r$, then from conditions (A), (B), (C) and Hölder's inequality it follows that

$$\begin{aligned}
&\big\|F\big(x_1(\cdot)\big)\big|(\xi) - F\big(x_2(\cdot)\big)\big|(\xi)\big\| \\
&\quad \leqslant h_0\big\|x_1(\xi) - x_2(\xi)\big\| + \lambda \int_{\Omega} l_1(\xi, s)\big\|x_1(s) - x_2(s)\big\| \,\mathrm{d}s \\
&\qquad + \lambda \int_{\Omega} h_2\big\|x_1(s) - x_2(s)\big\| \, \big\|u_*(s)\big\| \,\mathrm{d}s \\
&\quad \leqslant h_0\big\|x_1(\xi) - x_2(\xi)\big\| + \lambda\bigg(\int_{\Omega} l_1(\xi, s)^2 \,\mathrm{d}s\bigg)^{1/2}\bigg(\int_{\Omega} \big\|x_1(s) - x_2(s)\big\|^2 \,\mathrm{d}s\bigg)^{1/2} \\
&\qquad + \lambda h_2 r\bigg(\int_{\Omega} \big\|x_1(s) - x_2(s)\big\|^2 \,\mathrm{d}s\bigg)^{1/2} \\
&\quad = h_0\big\|x_1(\xi) - x_2(\xi)\big\| + \lambda\bigg(\int_{\Omega} l_1(\xi, s)^2 \,\mathrm{d}s\bigg)^{1/2} \big\|x_1(\cdot) - x_2(\cdot)\big\|_2 \\
&\qquad + \lambda h_2 r\big\|x_1(\cdot) - x_2(\cdot)\big\|_2
\end{aligned}$$

for a.a. $\xi \in \Omega$.

Proposition 2 yields

$$\begin{aligned}
&\big\|F\big(x_1(\cdot)\big)\big|(\xi) - F\big(x_2(\cdot)\big)\big|(\xi)\big\|^2 \\
&\quad \leqslant 3\bigg[h_0^2\big\|x_1(\xi) - x_2(\xi)\big\|^2 + \lambda^2 \int_{\Omega} l_1(\xi, s)^2 \,\mathrm{d}s \cdot \big\|x_1(\cdot) - x_2(\cdot)\big\|_2^2 \\
&\qquad + \lambda^2 h_2^2 r^2 \cdot \big\|x_1(\cdot) - x_2(\cdot)\big\|_2^2\bigg]
\end{aligned}$$

for a.a. $\xi \in \Omega$.

Integrating the last inequality, we get

$$\begin{aligned}
&\big\|F\big(x_1(\cdot)\big)\big|(\cdot) - F\big(x_2(\cdot)\big)\big|(\cdot)\big\|_2^2 \\
&\quad \leqslant 3\bigg[h_0^2\big\|x_1(\cdot) - x_2(\cdot)\big\|_2^2 + \lambda^2\big\|x_1(\cdot) - x_2(\cdot)\big\|_2^2 \int_{\Omega}\int_{\Omega} l_1(\xi, s)^2 \,\mathrm{d}s \,\mathrm{d}\xi \\
&\qquad + \lambda^2 h_2^2 r^2 \mu(\Omega)\big\|x_1(\cdot) - x_2(\cdot)\big\|_2^2\bigg] \\
&\quad = 3\big[h_0^2 + \lambda^2 h_1^2 + \lambda^2 h_2^2 r^2 \mu(\Omega)\big] \big\|x_1(\cdot) - x_2(\cdot)\big\|_2^2,
\end{aligned}$$

and hence

$$\begin{aligned} &\|F(x_1(\cdot))|(\cdot) - F(x_2(\cdot))|(\cdot)\|_2 \\ &\quad \leqslant \sqrt{3}\left[h_0^2 + \lambda^2 h_1^2 + \lambda^2 h_2^2 r^2 \mu(\Omega)\right]^{1/2} \|x_1(\cdot) - x_2(\cdot)\|_2. \end{aligned} \tag{6}$$

From inequality (6) and condition (D) it follows that the map $F(x(\cdot))|(\cdot) : L_2(\Omega;\mathbb{R}^n) \to L_2(\Omega;\mathbb{R}^n)$ is contractive. Since $L_2(\Omega;\mathbb{R}^n)$ is complete metric space, then according to the Banach fixed-point theorem, it has a unique fixed point $x_*(\cdot) \in L_2(\Omega;\mathbb{R}^n)$, which is unique trajectory of the equation

$$x_*(\xi) = f(\xi, x_*(\xi)) + \lambda \int_\Omega \left[K_1(\xi, s, x_*(s)) + K_2(\xi, s, x_*(s))u_*(s)\right] \mathrm{d}s. \qquad \square$$

4 Boundedness of the set of trajectories

Denote

$$k_* = \left(\frac{6[f_* + \lambda^2\mu(\Omega)k_1 + \lambda^2 r^2 k_2]}{1 - 6[h_0^2 + \lambda^2 h_1^2 + \lambda^2 h_2^2 r^2 \mu(\Omega)]}\right)^{1/2}, \tag{7}$$

where f_*, k_1 and k_2 are defined by relation (2).

The following proposition specifies boundedness of the set of trajectories of system (1) in the space $L_2(\Omega;\mathbb{R}^n)$.

Proposition 4. *For every $x(\cdot) \in \mathbf{X}_r$ the inequality $\|x(\cdot)\|_2 \leqslant k_*$ holds, where $k_* > 0$ is defined by relation* (7).

Proof. Let us choose an arbitrary $x(\cdot) \in \mathbf{X}_r$. Then there exists an admissible control function $u(\cdot) \in U_r$ such that

$$x(\xi) = f(\xi, x(\xi)) + \lambda \int_\Omega \left[K_1(\xi, s, x(s)) + K_2(\xi, s, x(s))u(s)\right] \mathrm{d}s \tag{8}$$

for a.a. $\xi \in \Omega$.

Similarly to (4), from (8), Proposition 1 and Hölder's inequality we have that

$$\begin{aligned} \|x(\xi)\| \leqslant{}& h_0\|x(\xi)\| + \|f(\xi,0)\| + \lambda\left(\int_\Omega l_1(\xi,s)^2\,\mathrm{d}s\right)^{1/2} \|x(\cdot)\|_2 \\ &+ \lambda\mu(\Omega)^{1/2}\left(\int_\Omega \|K_1(\xi,s,0)\|^2\,\mathrm{d}s\right)^{1/2} + \lambda h_2 r\|x(\cdot)\|_2 \\ &+ \lambda r\left(\int_\Omega \|K_2(\xi,s,0)\|^2\,\mathrm{d}s\right)^{1/2} \end{aligned} \tag{9}$$

for a.a. $\xi \in \Omega$. From (9) and Proposition 2 we obtain that

$$\begin{aligned} \|x(\xi)\|^2 \leqslant 6\bigg[& h_0^2\|x(\xi)\|^2 + \|f(\xi,0)\|^2 \\ & + \lambda^2\mu(\Omega)\int_\Omega \|K_1(\xi,s,0)\|^2\,\mathrm{d}s + \lambda^2 r^2\int_\Omega \|K_2(\xi,s,0)\|^2\,\mathrm{d}s \\ & + \lambda^2\int_\Omega l_1(\xi,s)^2\,\mathrm{d}s\,\|x(\cdot)\|_2^2 + \lambda^2 h_2^2 r^2\|x(\cdot)\|_2^2\bigg] \end{aligned}$$

for a.a. $\xi \in \Omega$. Integrating the last inequality, we have from (2) that

$$\begin{aligned} \|x(\cdot)\|_2^2 \leqslant{} & 6\big[f_* + \lambda^2\mu(\Omega)k_1 + \lambda^2 r^2 k_2\big] \\ & + 6\big[h_0^2 + \lambda^2 h_1^2 + \lambda^2 h_2^2 r^2\mu(\Omega)\big]\|x(\cdot)\|_2^2. \end{aligned} \tag{10}$$

According to condition (D), we have that $6[h_0^2 + \lambda^2 h_1^2 + \lambda^2 h_2^2 r^2\mu(\Omega)] < 1$. Finally, from (7) and (10) we get that $\|x(\cdot)\|_2 \leqslant k_*$. Since $x(\cdot) \in \mathbf{X}_r$ is arbitrarily chosen, the proof of the proposition is completed. □

5 Distance between trajectories

The following proposition gives us an evaluation between trajectories of system (1) generated by different admissible control functions. Denote

$$\gamma_* = \left(\frac{5\lambda^2 h_2^2 k_*^2\mu(\Omega) + 5\lambda^2 k_2}{1 - 5[h_0^2 + \lambda^2 h_1^2 + \lambda^2 h_2^2 r^2\mu(\Omega)]}\right)^{1/2}. \tag{11}$$

Proposition 5. *Let $x_1(\cdot) \in \mathbf{X}_r$ and $x_2(\cdot) \in \mathbf{X}_r$ be the trajectories of system* (1) *generated by the admissible control functions $u_1(\cdot) \in U_r$ and $u_2(\cdot) \in U_r$, respectively. Then*

$$\|x_2(\cdot) - x_1(\cdot)\|_2 \leqslant \gamma_*\|u_2(\cdot) - u_1(\cdot)\|_2.$$

Proof. Since $x_1(\cdot) \in \mathbf{X}_r$ and $x_2(\cdot) \in \mathbf{X}_r$ are the trajectories of system (1) generated by the admissible control functions $u_1(\cdot) \in U_r$ and $u_2(\cdot) \in U_r$, respectively, we have

$$x_1(\xi) = f\big(\xi, x_1(\xi)\big) + \lambda\int_\Omega \big[K_1\big(\xi,s,x_1(s)\big) + K_2\big(\xi,s,x_1(s)\big)u_1(s)\big]\,\mathrm{d}s, \tag{12}$$

$$x_2(\xi) = f\big(\xi, x_2(\xi)\big) + \lambda\int_\Omega \big[K_1\big(\xi,s,x_2(s)\big) + K_2\big(\xi,s,x_2(s)\big)u_2(s)\big]\,\mathrm{d}s \tag{13}$$

for a.a. $\xi \in \Omega$. Since $u_1(\cdot) \in U_r$, $u_2(\cdot) \in U_r$, then (12), (13), conditions (A), (B), (C), Hölder's inequality and Proposition 1 imply

$$\begin{aligned}
&\left\|x_2(\xi) - x_1(\xi)\right\| \\
&\quad\leqslant \left\|f\bigl(\xi, x_2(\xi)\bigr) - f\bigl(\xi, x_1(\xi)\bigr)\right\| \\
&\qquad + \lambda \int_\Omega \left\|K_1\bigl(\xi, s, x_2(s)\bigr) - K_1\bigl(\xi, s, x_1(s)\bigr)\right\| \mathrm{d}s \\
&\qquad + \lambda \int_\Omega \left\|K_2\bigl(\xi, s, x_2(s)\bigr) - K_2\bigl(\xi, s, x_1(s)\bigr)\right\| \left\|u_2(s)\right\| \mathrm{d}s \\
&\qquad + \lambda \int_\Omega \left\|K_2\bigl(\xi, s, x_1(s)\bigr)\right\| \left\|u_2(s) - u_1(s)\right\| \mathrm{d}s \\
&\quad\leqslant h_0\left\|x_2(\xi) - x_1(\xi)\right\| + \lambda \int_\Omega l_1(\xi, s)\left\|x_2(s) - x_1(s)\right\| \mathrm{d}s \\
&\qquad + \lambda h_2 \int_\Omega \left\|x_2(s) - x_1(s)\right\| \left\|u_2(s)\right\| \mathrm{d}s \\
&\qquad + \lambda \int_\Omega \bigl[h_2\left\|x_1(s)\right\| + \left\|K_2(\xi, s, 0)\right\|\bigr] \left\|u_2(s) - u_1(s)\right\| \mathrm{d}s \\
&\quad\leqslant h_0\left\|x_2(\xi) - x_1(\xi)\right\| + \lambda\biggl(\int_\Omega l_1(\xi, s)^2 \,\mathrm{d}s\biggr)^{1/2} \left\|x_2(\cdot) - x_1(\cdot)\right\|_2 \\
&\qquad + \lambda h_2 r\left\|x_2(\cdot) - x_1(\cdot)\right\|_2 + \lambda h_2\left\|x_1(\cdot)\right\|_2 \left\|u_2(\cdot) - u_1(\cdot)\right\|_2 \\
&\qquad + \lambda\biggl(\int_\Omega \left\|K_2(\xi, s, 0)\right\|^2 \mathrm{d}s\biggr)^{1/2} \left\|u_2(\cdot) - u_1(\cdot)\right\|_2
\end{aligned}$$

for a.a. $\xi \in \Omega$. The last inequality, Proposition 2 and Proposition 4 yield

$$\begin{aligned}
&\left\|x_2(\xi) - x_1(\xi)\right\|^2 \\
&\quad\leqslant 5\biggl[h_0^2\left\|x_2(\xi) - x_1(\xi)\right\|^2 + \lambda^2 \int_\Omega l_1(\xi, s)^2 \,\mathrm{d}s \cdot \left\|x_2(\cdot) - x_1(\cdot)\right\|_2^2 \\
&\qquad + \lambda^2 h_2^2 r^2\left\|x_2(\cdot) - x_1(\cdot)\right\|_2^2 + \lambda^2 h_2^2 k_*^2\left\|u_2(\cdot) - u_1(\cdot)\right\|_2^2 \\
&\qquad + \lambda^2 \int_\Omega \left\|K_2(\xi, s, 0)\right\|^2 \mathrm{d}s \cdot \left\|u_2(\cdot) - u_1(\cdot)\right\|_2^2\biggr]
\end{aligned} \tag{14}$$

for a.a. $\xi \in \Omega$, where k_* is defined by relation (7).

Integrating inequality (14), we have from (2) that

$$\begin{aligned} &\|x_2(\cdot) - x_1(\cdot)\|_2^2 \\ &\quad \leqslant 5\big[h_0^2\|x_2(\cdot) - x_1(\cdot)\|_2^2 + \lambda^2 h_1^2\|x_2(\cdot) - x_1(\cdot)\|_2^2 \\ &\qquad + \lambda^2 h_2^2 r^2 \mu(\Omega)\|x_2(\cdot) - x_1(\cdot)\|_2^2 + \lambda^2 h_2^2 k_*^2 \mu(\Omega)\|u_2(\cdot) - u_1(\cdot)\|_2^2 \\ &\qquad + \lambda^2 k_2\|u_2(\cdot) - u_1(\cdot)\|_2^2\big]. \end{aligned} \tag{15}$$

From condition (D), (11) and (15) we conclude that

$$\begin{aligned} &\|x_2(\cdot) - x_1(\cdot)\|_2 \\ &\quad \leqslant \left(\frac{5\lambda^2 h_2^2 k_*^2 \mu(\Omega) + 5\lambda^2 k_2}{1 - 5[h_0^2 + \lambda^2 h_1^2 + \lambda^2 h_2^2 r^2 \mu(\Omega)]}\right)^{1/2} \|u_2(\cdot) - u_1(\cdot)\|_2 \\ &\quad = \gamma_* \|u_2(\cdot) - u_1(\cdot)\|_2. \end{aligned}$$

□

6 Compactness of the set of trajectories and existence of optimal trajectories

Theorem 1. *The set of trajectories* $\mathbf{X}_r$ *of system* (1) *is a compact subset of the space* $L_2(\Omega;\mathbb{R}^n)$.

Proof. Let $\{x_k(\cdot)\}_{k=1}^{\infty}$ be a given sequence such that $x_k(\cdot) \in \mathbf{X}_r$ for every $k =, 1, 2, \dots$. Let us prove that the sequence $\{x_k(\cdot)\}_{k=1}^{\infty}$ has a subsequence converging in $\mathbf{X}_r$.

By virtue of definition of the set $\mathbf{X}_r$, there exists $u_k(\cdot) \in U_r$ such that

$$x_k(\xi) = f\big(\xi, x_k(\xi)\big) + \lambda \int_{\Omega} \big[K_1\big(\xi, s, x_k(s)\big) + K_2\big(\xi, s, x_k(s)\big)u_k(s)\big]\, \mathrm{d}s \tag{16}$$

for a.a. $\xi \in \Omega$. Since U_r is a weak compact subset of the space $L_2(\Omega;\mathbb{R}^m)$, then, without loss of generality, one can assume that the sequence $\{u_k(\cdot)\}_{k=1}^{\infty}$ weakly converges to a $u_*(\cdot)$ in the space $L_2(\Omega;\mathbb{R}^m)$, where $u_*(\cdot) \in U_r$. Let $x_*(\cdot)$ be the trajectory of system (1) generated by the admissible control function $u_*(\cdot) \in U_r$. Then $x_*(\cdot) \in \mathbf{X}_r$ and

$$x_*(\xi) = f\big(\xi, x_*(\xi)\big) + \lambda \int_{\Omega} \big[K_1\big(\xi, s, x_*(s)\big) + K_2\big(\xi, s, x_*(s)\big)u_*(s)\big]\, \mathrm{d}s \tag{17}$$

for a.a. $\xi \in \Omega$. Taking into consideration that $u_k(\cdot) \in U_r$ for every $k = 1, 2, \dots$, from (16), (17), conditions (A), (B), (C) and Hölder's inequality we obtain that

$$\begin{aligned} &\|x_k(\xi) - x_*(\xi)\| \\ &\quad \leqslant \|f\big(\xi, x_k(\xi)\big) - f\big(\xi, x_*(\xi)\big)\| + \lambda \int_{\Omega} \|K_1\big(\xi, s, x_k(s)\big) - K_1\big(\xi, s, x_*(s)\big)\|\, \mathrm{d}s \\ &\qquad + \lambda \int_{\Omega} \|K_2\big(\xi, s, x_k(s)\big) - K_2\big(\xi, s, x_*(s)\big)\| \|u_k(s)\|\, \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ \lambda \biggl\| \int_{\Omega} K_2\bigl(\xi, s, x_*(s)\bigr)\bigl[u_k(s) - u_*(s)\bigr] \,\mathrm{d}s \biggr\| \\
&\leqslant h_0 \bigl\|x_k(\xi) - x_*(\xi)\bigr\| + \lambda \int_{\Omega} l_1(\xi, s)\bigl\|x_k(s) - x_*(s)\bigr\| \,\mathrm{d}s \\
&+ \lambda h_2 \int_{\Omega} \bigl\|x_k(s) - x_*(s)\bigr\| \bigl\|u_k(s)\bigr\| \,\mathrm{d}s \\
&+ \lambda \biggl\| \int_{\Omega} K_2\bigl(\xi, s, x_*(s)\bigr)\bigl[u_k(s) - u_*(s)\bigr] \,\mathrm{d}s \biggr\| \\
&\leqslant h_0 \bigl\|x_k(\xi) - x_*(\xi)\bigr\| + \lambda \biggl(\int_{\Omega} l_1(\xi, s)^2 \,\mathrm{d}s \biggr)^{1/2} \bigl\|x_k(\cdot) - x_*(\cdot)\bigr\|_2 \\
&+ \lambda h_2 r \bigl\|x_k(\cdot) - x_*(\cdot)\bigr\|_2 \\
&+ \lambda \biggl\| \int_{\Omega} K_2\bigl(\xi, s, x_*(s)\bigr)\bigl[u_k(s) - u_*(s)\bigr] \,\mathrm{d}s \biggr\| \qquad (18)
\end{aligned}
$$

for a.a. $\xi \in \Omega$.

Denote $z_*(\xi, s) = K_2(\xi, s, x_*(s))$. Since $x_*(\cdot) \in L_2(\Omega; \mathbb{R}^n)$, then from condition (C) and Proposition 1 it follows that $z_*(\cdot, \cdot) \in L_2(\Omega \times \Omega; \mathbb{R}^{n \times m})$. Now let us denote

$$
\psi_k(\xi) = \biggl\| \int_{\Omega} K_2\bigl(\xi, s, x_*(s)\bigr)\bigl[u_k(s) - u_*(s)\bigr] \,\mathrm{d}s \biggr\|.
$$

Since the sequence $\{u_k(\cdot)\}_{k=1}^{\infty}$ weakly converges to a $u_*(\cdot)$ in the space $L_2(\Omega; \mathbb{R}^n)$, then we have that $\psi_k(\xi) \to 0$ as $k \to \infty$ for a.a. $\xi \in \Omega$, and hence $\psi_k^2(\xi) \to 0$ as $k \to \infty$ for a.a. $\xi \in \Omega$.

Let $\|x_*(\cdot)\|_2 = \alpha_*$. According to the Proposition 4, we have that $\alpha_* \leqslant k_*$, where k_* is defined by (7). Since $u_*(\cdot) \in U_r$, $u_k(\cdot) \in U_r$ for every $k = 1, 2, \dots$, then from Proposition 1 and Hölder's inequality we have that

$$
\begin{aligned}
\psi_k(\xi) &\leqslant \int_{\Omega} \bigl[h_2 \bigl\|x_*(s)\bigr\| + \bigl\|K_2(\xi, s, 0)\bigr\|\bigr] \bigl[\bigl\|u_k(s)\bigr\| + \bigl\|u_*(s)\bigr\|\bigr] \,\mathrm{d}s \\
&\leqslant h_2 \int_{\Omega} \bigl\|x_*(s)\bigr\| \bigl\|u_k(s)\bigr\| \,\mathrm{d}s + h_2 \int_{\Omega} \bigl\|x_*(s)\bigr\| \bigl\|u_*(s)\bigr\| \,\mathrm{d}s \\
&\quad + \int_{\Omega} \bigl\|K_2(\xi, s, 0)\bigr\| \bigl\|u_k(s)\bigr\| \,\mathrm{d}s + \int_{\Omega} \bigl\|K_2(\xi, s, 0)\bigr\| \bigl\|u_*(s)\bigr\| \,\mathrm{d}s \\
&\leqslant h_2 \bigl\|x_*(\cdot)\bigr\|_2 \bigl\|u_k(\cdot)\bigr\|_2 + h_2 \bigl\|x_*(\cdot)\bigr\|_2 \bigl\|u_*(\cdot)\bigr\|_2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\int_{\Omega}\left\|K_2(\xi,s,0)\right\|^2\mathrm{d}s\right)^{1/2}\left\|u_k(\cdot)\right\|_2\\
&+\left(\int_{\Omega}\left\|K_2(\xi,s,0)\right\|^2\mathrm{d}s\right)^{1/2}\left\|u_*(\cdot)\right\|_2\\
&\leqslant 2rh_2\alpha_*+2r\left(\int_{\Omega}\left\|K_2(\xi,s,0)\right\|^2\mathrm{d}s\right)^{1/2},
\end{aligned}
$$

and hence

$$
\left[\psi_k(\xi)\right]^2\leqslant 8r^2h_2^2\alpha_*^2+8r^2\int_{\Omega}\left\|K_2(\xi,s,0)\right\|^2\mathrm{d}s
$$

for a.a. $\xi\in\Omega$.

The function $\xi\to 8r^2h_2^2\alpha_*^2+8r^2\int_{\Omega}\|K_2(\xi,s,0)\|^2\,\mathrm{d}s$, $\xi\in\Omega$, is integrable. Thus, from Lebesgue convergence theorem we conclude that

$$
\int_{\Omega}\left[\psi_k(\xi)\right]^2\mathrm{d}\xi\to 0\quad\text{as } k\to\infty. \tag{19}
$$

From (18) it follows that

$$
\begin{aligned}
\left\|x_k(\xi)-x_*(\xi)\right\|\leqslant{}& h_0\left\|x_k(\xi)-x_*(\xi)\right\|\\
&+\lambda\left(\int_{\Omega}l_1(\xi,s)^2\,\mathrm{d}s\right)^{1/2}\left\|x_k(\cdot)-x_*(\cdot)\right\|_2\\
&+\lambda h_2r\left\|x_k(\cdot)-x_*(\cdot)\right\|_2+\lambda\psi_k(\xi)
\end{aligned}\tag{20}
$$

for a.a. $\xi\in\Omega$. (20), and Proposition 2 yield

$$
\begin{aligned}
\left\|x_k(\xi)-x_*(\xi)\right\|^2\leqslant{}& 4h_0^2\left\|x_k(\xi)-x_*(\xi)\right\|^2\\
&+4\lambda^2\int_{\Omega}l_1(\xi,s)^2\,\mathrm{d}s\cdot\left\|x_k(\cdot)-x_*(\cdot)\right\|_2^2\\
&+4\lambda^2h_2^2r^2\left\|x_k(\cdot)-x_*(\cdot)\right\|_2^2+4\lambda^2\left[\psi_k(\xi)\right]^2
\end{aligned}
$$

for a.a. $\xi\in\Omega$.

Integrating the last inequality, we get

$$
\begin{aligned}
\left\|x_k(\cdot)-x_*(\cdot)\right\|_2^2\leqslant{}& 4h_0^2\left\|x_k(\cdot)-x_*(\cdot)\right\|_2^2+4\lambda^2h_1^2\left\|x_k(\cdot)-x_*(\cdot)\right\|_2^2\\
&+4\lambda^2h_2^2r^2\mu(\Omega)\left\|x_k(\cdot)-x_*(\cdot)\right\|_2^2\\
&+4\lambda^2\int_{\Omega}\left[\psi_k(\xi)\right]^2\mathrm{d}\xi.
\end{aligned}\tag{21}
$$

Using condition (D), we have from (21) that

$$\left\|x_k(\cdot) - x_*(\cdot)\right\|_2 \leqslant \frac{2\lambda(\int_\Omega [\psi_k(\xi)]^2 d\xi)^{1/2}}{[1 - (4h_0^2 + 4\lambda^2 h_1^2 + 4\lambda^2 h_2^2 r^2 \mu(\Omega))]^{1/2}}. \tag{22}$$

(19) implies that for given $\varepsilon > 0$, there exists $K(\varepsilon)$ such that the inequality

$$\int_\Omega \left[\psi_k(\xi)\right]^2 \mathrm{d}\xi \leqslant \frac{1 - (4h_0^2 + 4\lambda^2 h_1^2 + 4\lambda^2 h_2^2 r^2 \mu(\Omega))}{4\lambda^2} \varepsilon^2 \tag{23}$$

is satisfied for every $k > K(\varepsilon)$. (22) and (23) yield that for given $\varepsilon > 0$, there exists $K(\varepsilon)$ such that

$$\left\|x_k(\cdot) - x_*(\cdot)\right\|_2 \leqslant \varepsilon$$

for every $k > K(\varepsilon)$. This means that $x_k(\cdot) \to x_*(\cdot)$ in the space $L_2(\Omega; \mathbb{R}^n)$ as $k \to \infty$, where $x_*(\cdot) \in \mathbf{X}_r$. □

Let $J(\cdot) : L_2(\Omega; \mathbb{R}^n) \to \mathbb{R}$ be a lower semicontinuous functional. Consider optimal control problem

$$J\bigl(x(\cdot)\bigr) \to \inf, \quad x(\cdot) \in \mathbf{X}_r. \tag{24}$$

The trajectory $x_*(\cdot) \in \mathbf{X}_r$ is called an optimal one iff $J(x_*(\cdot)) \leqslant J(x(\cdot))$ for every $x(\cdot) \in \mathbf{X}_r$.

Proposition 6. *Problem* (24) *has an optimal trajectory.*

The proof of the proposition immediately follows from compactness of the set of trajectories $\mathbf{X}_r$ and lower semicontinuity of the functional $J(\cdot)$.

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