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Robust set stabilization of Boolean control networks with impulsive effects*

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Abstract. This paper addresses the robust set stabilization problem of Boolean control networks (BCNs) with impulsive effects via the semi-tensor product method. Firstly, the closed-loop system consisting of a BCN with impulsive effects and a given state feedback control is converted into an algebraic form. Secondly, based on the algebraic form, some necessary and sufficient conditions are presented for the robust set stabilization of BCNs with impulsive effects under a given state feedback control and a free-form control sequence, respectively. Thirdly, as applications, some necessary and sufficient conditions are presented for robust partial stabilization and robust output tracking of BCNs with impulsive effects, respectively. The study of two illustrative examples shows that the obtained new results are effective.

Keywords: Boolean control network, impulsive effects, set stabilization, robust control, semi-tensor product of matrices.

1 Introduction

Impulsive phenomenon is usually caused by changes in the interconnections of subsystems, sudden changes of external environment, and so on. Recently, the study of nonlinear systems with impulsive effects has attracted many scholars' interest, and a great deal of excellent results have been established [26–28, 37, 39, 43]. As a special kind of nonlinear systems, Boolean control network (BCN) is an important model of gene

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regulatory networks (GRNs) [13, 14]. It is noted that impulsive phenomenon often occurs in GRNs due to the environment fluctuations between intracellular and extracellular, which may prohibit the finite-time stability of GRNs [38]. Therefore, it is meaningful to study BCNs with impulsive effects. Recently, by using the semi-tensor product (STP) of matrices [5, 7, 8, 15, 33, 34, 41, 42, 44, 45], many scholars have presented some basic results on BCNs with impulsive effects, including controllability [30,32,35], observability [17,46], stability and stabilization [1,3,9,18,39], set stability [10,16,19], optimal control [4,40], function perturbations [31], and synchronization [2,45].

It is worth noting that stability and stabilization [25, 27, 29] are significant and meaningful issues in the study of BCNs because of their applications in the design of therapeutic interventions and the explanation of some living phenomena [6, 11, 12, 20, 25, 36]. The stability and stabilization of BCNs were firstly presented in [6]. Then Li et al. [25] presented a novel procedure to design state feedback stabilizers for BCNs. Guo et al. [11] further investigated the set stability and set stabilization of BCNs. Li and Wang [21] established some new results on the robust stability and robust stabilization of BCNs with disturbance inputs. However, there exist fewer results to study robust set stabilization of BCNs with impulsive effects.

In this paper, using STP, we investigate the robust set stabilization problem of BCNs with impulsive effects. The main contributions of this paper are as follows.

(i) The closed-loop system consisting of a BCN with impulsive effects and a given state feedback control is converted into an algebraic form via STP, which facilitates the study of BCNs with impulsive effects.

(ii) Two necessary and sufficient conditions are presented for the robust set stabilization of BCNs with impulsive effects under a given state feedback control and a free-form control sequence, respectively, which are easily verified via MATLAB toolbox. Our results generalize the existing set stabilization results [10,11,16,19] to BCNs with impulsive effects. In addition, our results are applicable to BCNs with disturbance inputs, while the results in [10, 11, 16, 19] just consider BCNs without disturbance inputs.

(iii) As applications of robust set stabilization, some necessary and sufficient conditions are presented for robust partial stabilization and robust output tracking [22–24] of BCNs with impulsive effects.

The rest of this paper is arranged as follows. Section 2 recalls some preliminary results on STP. In Section 3, we investigate robust set stabilization, robust partial stabilization and robust output tracking of BCNs with impulsive effects, and present the main results of this paper. Two illustrative examples are given to support our new results in Section 4, which is followed by a brief conclusion in Section 5.

Notations. $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}_+$ denote the sets of real numbers, natural numbers, and positive integers, respectively. $\mathcal{D} := \{1, 0\}$. $\Delta_n := \{\delta_n^k\colon k = 1, \dots, n\}$, where δ_n^k denotes the kth column of the identity matrix I_n. When $n = 2$, Δ_2 is briefly denoted by Δ. An $n \times t$ logical matrix $M = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_t}]$ is briefly expressed as $M = \delta_n[i_1, i_2, \dots, i_t]$. Denote the set of $n \times t$ logical matrices by $\mathcal{L}_{n\times t}$. $\mathrm{Col}_i(A)$ denotes the ith column of the matrix A. Denote by $\mathrm{Blk}_i(A)$ the ith block of the matrix A.

2 Preliminaries

In this section, we recall some necessary preliminaries on the semi-tensor product of matrices.

Definition 1. (See [5].) The semi-tensor product of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is

$$A \ltimes B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}),$$

where $\alpha = \operatorname{lcm}(n,p)$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

When $n = p$, the semi-tensor product of A and B becomes the conventional matrix product. Thus, it is a generalization of the conventional matrix product. We omit the symbol "$\ltimes$" if no confusion arises in the following. The semi-tensor product of matrices has the following properties.

Proposition. (*See* [5].)

(i) *Let $X \in \mathbb{R}^{t\times 1}$ be a column vector and $A \in \mathbb{R}^{m\times n}$. Then*

$$X \ltimes A = (I_t \otimes A) \ltimes X.$$

(ii) *Given $X_1 \in \mathbb{R}^{p\times 1}$ and $X_2 \in \mathbb{R}^{q\times 1}$. Then*

$$X_2 \ltimes X_1 = W_{[p,q]} \ltimes X_1 \ltimes X_2,$$

where

$$\begin{aligned} W_{[p,q]} = \delta_{pq} \bigl[&1 \quad p+1 \quad \dots \quad (q-1)p+1 \\ &2 \quad p+2 \quad \dots \quad (q-1)p+2 \\ &\qquad\qquad \dots \\ &p \quad p+p \quad \dots \quad (q-1)p+p \bigr] \end{aligned}$$

is called swap matrix.

(iii) *Let $X \in \mathbb{R}^{n\times 1}$ be a column vector. Then*

$$X^2 = M_{r,n} X,$$

where

$$M_{r,n} = \operatorname{diag}\bigl[\delta_n^1,\ \delta_n^2,\ \dots,\ \delta_n^n\bigr].$$

Identifying $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$, then $\Delta \sim \mathcal{D}$, where "$\sim$" denotes two different forms of the same object. In most places of this work, we use δ_2^1 and δ_2^2 to express logical variables and call them the vector form of logical variables. The following lemma is fundamental for the matrix expression of logical functions.

Lemma 1. (*See* [5].) *Let $f(x_1, x_2, \dots, x_s) : \mathcal{D}^s \to \mathcal{D}$ be a logical function. Then there exists a unique matrix $M_f \in \mathcal{L}_{2\times 2^s}$, called the structural matrix of f, such that*

$$f(x_1, x_2, \dots, x_s) = M_f \ltimes_{i=1}^s x_i,$$

where $x_i \in \Delta$ and $\ltimes_{i=1}^s x_i = x_1 \ltimes \cdots \ltimes x_s$.

For example, the structural matrices for negation ($\neg$), conjunction ($\wedge$), and disjunction ($\vee$) are $M_n = \delta_2[2,\ 1]$, $M_c = \delta_2[1,\ 2,\ 2,\ 2]$, and $M_d = \delta_2[1,\ 1,\ 1,\ 2]$, respectively.

Example 1. Calculate the structural matrix of the following Boolean function:

$$f(x_1, x_2, x_3) = x_1 \wedge (\neg x_2 \vee x_3).$$

According to Lemma 1, one can see that

$$\begin{aligned} f(x_1, x_2, x_3) &= M_c \ltimes x_1 \ltimes M_d \ltimes M_n \ltimes x_2 \ltimes x_3 \\ &= M_c \ltimes \left[I_2 \otimes (M_d \ltimes M_n)\right] \ltimes x_1 \ltimes x_2 \ltimes x_3 \\ &= \delta_2[1,\ 2,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2] \ltimes x_1 \ltimes x_2 \ltimes x_3. \end{aligned}$$

3 Main results

In this section, we study the robust set stabilization of Boolean control networks with impulsive effects, and present the main results of this paper. We firstly present a necessary and sufficient condition for the robust set stabilization of Boolean control networks with impulsive effects, and then apply the obtained results to the investigation of robust partial stabilization and robust output tracking.

3.1 Robust set stabilization

Consider the following Boolean control network with impulsive effects:

$$\begin{aligned} x_i(t+1) &= f_{1i}\big(X(t), U(t), \Xi(t)\big), \quad i = 1, \dots, n,\ t_{k-1} \leqslant t < t_k - 1; \\ x_i(t_k) &= f_{2i}\big(X(t_k - 1), \Xi(t_k - 1)\big), \quad i = 1, \dots, n,\ k \in \mathbb{Z}; \\ y_j(t) &= h_j\big(X(t)\big), \quad j = 1, \dots, p, \end{aligned} \tag{1}$$

where $t_0 = 0$, $\{t_k\colon k \in \mathbb{Z}_+\} \subseteq \mathbb{Z}_+$ satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ is the impulsive time sequence, $X(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in \mathcal{D}^n$, $U(t) = (u_1(t), \dots, u_m(t)) \in \mathcal{D}^m$, $\Xi(t) = (\xi_1(t), \dots, \xi_q(t)) \in \mathcal{D}^q$ and $Y(t) = (y_1(t), \dots, y_p(t)) \in \mathcal{D}^p$ denote the state variables, the control inputs, the disturbance inputs and the outputs of system (1), respectively, and $f_{1i} : \mathcal{D}^{n+m+q} \to \mathcal{D}$, $f_{2i} : \mathcal{D}^{n+q} \to \mathcal{D}$, $i = 1, \dots, n$, and $h_j : \mathcal{D}^n \to \mathcal{D}$, $j = 1, \dots, p$, are logical functions.

Definition 2. Given a nonempty set $A \subseteq \mathcal{D}^n$. System (1) is said to be robustly stabilizable to the set A if there exist a control sequence $\{U(t)\colon t \in \mathbb{N}\}$ and a positive integer τ such that

$$X\big(t; X(0), U, \Xi\big) \in A$$

holds for any integer $t \geqslant \tau$, any initial state $X(0) \in \mathcal{D}^n$, and any disturbance sequence $\{\Xi(t)\colon t \in \mathbb{N}\} \subseteq D^q$.

In this part, we firstly consider state feedback control in the form of

$$u_i(t) = \varphi_i\big(X(t)\big), \quad i = 1, \dots, m, \tag{2}$$

where $\varphi_i : \mathcal{D}^n \to \mathcal{D}$, $i = 1, \dots, m$, are logical functions.

In the following, we convert system (1) and the state feedback control (2) into algebraic forms, respectively.

Using the vector form of logical variables and setting $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{2^n}$, $u(t) = \ltimes_{i=1}^{m} u_i(t) \in \Delta_{2^m}$, $\xi(t) = \ltimes_{i=1}^{q} \xi_i(t) \in \Delta_{2^q}$, and $y(t) = \ltimes_{i=1}^{p} y_i(t) \in \Delta_{2^p}$, by Lemma 1, one can convert system (1) into the following algebraic form:

$$\begin{aligned} x(t+1) &= L_1\xi(t)u(t)x(t), \quad t_{k-1} \leqslant t < t_k - 1; \\ x(t_k) &= L_2\xi(t_k - 1)x(t_k - 1); \\ y(t) &= Hx(t), \end{aligned} \tag{3}$$

where $L_1 \in \mathcal{L}_{2^n \times 2^{n+m+q}}$, $L_2 \in \mathcal{L}_{2^n \times 2^{n+q}}$, and $H \in \mathcal{L}_{2^p \times 2^n}$.

Similarly, the state feedback control (2) can be converted into

$$u(t) = Gx(t), \tag{4}$$

where $G \in \mathcal{L}_{2^m \times 2^n}$ is called the state feedback gain matrix.

Now, based on the algebraic form (3), we study how to verify whether or not system (1) is robustly stabilizable to the set A under a given state feedback control $u(t) = Gx(t)$.

Substituting (4) into (3), one can obtain the following closed-loop system:

$$\begin{aligned} x(t+1) &= \bar{L}_1\xi(t)x(t), \quad t_{k-1} \leqslant t < t_k - 1; \\ x(t_k) &= L_2\xi(t_k - 1)x(t_k - 1); \\ y(t) &= Hx(t), \end{aligned} \tag{5}$$

where $\bar{L}_1 = L_1[I_{2^q} \otimes (GM_{r,2^n})]$.

According to system (5), we have

$$\begin{aligned} x(1) &= \bar{L}_1\xi(0)x(0) := \tilde{L}_1\xi(0)x(0), \\ x(2) &= \bar{L}_1\xi(1)x(1) = \bar{L}_1(I_{2^q} \otimes \tilde{L}_1)\xi(1)\xi(0)x(0) := \tilde{L}_2\xi(1)\xi(0)x(0), \\ &\dots, \\ x(t_1 - 1) &= \bar{L}_1\xi(t_1 - 2)x(t_1 - 2) = \bar{L}_1(I_{2^q} \otimes \tilde{L}_{t_1-2}) \ltimes_{i=t_1-2}^{0} \xi(i)x(0) \\ &:= \tilde{L}_{t_1-1} \ltimes_{i=t_1-2}^{0} \xi(i)x(0), \\ x(t_1) &= L_2\xi(t_1 - 1)x(t_1 - 1) = L_2(I_{2^q} \otimes \tilde{L}_{t_1-1}) \ltimes_{i=t_1-1}^{0} \xi(i)x(0) \\ &:= \tilde{L}_{t_1} \ltimes_{i=t_1-1}^{0} \xi(i)x(0), \\ &\dots . \end{aligned}$$

Keep this procedure going, for an arbitrarily given positive integer τ, we have

$$x(\tau) = \tilde{L}_\tau \ltimes_{i=\tau-1}^{0} \xi(i)x(0), \tag{6}$$

where

$$\tilde{L}_\tau = \begin{cases} \bar{L}_1(I_{2^q} \otimes \tilde{L}_{\tau-1}), & t_k < \tau < t_{k+1}, \\ L_2(I_{2^q} \otimes \tilde{L}_{\tau-1}), & \tau = t_{k+1}, \end{cases} \tag{7}$$

and $k \in \mathbb{N}$.

Definition 3. Consider the system $x(t+1) = L\xi(t)x(t)$ with $x(t) \in \Delta_{2^n}$, $\xi(t) \in \Delta_{2^q}$, and $L \in \mathcal{L}_{2^n \times 2^{n+q}}$. Given a nonempty set $S \subseteq \Delta_{2^n}$. S is said to be a robust L-invariant set if $L\xi x \in S$ holds for any $x \in S$ and any $\xi \in \Delta_{2^q}$.

Lemma 2. *Given a nonempty set $S = \{\delta_{2^n}^{\alpha_1}, \dots, \delta_{2^n}^{\alpha_r}\}$, where $\alpha_1 < \alpha_2 < \dots < \alpha_r$. Split L into 2^q equal blocks as $[\mathrm{Blk}_1(L), \dots, \mathrm{Blk}_{2^q}(L)]$. Then S is a robust L-invariant set if and only if $\mathrm{Blk}_i(L)|_S \in \mathcal{L}_{r\times r}$ holds for any $i \in \{1, \dots, 2^q\}$, where*

$$\mathrm{Blk}_i(L)|_S = \begin{bmatrix} (\mathrm{Blk}_i(L))_{\alpha_1,\alpha_1} & \cdots & (\mathrm{Blk}_i(L))_{\alpha_1,\alpha_r} \\ \vdots & \vdots & \vdots \\ (\mathrm{Blk}_i(L))_{\alpha_r,\alpha_1} & \cdots & (\mathrm{Blk}_i(L))_{\alpha_r,\alpha_r} \end{bmatrix}, \tag{8}$$

and $(\mathrm{Blk}_i(L))_{j,k}$ denotes the (j,k)th element of the matrix $\mathrm{Blk}_i(L)$.

Based on (7) and Definition 3, we have the following result.

Theorem 1. *Given a nonempty set $A \subseteq \Delta_{2^n}$. Assume that A is a robust $\bar{L}_1$-invariant set as well as a robust L_2-invariant set. System* (1) *is robustly stabilizable to A under the state feedback control $u(t) = Gx(t)$ if and only if there exists a positive integer τ such that*

$$\mathrm{Col}(\tilde{L}_\tau) \subseteq A, \tag{9}$$

where $\tilde{L}_\tau$ is given in (7)*, and $\mathrm{Col}(\cdot)$ denotes the set of columns of a matrix.*

Proof. Sufficiency. Assume that (9) holds. We prove that $\mathrm{Col}(\tilde{L}_t) \subseteq A$ holds for any integer $t \geqslant \tau$ by induction.

Obviously, $\mathrm{Col}(\tilde{L}_t) \subseteq A$ holds for $t = \tau$.

Assuming that $\mathrm{Col}(\tilde{L}_t) \subseteq A$ holds for some integer $t = \lambda > \tau$, that is $x(\lambda) = \tilde{L}_\lambda \ltimes_{i=\lambda-1}^{0} \xi(i)x(0) \in A$ holds for any $\{\xi(0), \dots, \xi(\lambda-1)\} \subseteq \Delta_{2^q}$ and any $x(0) \in \Delta_{2^n}$. Now, we prove the case of $t = \lambda + 1$.

- If $t_k < \lambda + 1 < t_{k+1}$, we have $x(\lambda+1; u, \xi, x(0)) = \bar{L}_1\xi(\lambda)x(\lambda)$. Since A is a robust $\bar{L}_1$-invariant set and $x(\lambda) \in A$, one can see that $x(\lambda+1) = \tilde{L}_{\lambda+1} \ltimes_{i=\lambda}^{0} \xi(i)x(0) \in A$. From the arbitrariness of $\{\xi(0), \dots, \xi(\lambda)\} \subseteq \Delta_{2^q}$ and $x(0)$ we have $\mathrm{Col}(\tilde{L}_{\lambda+1}) \subseteq A$.
- If $\lambda + 1 = t_k$, we have $x(\lambda+1; u, \xi, x(0)) = L_2\xi(\lambda)x(\lambda)$. Noticing that A is a robust L_2-invariant set and $x(\lambda) \in A$, one can see that $x(\lambda+1) = \tilde{L}_{\lambda+1} \ltimes_{i=\lambda}^{0} \xi(i)x(0) \in A$. From the arbitrariness of $\{\xi(0), \dots, \xi(\lambda)\} \subseteq \Delta_{2^q}$ and $x(0)$ we also have $\mathrm{Col}(\tilde{L}_{\lambda+1}) \subseteq A$.

By induction, $\mathrm{Col}(\tilde{L}_t) \subseteq A$ holds for any integer $t \geqslant \tau$. Therefore, $x(t; u, \xi, x(0)) \in A$ holds for any integer $t \geqslant \tau$, any $x(0) \in \Delta_{2^n}$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \Delta_{2^q}$. By Definition 2, system (1) is robustly stabilizable to A under the state feedback control $u(t) = Gx(t)$.

Necessity. Suppose that system (1) is robustly stabilizable to A under the state feedback control $u(t) = Gx(t)$. From Definition 2 there exists a positive integer τ such that

$x(t; u, \xi, x(0)) \in A$ holds for any initial state $x(0) \in \Delta_{2^n}$, any integer $t \geqslant \tau$, and any $\{\xi(t)\colon\ t \in \mathbb{N}\} \subseteq \Delta_{2^q}$. Thus, it is easy to see from (7) that

$$x\big(\tau; u, \xi, x(0)\big) = \tilde{L}_\tau \ltimes_{i=\tau-1}^{0} \xi(i)x(0) \in A,$$

which, together with the arbitrariness of $\{\xi(t)\colon\ t \in \mathbb{N}\} \subseteq \Delta_{2^q}$ and $x(0) \in \Delta_{2^n}$, shows that $\mathrm{Col}(\tilde{L}_\tau) \subseteq A$. □

Remark 1.

(i) The assumption "A is a robust L_1-invariant set as well as a robust L_2-invariant set" is used to make the state trajectory of system (1) stay at A forever after the state starting from any initial state reaches the set A in the time τ.

(ii) It should be pointed out that the positive integer τ, which satisfies (9), is determined by $\bar{L}_1$, L_2 and impulsive time sequence $\{t_k\colon\ k \in \mathbb{Z}_+\}$. Although system (1) has only 2^n different states, the lower bound of τ, which satisfies (9), may be greater than 2^n because of the impulsive effects, which is very different from BCNs without impulsive effects.

Next, we consider the robust set stabilization of system (1) under free-form control sequence.

Using swap matrix, one can convert (3) into the following algebraic form:

$$\begin{aligned} x(t+1) &= \hat{L}_1 u(t)\xi(t)x(t), \quad t_{k-1} \leqslant t < t_k - 1; \\ x(t_k) &= L_2\xi(t_k - 1)x(t_k - 1); \\ y(t) &= Hx(t), \end{aligned} \tag{10}$$

where $\hat{L}_1 = L_1 W_{[2^m, 2^q]}$.

Given a positive integer t, there exist unique integers t_{k-1} and j such that $t = t_{k-1} + j$, where $1 \leqslant j \leqslant t_k - t_{k-1}$. Let $\Lambda(t) = \{t_1 - 1, \dots, t_{k-1} - 1\}$. Then a free-form control sequence with length t can be expressed as

$$\big\{u(i)\colon\ i \in \{0, 1, \dots, t-1\} \setminus \Lambda(t)\big\}. \tag{11}$$

Under the free-form control sequence (11), along the trajectory starting form any initial state $x(0) \in \Delta_{2^n}$, we have

$$\begin{aligned} x(1) &= \hat{L}_1 u(0)\xi(0)x(0), \\ x(2) &= \hat{L}_1 u(1)\xi(1)x(1) = \hat{L}_1(I_{2^{m+q}} \otimes \hat{L}_1)\big(I_{2^m} \otimes W_{[2^m, 2^q]}\big) \\ &\quad \ltimes u(1)u(0)\xi(1)\xi(0)x(0) \\ &:= \hat{L}_2 u(1)u(0)\xi(1)\xi(0)x(0), \\ &\dots, \\ x(t_1 - 1) &= \hat{L}_1 u(t_1 - 2)\xi(t_1 - 2)x(t_1 - 2) = \hat{L}_1\big(I_{2^{m+q}} \otimes \hat{L}_{t_1 - 2}\big) \\ &\quad \ltimes \big(I_{2^m} \otimes W_{[2^{(t_1-2)m}, 2^q]}\big) \ltimes_{i=t_1-2}^{0} u(i) \ltimes_{i=t_1-2}^{0} \xi(i)x(0) \\ &:= \hat{L}_{t_1 - 1} \ltimes_{i=t_1-2}^{0} u(i) \ltimes_{i=t_1-2}^{0} \xi(i)x(0), \end{aligned}$$

$$x(t_1) = L_2\xi(t_1-1)x(t_1-1) = L_2(I_{2^q}\otimes \hat{L}_{t_1-1})W_{[2^{(t_1-1)m},2^q]} \\ \ltimes_{i=t_1-2}^{0} u(i) \ltimes_{i=t_1-1}^{0} \xi(i)x(0) \\ := \hat{L}_{t_1} \ltimes_{i=t_1-2}^{0} u(i) \ltimes_{i=t_1-1}^{0} \xi(i)x(0), \\ \dots .$$

Keep this procedure going, for an arbitrarily given positive integer τ, we have

$$x(\tau) = \begin{cases} \hat{L}_\tau \ltimes_{i=\tau-1,\, i\notin\Lambda(\tau)}^{0} u(i) \ltimes_{i=\tau-1}^{0} \xi(i)x(0), & t_{k-1}+1 \leqslant \tau < t_k, \\ \hat{L}_\tau \ltimes_{i=\tau-2,\, i\notin\Lambda(\tau)}^{0} u(i) \ltimes_{i=\tau-1}^{0} \xi(i)x(0), & \tau = t_k, \end{cases}$$

where

$$\hat{L}_\tau = \begin{cases} \hat{L}_1(I_{2^{m+q}}\otimes \hat{L}_{\tau-1})(I_{2^m}\otimes W_{[2^{(\tau-k)m},2^q]}), & t_{k-1}+1 \leqslant \tau < t_k, \\ L_2(I_{2^q}\otimes \hat{L}_{\tau-1})W_{[2^{(\tau-k)m},2^q]}, & \tau = t_k. \end{cases} \tag{12}$$

For $t_{k-1}+1 \leqslant \tau < t_k$, split $\hat{L}_\tau$ into $2^{(\tau-k+1)m}$ blocks as

$$\hat{L}_\tau = \big[\operatorname{Blk}_1(\hat{L}_\tau),\ \dots,\ \operatorname{Blk}_{2^{(\tau-k+1)m}}(\hat{L}_\tau)\big];$$

for $\tau = t_k$, split $\hat{L}_\tau$ into $2^{(\tau-k)m}$ blocks as $\hat{L}_\tau = [\operatorname{Blk}_1(\hat{L}_\tau), \dots, \operatorname{Blk}_{2^{(\tau-k)m}}(\hat{L}_\tau)]$, where $\operatorname{Blk}_i(\hat{L}_\tau) \in \mathcal{L}_{2^n\times 2^{n+\tau q}}$. Then we have the following result on the robust set stabilization of system (1) under free-form control sequence.

Theorem 2. *Given a nonempty set $A \subseteq \Delta_{2^n}$. Assume that there exists a positive integer $\alpha \leqslant 2^m$ such that A is a robust $\operatorname{Blk}_\alpha(\hat{L}_1)$-invariant set as well as a robust L_2-invariant set. System* (1) *is robustly stabilizable to A under a free-form control sequence if and only if there exist two positive integers τ and β such that*

$$\operatorname{Col}\big(\operatorname{Blk}_\beta(\hat{L}_\tau)\big) \subseteq A. \tag{13}$$

Moreover, if (13) *holds, then the free-form control sequence is given as*

$$u(t) = \begin{cases} u^*(t), & t \in ([0,\tau-1]\cap\mathbb{N}) \setminus \Lambda(t), \\ \delta_{2^m}^{\alpha}, & t \in ([\tau,+\infty)\cap\mathbb{N}) \setminus \Lambda(t), \end{cases} \tag{14}$$

where

$$u^* = \begin{cases} \ltimes_{i=\tau-1,\, i\notin\Lambda(\tau)}^{0} u^*(i) = \delta_{2^{(\tau-k+1)m}}^{\beta}, & t_{k-1}+1 \leqslant \tau < t_k; \\ \ltimes_{i=\tau-2,\, i\notin\Lambda(\tau)}^{0} u^*(i) = \delta_{2^{(\tau-k)m}}^{\beta}, & \tau = t_k. \end{cases}$$

Proof. Sufficiency. Assume that (13) holds. We prove that $x(t) \in A$ holds for any integer $t \geqslant \tau$ by induction.

For $t = \tau$, if $t_{k-1}+1 \leqslant \tau < t_k$, we have

$$x(\tau) = \hat{L}_\tau \ltimes_{i=\tau-1,\, i\notin\Lambda(\tau)}^{0} u(i) \ltimes_{i=\tau-1}^{0} \xi(i)x(0) \\ = \hat{L}_\tau \delta_{2^{(\tau-k+1)m}}^{\beta} \ltimes_{i=\tau-1}^{0} \xi(i)x(0) \\ = \operatorname{Blk}_\beta(\hat{L}_\tau) \ltimes_{i=\tau-1}^{0} \xi(i)x(0)$$

Similarly, if $\tau = t_k$, we have

$$\begin{aligned} x(\tau) &= \hat{L}_\tau \ltimes_{i=\tau-2,\, i\notin\Lambda(\tau)}^{0} u(i) \ltimes_{i=\tau-1}^{0} \xi(i)x(0) \\ &= \hat{L}_\tau \delta_{2^{(\tau-k)m}}^{\beta} \ltimes_{i=\tau-1}^{0} \xi(i)x(0) \\ &= \mathrm{Blk}_\beta(\hat{L}_\tau) \ltimes_{i=\tau-1}^{0} \xi(i)x(0). \end{aligned}$$

Since $\mathrm{Col}(\mathrm{Blk}_\beta(\hat{L}_\tau)) \subseteq A$, one can see that $x(\tau) \in A$ holds for any $\{\xi(0), \dots, \xi(\tau-1)\} \subseteq \Delta_{2^q}$ and any $x(0) \in \Delta_{2^n}$.

Assuming that $x(t) \in A$ holds for some integer $t = \lambda \geqslant \tau$.

- If $t_k < \lambda + 1 < t_{k+1}$, we have $x(\lambda+1; u, \xi, x(0)) = \hat{L}_1 u(\lambda)\xi(\lambda)x(\lambda) = \hat{L}_1 \delta_{2^m}^{\alpha} \xi(\lambda)x(\lambda) = \mathrm{Blk}_\alpha(\hat{L}_1)\xi(\lambda)x(\lambda)$. Since A is a robust $\mathrm{Blk}_\alpha(\hat{L}_1)$-invariant set and $x(\lambda) \in A$, one can see that $x(\lambda+1) \in A$.
- If $\lambda + 1 = t_k$, we have $x(\lambda+1; u, \xi, x(0)) = L_2\xi(\lambda)x(\lambda)$. Since A is a robust L_2-invariant set and $x(\lambda) \in A$, one can obtain that $x(\lambda+1) \in A$.

By induction, $x(t) \in A$ holds for any integer $t \geqslant \tau$. Therefore, $x(t; u, \xi, x(0)) \in A$ holds for any integer $t \geqslant \tau$, any $x(0) \in \Delta_{2^n}$ and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \Delta_{2^q}$. By Definition 2, system (1) is robustly stabilizable to A under the free-form control sequence (14).

Necessity. Suppose that system (1) is robustly stabilizable to A under a free-form control sequence. From Definition 2 there exists a positive integer τ such that $x(t; u, \xi, x(0)) \in A$ holds for any initial state $x(0) \in \Delta_{2^n}$, any integer $t \geqslant \tau$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \Delta_{2^q}$.

Let

$$\ltimes_{i=\tau-1,\, i\notin\Lambda(\tau)}^{0} u^*(i) = \begin{cases} \delta_{2^{(\tau-k+1)m}}^{\beta}, & \tau \neq t_k, \\ \delta_{2^{(\tau-k)m}}^{\beta}, & \tau = t_k. \end{cases}$$

Then it is easy to see from (8) that

$$x\bigl(\tau; u, \xi, x(0)\bigr) = \mathrm{Blk}_\beta(\hat{L}_\tau) \ltimes_{i=\tau-1}^{0} \xi(i)x(0) \in A$$

holds for any $\{\xi(0), \dots, \xi(\tau-1)\} \subseteq \Delta_{2^q}$ and any $x(0) \in \Delta_{2^n}$, which, together with the arbitrariness of $\{\xi(t)\colon t \in \{0, \dots, \tau-1\}\} \subseteq \Delta_{2^q}$ and $x(0) \in \Delta_{2^n}$, shows that $\mathrm{Col}(\mathrm{Blk}_\beta(\hat{L}_\tau)) \subseteq A$. $\square$

3.2 Robust partial stabilization

In the following, we study the robust partial stabilization of system (1). Given $(x_1^*, \dots, x_r^*) \in \mathcal{D}^r$ with $r \leqslant n$, one can obtain its canonical vector form, denoted by $x^r = \ltimes_{i=1}^{r} x_i^* = \delta_{2^r}^{\theta}$.

Definition 4. System (1) is said to be robustly partial stabilizable to x^r, if there exist a control sequence $\{u(t)\colon t \in \mathbb{N}\} \subseteq \mathcal{D}^m$ and a positive integer τ such that

$$x_i(t; x(0), u, \xi) = x_i^*, \quad i = 1, \dots, r,$$

holds for any integer $t \geqslant \tau$, any $x(0) \in \mathcal{D}^n$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \mathcal{D}^q$.

Set

$$A = \{\delta_{2^r}^{\theta} \ltimes \delta_{2^{n-r}}^{\eta}\colon \eta = 1, \dots, 2^{n-r}\}.$$

Assume that A is a robust $\bar{L}_1$-invariant set as well as a robust L_2-invariant set. Based on the proof of Theorem 1, we have the following result.

Theorem 3. *For system* (1)*, the following statements are equivalent*:

(i) *System* (1) *is robustly partial stabilizable to* x^r *under the state feedback control* $u(t) = Gx(t)$;
(ii) *System* (1) *is robustly stabilizable to the set* A *under the state feedback control* $u(t) = Gx(t)$;
(iii) *There exists a positive integer* τ *such that* $\mathrm{Col}(\tilde{L}_\tau) \subseteq A$.

Proof. (i) ⇒ (ii). Assume that system (1) is robustly partial stabilizable to x^r under the state feedback control $u(t) = Gx(t)$. By Definition 4, there exists an integer $\tau > 0$ such that $x_i(t; x(0), u, \xi) = x_i^*$, $i = 1, \dots, r$, holds for any integer $t \geqslant \tau$, any $x(0) \in \mathcal{D}^n$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \mathcal{D}^q$. Then one can see that $x(t) = \ltimes_{i=1}^r x_i(t) \ltimes_{j=r+1}^n x_j(t) = \ltimes_{i=1}^r x_i^* \ltimes_{j=r+1}^n x_j(t) = \delta_{2^r}^{\theta} \ltimes_{j=r+1}^n x_j(t) \in A$ holds for any $t \geqslant \tau$, any $x(0) \in \mathcal{D}^n$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \mathcal{D}^q$. By Definition 2, system (1) is robustly stabilizable to the set A under the state feedback control $u(t) = Gx(t)$.

(i) ⇒ (ii). Assume that system (1) is robustly stabilizable to the set A under the state feedback control $u(t) = Gx(t)$. Then there exists an integer $\tau > 0$ such that $x(t; x(0), u, \xi) \in A$ holds for any $t \geqslant \tau$, any $x(0) \in \mathcal{D}^n$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \mathcal{D}^q$. Hence, one can obtain that $x(t) = \delta_{2^r}^{\theta} \ltimes \delta_{2^{n-r}}^{\eta} = \ltimes_{i=1}^r x_i^* \ltimes \delta_{2^{n-r}}^{\eta}$, which shows that $x_i(t; x(0), u, \xi) = x_i^*$, $i = 1, \dots, r$, holds for any integer $t \geqslant \tau$, any $x(0) \in \mathcal{D}^n$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \mathcal{D}^q$. By Definition 4, system (1) is robustly partial stabilizable to x^r under the state feedback control $u(t) = Gx(t)$.

The proof of (ii) ⇔ (iii) is similar to the proof of Theorem 1. □

Remark 2. When $r = n$, one can check the robust stabilization of system (1) by using Theorem 3.

3.3 Robust output tracking

In this part, we consider the robust output tracking control of system (1).

Definition 5. Given a reference signal $Y_r = (y_1^r, \dots, y_p^r) \in \mathcal{D}^p$. The output of system (1) is said to robustly track Y_r if there exist a control sequence $\{U(t)\colon t \in \mathbb{N}\} \subseteq \mathcal{D}^m$ and an integer $\tau > 0$ such that

$$Y(t; X(0), U, \Xi) = Y_r$$

holds for any initial state $X(0) \in \mathcal{D}^n$, any integer $t \geqslant \tau$, and any $\{\Xi(t)\colon t \in \mathbb{N}\} \subseteq \mathcal{D}^q$.

Given a state feedback control $u(t) = Gx(t)$, according to (6), we have

$$y(\tau) = H\tilde{L}_\tau \ltimes_{i=\tau-1}^{0} \xi(i)x(0) \quad \forall\, \tau \in \mathbb{Z}_+. \tag{15}$$

Let $\mathcal{O}(\beta)$ be the set containing all the states of system (1) whose outputs correspond to the reference signal $y_r = \delta_{2^p}^{\beta}$. Then one can easily see that

$$\mathcal{O}(\beta) = \left\{\delta_{2^n}^{r}\colon\ \mathrm{Col}_r(H) = \delta_{2^p}^{\beta},\ 1 \leqslant r \leqslant 2^n\right\}.$$

We should assume $\mathcal{O}(\beta) \neq \emptyset$ in the following. Otherwise, the state feedback based output tracking control problem is not solvable.

Theorem 4. *Assume that $\mathcal{O}(\beta)$ is a robust $\bar{L}_1$-invariant set as well as a robust L_2-invariant set. The output of system* (1) *can robustly track $y_r = \delta_{2^p}^{\beta}$ under the state feedback control $u(t) = Gx(t)$ if and only if there exists a positive integer τ such that*

$$\mathrm{Col}(H\tilde{L}_\tau) = \left\{\delta_{2^p}^{\beta}\right\}. \tag{16}$$

Proof. Sufficiency. Assume that (16) holds. We prove that $\mathrm{Col}(H\tilde{L}_t) = \{\delta_{2^p}^{\beta}\}$ holds for any integer $t \geqslant \tau$ by induction.

Obviously, $\mathrm{Col}(H\tilde{L}_t) = \{\delta_{2^p}^{\beta}\}$ holds for $t = \tau$.

Assuming that $\mathrm{Col}(H\tilde{L}_t) = \{\delta_{2^p}^{\beta}\}$ holds for some integer $t = \lambda > \tau$, that is,

$$y(\lambda) = Hx(\lambda) = H\tilde{L}_\lambda \ltimes_{i=\lambda-1}^{0} \xi(i)x(0) = \delta_{2^p}^{\beta}.$$

Thus, $x(\lambda) \in \mathcal{O}(\beta)$.

Now we prove the case of $t = \lambda + 1$.

- If $t_k < \lambda+1 < t_{k+1}$, we have $y(\lambda+1; u, \xi, x(0)) = Hx(\lambda+1) = H\bar{L}_1\xi(\lambda)x(\lambda)$. Since $\mathcal{O}(\beta)$ is a robust $\bar{L}_1$-invariant set and $x(\lambda) \in \mathcal{O}(\beta)$, we have $\bar{L}_1\xi(\lambda)x(\lambda) \in \mathcal{O}(\beta)$, which shows that $y(\lambda + 1) = H\tilde{L}_{\lambda+1} \ltimes_{i=\lambda}^{0} \xi(i)x(0) = \delta_{2^p}^{\beta}$. From the arbitrariness of $\xi(i)$ and $x(0)$ we have $\mathrm{Col}(H\tilde{L}_{\lambda+1}) = \{\delta_{2^p}^{\beta}\}$.
- If $\lambda + 1 = t_k$, we have $y(\lambda + 1; u, \xi, x(0)) = Hx(\lambda + 1) = HL_2\xi(\lambda)x(\lambda)$. Noticing that $\mathcal{O}(\beta)$ is a robust L_2-invariant set and $x(\lambda) \in \mathcal{O}(\beta)$, one can see that $L_2\xi(\lambda)x(\lambda) \in \mathcal{O}(\beta)$. Hence, $y(\lambda + 1) = H\tilde{L}_{\lambda+1} \ltimes_{i=\lambda}^{0} \xi(i)x(0) = \delta_{2^p}^{\beta}$, which together with the arbitrariness of $\xi(i)$ and $x(0)$ implies that $\mathrm{Col}(H\tilde{L}_{\lambda+1}) = \{\delta_{2^p}^{\beta}\}$.

By induction, $\mathrm{Col}(H\tilde{L}_t) = \{\delta_{2^p}^{\beta}\}$ holds for any integer $t \geqslant \tau$. Therefore, $y(t; u, \xi, x(0)) = \delta_{2^p}^{\beta}$ holds for any integer $t \geqslant \tau$, any $x(0) \in \Delta_{2^n}$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \Delta_{2^q}$. By Definition 5, the output of system (1) can robustly track $y_r = \delta_{2^p}^{\beta}$ under the state feedback control $u(t) = Gx(t)$.

Necessity. Suppose that the output of system (1) can robustly track $y_r = \delta_{2^p}^{\beta}$ under the state feedback control $u(t) = Gx(t)$. From Definition 5 there exists a positive integer τ such that $y(t; u, \xi, x(0)) = \delta_{2^p}^{\beta}$ holds for any initial state $x(0) \in \Delta_{2^n}$, any integer $t \geqslant \tau$, and any $\{\xi(t)\colon t \in \mathbb{N}\} \subseteq \Delta_{2^q}$. One can see from (15) that $y(\tau) = H\tilde{L}_\tau \ltimes_{i=\tau-1}^{0} \xi(i)x(0) = \delta_{2^p}^{\beta}$. From the arbitrariness of $\xi(i)$ and $x(0)$ we have $\mathrm{Col}(H\tilde{L}_\tau) = \{\delta_{2^p}^{\beta}\}$. $\square$

Remark 3. From Theorems 3 and 4 one can see that robust partial stabilization and robust output tracking are special cases of robust set stabilization for system (1).

4 Illustrative examples

Example 2. Consider the following BCN with impulsive effects:

$$\begin{aligned}
x_1(t+1) &= \xi(t) \wedge \big[\big(u(t) \wedge \big(x_1(t) \vee x_2(t)\big)\big) \\
&\quad \vee \big(\neg u(t) \wedge \big(x_1(t) \vee x_2(t)\big)\big)\big] \vee \big(\neg\xi(t) \wedge \neg u(t)\big), \\
x_2(t+1) &= \xi(t) \wedge \big[u(t) \vee \big(\neg u(t) \wedge \neg x_1(t) \wedge x_2(t)\big)\big] \\
&\quad \vee \big(\neg\xi(t) \wedge \big[u(t) \wedge \neg x_1(t) \wedge \neg x_2(t) \\
&\quad \vee \big(\neg u(t) \wedge \big(x_1(t) \vee \neg x_2(t)\big)\big)\big]\big); \\
x_1(t_k) &= \big[\xi(t) \wedge x_1(t) \wedge x_2(t)\big] \vee \big[\neg\xi(t) \wedge \big(x_1(t) \bar{\vee} x_2(t)\big)\big], \\
x_2(t_k) &= \xi(t) \wedge \big(x_1(t) \vee x_2(t)\big); \\
y(t) &= x_1(t) \leftrightarrow x_2(t),
\end{aligned} \tag{17}$$

where $t_{k-1} \leqslant t < t_k - 1$, $t_0 = 0$, and $t_k = k^2 + 1$, $k \in \mathbb{Z}_+$.

Given the following state feedback control

$$u(t) = x_1(t) \vee x_2(t). \tag{18}$$

Our objective is to verify whether or not the output of system (17) can robustly track $y_r = 1$ under the state feedback control (18)?

We firstly convert (17) and (18) into algebraic forms. Letting $x(t) = x_1(t) \ltimes x_2(t)$, system (17) can be expressed as

$$\begin{aligned}
x(t+1) &= L_1\xi(t)u(t)x(t), \quad t_{k-1} \leqslant t < t_k - 1; \\
x(t_k) &= L_2\xi(t_k - 1)x(t_k - 1); \\
y(t) &= Hx(t),
\end{aligned}$$

where $L_1 = \delta_4[1, 1, 1, 3, 2, 2, 1, 4, 4, 4, 4, 3, 1, 1, 2, 1]$, $L_2 = \delta_4[1, 3, 3, 4, 4, 2, 2, 4]$, and $H = \delta_2[1, 2, 2, 1]$. In addition, $y_r = \delta_2^1$.

Similarly, the control (18) has the following algebraic form:

$$u(t) = Gx(t),$$

where $G = \delta_2[1, 1, 1, 2]$.

A simple calculation shows that $\bar{L}_1 = \delta_4[1, 1, 1, 4, 4, 4, 4, 1]$ and $\mathcal{O}(1) = \{\delta_4^1, \delta_4^4\}$. Obviously, $\mathcal{O}(1)$ is a robust $\bar{L}_1$-invariant set as well as a robust L_2-invariant set. By (7), when $\tau = 1$, we have $\tilde{L}_1 = \delta_4[1, 1, 1, 4, 4, 4, 4, 1]$ and $\mathrm{Col}(H\tilde{L}_1) = \{\delta_2^1\}$. By Theorem 4, the output of system (17) can robustly track y_r under the state feedback control (18).

Example 3. Consider the following BCN with impulsive effects:

$$\begin{aligned}
x(t+1) &= L_1\xi(t)u(t)x(t), \quad t_{k-1} \leqslant t < t_k - 1; \\
x(t_k) &= L_2\xi(t_k - 1)x(t_k - 1); \\
y(t) &= Hx(t),
\end{aligned} \tag{19}$$

where $t_0 = 0$, $t_k = k^2+1$, $k \in \mathbb{Z}_+$, $L_1 = \delta_4[1,\ 1,\ 1,\ 4,\ 2,\ 2,\ 1,\ 4,\ 4,\ 4,\ 4,\ 1,\ 1,\ 1,\ 2,\ 1]$, $L_2 = \delta_4[1,\ 3,\ 3,\ 4,\ 4,\ 2,\ 2,\ 4]$, and $H = \delta_2[1,\ 2,\ 2,\ 1]$.

We aim to design a free-form control sequence under which system (19) is robustly stabilizable to the set $A = \{\delta_4^1, \delta_4^4\}$.

According to (10), we have $\hat{L} = L_1 W_{[2,2]} = \delta_4[1,\ 1,\ 1,\ 4,\ 4,\ 4,\ 4,\ 1,\ 2,\ 2,\ 1,\ 4,\ 1,\ 1,\ 2,\ 1]$. Obviously, the set A is a robust $\mathrm{Blk}_1(\hat{L}_1)$-invariant set and a robust L_2-invariant set. Setting $\tau = 2$, from (12) we have $\hat{L}_2 = \delta_4[1,\ 1,\ 1,\ 4,\ 4,\ 4,\ 4,\ 1,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 3,\ 3,\ 1,\ 4,\ 1,\ 1,\ 3,\ 1,\ 2,\ 2,\ 4,\ 4,\ 4,\ 4,\ 2,\ 4]$. It is easy to see that $\mathrm{Col}(\mathrm{Blk}_1(\hat{L}_2)) \subseteq A$. Therefore, by Theorem 2, system (19) is robustly stabilizable to A under the following free-form control sequence:

$$u(t) = \delta_2^1, \quad t \in \mathbb{N} \setminus \{t_k - 1\colon k \in \mathbb{Z}_+\}.$$

Remark 4. For the above free-form control sequence, we can obtain the state feedback gain matrix $G = \delta_2[1,\ 1,\ 1,\ 1]$ under which system (19) is robustly stabilizable to A. This is one possible way to design state feedback gain matrix for the stabilization of BCNs with impulsive effects.

5 Conclusion

In this paper, we have studied the robust set stabilization of BCNs with impulsive effects, and presented some new results. We have converted the closed-loop system consisting of a BCN with impulsive effects and a given state feedback control into an algebraic form. Based on the algebraic form, we have presented a necessary and sufficient condition for the robust set stabilization of BCNs with impulsive effects. As applications of robust set stabilization, we have studied the robust partial stabilization and the robust output tracking of BCNs with impulsive effects. Future works will focus on two issues: (i) design of state feedback gains for the robust set stabilization of BCNs with impulsive effects; (ii) robust set stabilization of BCNs with complex impulses [26–28].

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