Nonlinear Analysis: Modelling and Control, Vol. 23, No. 4, 533–552 ISSN 1392-5113
https://doi.org/10.15388/NA.2018.4.5

Maximum principle for a nonlinear size-structured model of fish and fry management*

Rong Liu, Guirong Liu[1]

School of Mathematical Sciences, Shanxi University,
Taiyuan, Shanxi 030006, China
liurong2006060126@163.com; lgr5791@sxu.edu.cn

Received: September 27, 2017 / **Revised:** March 5, 2018 / **Published online:** June 15, 2018

Abstract. This paper investigates the maximum principle for a nonlinear size-structured model that describes the optimal management of the fish resources taking into account harvesting the fish and putting the fry. We establish the well-posedness of the state system by Banach fixed-point theorem. Necessary conditions for optimality are established via the normal cone technique and adjoint system. The existence of a unique optimal policy is proved via Ekeland's variational principle and fixed-point reasoning. Finally, some examples and numerical results demonstrate the effectiveness of the theoretical results in our paper.

Keywords: size-structure, maximum principle, well-posedness, uniqueness, Ekeland's variational principle.

1 Introduction

The topic of the population dynamic system with structural differences has attracted the interest of many researchers, and important progresses have been made during the last century. Among the individual structure, there are many structural differences, such as age, body size, gender, gene, and life stage. During the past few decades, the well-posedness, asymptotic behavior, and optimal control of partial differential equations have been frequently used to explain biological and chemical evolution process, see, e.g., [10, 15]. Especially, age-structured first-order partial differential equations provide a main tool for modeling population systems [19] and are recently employed in economics. Models with age structure have been proposed about the well-posedness, asymptotic behavior in the mathematical analysis, and optimal control of populations in biology. To name a few, see [1,2,8,9,16–18] and the references therein. In [1], the authors investigated two optimal harvesting problems for the following age-structured population dynamics

*This research was supported by the National Natural Science Foundation of China (No. 11471197).
[1]Corresponding author.

with logistic term and time-periodic vital rates:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial(u)}{\partial a}=-\mu(a,t)u(a,t)-\mathcal{M}\bigl(\Gamma(t)\bigr)u(a,t)-\alpha(t)u(a,t),\\
&\quad(x,t)\in Q,\\
&u(0,t)=\int\limits_0^A\beta(a,t)u(a,t)\,\mathrm{d}a,\quad t\in\mathbb{R}^+,\\
&u(a,0)=u_0(a),\quad x\in[0,A),\\
&\Gamma(t)=\int\limits_0^A\gamma(a)u(a,t)\,\mathrm{d}a,\quad t\in\mathbb{R}^+,
\end{aligned}\tag{1}$$

where $Q=[0,A)\times\mathbb{R}^+$, and $A\in(0,\infty)$ is a maximal age of the population species. Here $u(a,t)$ is the population density for age a at time t; β and μ are natural fertility rate and mortality rate, both time-periodic of period T; $\mathcal{M}(\Gamma(t))$ represents an external mortality rate and is due to the overpopulation; $u(t)$ is the harvesting effort. In [17], the authors presented necessary optimality conditions of Pontryagin's type for infinite-horizon optimal control problems for age-structured systems with state and control-dependent boundary conditions.

Long-term ecological researches show that the vital parameters of individual are closely connected with its body size for many populations. Size of an individual has a strong influence upon dynamical processes like its feeding, growth, and reproduction [20], which in turn affect the dynamics of the population as a whole. In [5], Caswell pointed out that size-dependent demography is probably the rule rather than the exception. Here by size we mean some indices displaying the physiological or statistical characteristics of population individuals. Sizes can be mass, length, diameter, and so on. As a result, modelling population dynamics, it is natural to assume that the vital rates, such as fertility, mortality, and growth rates of individuals depend on their body size and time (see [3, 7, 11–14, 21]). In [21], the authors studied the optimal harvesting problem for a food chain of three species in a periodic environment with size structures in the predators. In [14], we investigated the least cost-size problem and the least cost-deviation problem for the following nonlinear vermin population model with size-structure:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial(V(x,t)u)}{\partial x}=f(x,t)-\mu(x,t)u(x,t)-\Phi\bigl(I(t)\bigr)u(x,t),\\
&\quad(x,t)\in Q,\\
&V(0,t)u(0,t)=\int\limits_0^l\beta(x,t)\omega(x,t)u(x,t)\,\mathrm{d}x,\quad t\in(0,T],\\
&u(x,0)=u_0(x),\quad x\in[0,l),\\
&I(t)=\int\limits_0^l m(x)u(x,t)\,\mathrm{d}x,\quad t\in[0,T].
\end{aligned}\tag{2}$$

where $Q = [0, l) \times [0, T]$ and $\beta \in \Omega = \{h \in L^\infty(Q)\colon \underline{\beta} \leqslant h(x,t) \leqslant \bar{\beta}\ \forall (x,t) \in Q\}$. Here $u(x,t)$ is the population density of size $x \in [0,l)$ at time $t \in [0,T]$; $l \in (0,\infty)$ is a maximal size and T is a given time. In [18], a general model of a heterogeneous control system is introduced in the form of a first-order distributed system with nonlocal dynamics and exogenous side-conditions. The author gave the necessary optimality conditions in the form of the Pontryagin maximum principle on a finite time-horizon $[0,T]$, which can include the age-structured and size-structured problems.

However, in the ecological environment, only some of the newborns can survive for some populations especially for fish resources. For fish, size is a more important parameter than age because it is easy to get the quality of the fish, but it is hard to know the age of the fish. To the best of our knowledge, so far there is no investigation on the optimal control of size structured population models by taking the survival rate of the new born individuals into account. Motivated by the above discussion, this paper considers a nonlinear size-structured model that describes the optimal management of the fish resources taking into account harvesting the fish and putting the fry.

To build our model, we assume that $u(x,t)$ denotes the fish density of individuals at time t with respect to size x; $\mu(x,t)$ and $\beta(x,t)$ are, respectively, the natural mortality and egg-laying rate; $g(x)$ stands for the growth rate of individual's size, that is, $\mathrm{d}x/\mathrm{d}t = g(x)$; ϕ is the amount of artificial stocking fry, which depends on the total population $P(t)$; $I(t)$ corresponds to the number of eggs deposited at time t; f is the proportion of eggs that survive to become fishes, which depend on the parental egg production at time t. Then we propose the following size-structured model for the fish dynamics in a control period:

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \frac{\partial (g(x)u(x,t))}{\partial x} = -\mu(x,t)u(x,t) - \alpha(x,t)u(x,t), \\ &\quad (x,t) \in Q, \\ &g(0)u(0,t) = \phi\bigl(P(t)\bigr) + f\bigl(I(t)\bigr)I(t), \quad t \in [0,T], \\ &u(x,0) = u_0(x), \quad x \in [0,l), \end{aligned} \tag{3}$$

where

$$P(t) = \int_0^l u(x,t)\,\mathrm{d}x, \quad I(t) = \int_0^l \beta(x,t)u(x,t)\,\mathrm{d}x, \quad t \in [0,T].$$

Here $Q = [0,l) \times [0,T]$, $l \in (0,+\infty)$ is the maximum size of any individual in the population, and $T \in (0,+\infty)$ is a given time. The control variable $\alpha(x,t)$ is the harvesting efforts, which belongs to

$$\mathcal{U} = \bigl\{\alpha \in L^\infty(Q)\colon 0 \leqslant \underline{\alpha} \leqslant \alpha(x,t) \leqslant \overline{\alpha} \text{ a.e. } (x,t) \in Q\bigr\},$$

where $\underline{\alpha}$, $\overline{\alpha}$ are positive constants. Let $u^\alpha(x,t)$ be the solution of system (3) corresponding to $\alpha \in \mathcal{U}$. In this paper, we investigate the following optimization problem:

$$\max_{\alpha \in \mathcal{U}} J(\alpha), \tag{4}$$

where

$$J(\alpha) = \int_0^T \int_0^l \omega(x,t)\alpha(x,t)u^\alpha(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T r_1(t)\phi\big(P^\alpha(t)\big)\,\mathrm{d}t - \int_0^T \int_0^l r_2(t)h(x,t)u^\alpha(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^T \int_0^l \rho\alpha^2(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Here the function $\omega(x,t)$ is the economic values of an individual of fish with size x at time t; $h(x,t)$ represents the average amount of fish food eaten by a single fish individual with size x at time t; $r_1(t)$ is the price of the fish fry at time t; $r_2(t)$ is the price of the fish food at time t; $\rho > 0$ is the weight factor of the costs for implementing the controls. Thus, the 1st term of functional J represents the benefits from harvesting the fish, the 2nd term represents the cost of restocking fish fry, the 3rd term is the cost of fish feed, and the 4th term is the cost of controls. Therefore, $J(\alpha)$ represents the total net economic benefit yielded from harvesting during a time of T.

Denote $\mathbb{R}_+ := [0,\infty)$, $L^1_+ := L^1(0,l;\mathbb{R}_+)$, and $L^\infty_+ := L^\infty(0,l;\mathbb{R}_+)$. In the following discussion, we make the following assumptions:

- (A1) $g \in C^1[0,l)$ is a bounded function, $g(0) = 1$, $\lim_{x\uparrow l} g(x) = 0$, and $g(x) > 0$, $g'(x) \leqslant 0$ for $x \in [0,l)$. Moreover, there is a positive constant L_V such that for $x_1, x_2 \in [0,l)$, $|g(x_1) - g(x_2)| \leqslant L_V|x_1 - x_2|$.
- (A2) There exists $\bar{\beta} \in \mathbb{R}_+$ such that $0 < \beta(x,t) \leqslant \bar{\beta}$ for $(x,t) \in Q$.
- (A3) $\mu : [0,l) \times [0,T] \to \mathbb{R}_+$ is a measurable function, and $\mu(x,t) + g'(x) \geqslant 0$.
- (A4) $u_0 \in L^1_+$ and there exists $\bar{u} \in \mathbb{R}_+$ such that $0 \leqslant u_0(x) \leqslant \bar{u}$.
- (A5) $f(s) \geqslant 0$ is continuous and bounded on $\mathbb{R}_+$, and $F(s) := sf(s) \in C^1$ is increasing on $\mathbb{R}_+$. Moreover, there exists a constant k_1 such that for $s_1, s_2 \in \mathbb{R}_+$, $|F(s_1) - F(s_2)| \leqslant k_1|s_1 - s_2|$.
- (A6) $\phi(s) \geqslant 0$ is continuous decreasing on $\mathbb{R}_+$, $\phi(0) = 0$, and $\phi \in C^1$. Moreover, there exists $k_2 > 0$ such that for $s_1, s_2 \in \mathbb{R}_+$, $|\phi(s_1) - \phi(s_2)| \leqslant k_2|s_1 - s_2|$.

The remainder of the present paper is organized as follows. First, in Section 2, we show the existence and uniqueness of solutions to the basic system by Banach fixed-point theorem and make several estimates. In Section 3, we study the adjoint system, which will be used latter. Necessary conditions for optimality will be established in Section 4, while Section 5 is devoted to the existence of a unique optimal control. Section 6 contains example and numerical results, which are used to demonstrate the effectiveness of the theoretical results in our paper. The paper ends with a conclusion section.

2 Well-posedness of the state system

In this section, we provide some properties of the solutions, which include the boundedness and the continuous dependence of the population density.

Definition 1. The unique solution $x = \varphi(t; t_0, x_0)$ of the initial-valued problem $x'(t) = g(x),\ x(t_0) = x_0$ is said to be a characteristic curve of system (3). Let $z(t) = \varphi(t; 0, 0)$ be the characteristic curve through $(0, 0)$ in the xt-plane.

Definition 2. The derivative of the function $u(x, t)$ at (x, t) along the characteristic curve φ is given by

$$D_\varphi u(x,t) = \lim_{h\to 0} \frac{u(\varphi(t+h;t,x),\, t+h) - u(x,t)}{h}.$$

Without loss of generality, we assume that $\alpha(x, t) \equiv 0$. For an arbitrary point (x, t) in the first quadrant of the xt-plane such that $x \leqslant z(t)$, that is, $\varphi(t; t, x) \leqslant z(t)$, define the initial time $\tau := \tau(x, t)$ implicitly by the relation that $\varphi(t; \tau, 0) = x$ if and only if $\varphi(\tau; t, x) = 0$. It is clear that $\tau = \varphi^{-1}(0; t, x)$. Utilizing the characteristic curve technique, the solution of system (3) can be given as

$$u(x,t) = \begin{cases} u(0, \varphi^{-1}(0;t,x)) \exp\{-\int_\tau^t [\mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x))]\,\mathrm{d}s\}, \\ \quad x \leqslant z(t), \\ u_0(\varphi(0;t,x)) \exp\{-\int_0^t [\mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x))]\,\mathrm{d}s\}, \\ \quad x > z(t). \end{cases} \tag{5}$$

We then show the existence, uniqueness, and boundedness of solutions to system (3). Let $\mathbf{X} = L^\infty(Q)$ and define a new norm in $\mathbf{X}$ by

$$\|u\|_* = \operatorname*{ess\,sup}_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \int_0^l \big|u(x,t)\big|\,\mathrm{d}x \right\}$$

for some $\lambda > 0$, which is equivalent to the usual norm on the space $\mathbf{X}$. Thus, $(\mathbf{X}, \|\cdot\|_*)$ is a Banach space. Let $M := l\bar{u} + l\bar{u}(k_1\bar{\beta} + k_2)T\mathrm{e}^{(k_1\bar{\beta}+k_2)T}$ and define the solution space as follows:

$$\mathcal{X} = \left\{ u \in \mathbf{X} \;\middle|\; u(x,t) \geqslant 0 \text{ a.e. } (x,t) \in Q \text{ and } \int_0^l u(x,t)\,\mathrm{d}x \leqslant M \right\}.$$

It is clear that $\mathcal{X}$ is a nonempty closed subset in $\mathbf{X}$. Define $\mathcal{A} : \mathcal{X} \to \mathbf{X}$ by

$$(\mathcal{A}u)(x,t) = u(x,t), \tag{6}$$

where $u(x, t)$ is given as (5). Obviously, if $u(x, t)$ is a fixed point of the map $\mathcal{A}$, then it is a solution of system (3) and vice versa.

In this section, assume that $T > z^{-1}(l)$. When $T \leqslant z^{-1}(l)$, we can get the same results by the same method.

Next, we verify that the mapping $\mathcal{A}$ satisfies the conditions of the Banach fixed-point theorem. For any $u, v \in \mathbf{X}$, denote $d(u, v) = \|u - v\|_*$.

Lemma 1. $(\mathcal{X}, d)$ *is a complete metric space.*

Proof. It is easy to show that d is a metric in $\mathbf{X}$. Note that $(\mathbf{X}, \|\cdot\|_*)$ is a Banach space. Then $(\mathbf{X}, d)$ is a complete metric space. In addition, $\mathcal{X}$ is a nonempty closed subset in $\mathbf{X}$. Hence, $(\mathcal{X}, d)$ is a complete metric space. The proof is complete. □

Lemma 2. *The mapping $\mathcal{A}$ maps $\mathcal{X}$ into $\mathcal{X}$.*

Proof. Note that $g(0) = 1$. Denoting $b(t) := g(0)u(0,t) = \phi(P(t)) + F(I(t))$, then by (A5) and (A6) we have $F(I(t)) \leqslant k_1 I(t)$ and $\phi(P(t)) \leqslant k_2 P(t)$. Thus,

$$\begin{aligned} b(t) &= \phi\big(P(t)\big) + F\big(I(t)\big) \leqslant k_1 I(t) + k_2 P(t) \\ &= k_1 \int_0^l \beta(x,t)u(x,t)\,\mathrm{d}x + k_2 \int_0^l u(x,t)\,\mathrm{d}x \leqslant (k_1\bar\beta + k_2)\int_0^l u(x,t)\,\mathrm{d}x \\ &= (k_1\bar\beta + k_2)\left[\int_0^{z(t)} u(x,t)\,\mathrm{d}x + \int_{z(t)}^l u(x,t)\,\mathrm{d}x\right] \\ &\leqslant (k_1\bar\beta + k_2)\left[l\bar u + \int_0^{z(t)} b\big(\varphi^{-1}(0;t,x)\big)\,\mathrm{d}x\right]. \end{aligned} \tag{7}$$

Denote $I_1 := \int_0^{z(t)} b(\varphi^{-1}(0;t,x))\,\mathrm{d}x$. Let $s = \varphi^{-1}(0;t,x)$. Then by Definition 1, $s = t$ when $x = 0$, while $s = 0$ when $x = z(t)$. It follows from $s = \varphi^{-1}(0;t,x)$ that $x = \varphi(t;s,0)$. Note that $\mathrm{d}x/\mathrm{d}s$ is the solution of the initial-valued problem

$$\frac{\mathrm{d}z}{\mathrm{d}t} = g'\big(\varphi(t;s,0)\big)z, \qquad z(s) = -g(0) = -1. \tag{8}$$

It is clear that the solution of (8) $z(t) = -\exp\{\int_s^t g'(\varphi(r;s,0))\,\mathrm{d}r\}$, that is,

$$\mathrm{d}x = -\exp\left\{\int_s^t g'\big(\varphi(r;s,0)\big)\,\mathrm{d}r\right\}\mathrm{d}s.$$

Thus, by (A1), we have

$$I_1 = -\int_t^0 b(s)\exp\left\{\int_s^t g'\big(\varphi(r;s,0)\big)\,\mathrm{d}r\right\}\mathrm{d}s \leqslant \int_0^t b(s)\mathrm{d}s, \tag{9}$$

which, together with (7), yields

$$b(t) \leqslant (k_1\bar\beta + k_2)\left[l\bar u + \int_0^t b(s)\,\mathrm{d}s\right].$$

It follows from Grönwall's inequality that

$$b(t) \leqslant l\bar{u}(k_1\bar{\beta} + k_2)\mathrm{e}^{(k_1\bar{\beta}+k_2)t} \leqslant l\bar{u}(k_1\bar{\beta} + k_2)\mathrm{e}^{(k_1\bar{\beta}+k_2)T}. \tag{10}$$

Next, we consider $(\mathcal{A}u)(x,t)$. From (9) and (10) it is clear that

$$\begin{aligned}
&\int_0^l |\mathcal{A}u|(x,t)\,\mathrm{d}x \\
&\quad = \int_0^{z(t)} u(0,\tau)\exp\left\{-\int_\tau^t \big[\mu\big(\varphi(s;t,x),s\big) + g'\big(\varphi(s;t,x)\big)\big]\mathrm{d}s\right\}\mathrm{d}x \\
&\qquad + \int_{z(t)}^l u_0\big(\varphi(0;t,x)\big)\exp\left\{-\int_0^t \big[\mu\big(\varphi(s;t,x),s\big) + g'\big(\varphi(s;t,x)\big)\big]\,\mathrm{d}s\right\}\mathrm{d}x \\
&\quad \leqslant \int_0^{z(t)} u(0,\tau)\,\mathrm{d}x + \int_0^l u_0\big(\varphi(0;t,x)\big)\,\mathrm{d}x \\
&\quad = \int_0^{z(t)} b\big(\varphi^{-1}(0;t,x)\big)\,\mathrm{d}x + \int_0^l u_0\big(\varphi(0;t,x)\big)\,\mathrm{d}x \\
&\quad \leqslant l\bar{u} + l\bar{u}(k_1\bar{\beta} + k_2)T\mathrm{e}^{(k_1\bar{\beta}+k_2)T}.
\end{aligned}$$

It follows that $\mathcal{A}$ is a mapping from $\mathcal{X}$ to $\mathcal{X}$. The proof is complete. □

Lemma 3. *$\mathcal{A}$ is a contraction mapping on the complete metric space $(\mathcal{X}, d)$.*

Proof. For any $u_1, u_2 \in \mathcal{X}$, by the definition of mapping $\mathcal{A}$, it is easy to show that $(\mathcal{A}u_1)(x,t) - (\mathcal{A}u_2)(x,t) = 0$, $x > z(t)$, for $t \in [0,T]$. It follows from the definition of mapping $\mathcal{A}$ and (9) that

$$\begin{aligned}
&\int_0^l |\mathcal{A}u_1 - \mathcal{A}u_2|(x,t)\,\mathrm{d}x \\
&\quad \leqslant \int_0^{z(t)} \big|u_1\big(0,\varphi^{-1}(0;t,x)\big) - u_2\big(0,\varphi^{-1}(0;t,x)\big)\big|\,\mathrm{d}x \leqslant \int_0^t \big|b_1(s) - b_2(s)\big|\,\mathrm{d}s \\
&\quad = \int_0^t \big|\phi\big(P_1(s)\big) + F\big(I_1(s)\big) - \phi\big(P_2(s)\big) - F\big(I_2(s)\big)\big|\,\mathrm{d}s \\
&\quad \leqslant (k_1\bar{\beta} + k_2)\int_0^t\int_0^l \big|u_1(x,s) - u_2(x,s)\big|\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Then

$$d(\mathcal{A}u_1, \mathcal{A}u_2) = \operatorname*{ess\,sup}_{t\in[0,T]}\left\{ \mathrm{e}^{-\lambda t} \int\limits_0^l \bigl|(\mathcal{A}u_1)(x,t) - (\mathcal{A}u_2)(x,t)\bigr| \,\mathrm{d}x \right\}$$
$$\leqslant (k_1\bar{\beta} + k_2) \operatorname*{ess\,sup}_{t\in[0,T]}\left\{ \mathrm{e}^{-\lambda t} \int\limits_0^t \int\limits_0^l \bigl|u_1(x,s) - u_2(x,s)\bigr| \,\mathrm{d}x\,\mathrm{d}s \right\}$$
$$= \frac{(k_1\bar{\beta} + k_2)}{\lambda} d(u_1, u_2).$$

Choose λ such that $\lambda > (k_1\bar{\beta} + k_2)$. Then $\mathcal{A}$ becomes a contraction on the complete metric space $(\mathcal{X}, \|\cdot\|_*)$. The proof is complete. □

By Lemmas 1–3 and the Banach fixed-point theorem, we can see that $\mathcal{A}$ has a unique fixed point, which is a nonnegative bounded solution for system (3). Hence, we have the following theorem.

Theorem 1. *Let assumptions* (A1)–(A6) *hold and* $\alpha \equiv 0$. *Then system* (3) *has one and only one nonnegative bounded solution* $u(x,t) \in L^\infty(Q)$.

Just for completeness, we state the following result for the case $\alpha \not\equiv 0$.

Theorem 2. *Let assumptions* (A1)–(A6) *hold. Then for any* $\alpha \in \mathcal{U}$, *system* (3) *has a unique solution* $u \in L^\infty(Q)$, *which is nonnegative and bounded,* $|u(x,t)| \leqslant B_1$. *Moreover, for any* $t \in [0,T]$ *and* $\alpha_1, \alpha_2 \in \mathcal{U}$, *there exists positive constants* B_2 *such that*

$$\bigl\|u_1(\cdot,t) - u_2(\cdot,t)\bigr\|_{L^1[0,l)} \leqslant B_2 \int\limits_0^t \bigl\|\alpha_1(\cdot,s) - \alpha_2(\cdot,s)\bigr\|_{L^1[0,l)} \,\mathrm{d}s, \tag{11}$$

where u_1 *and* u_2 *are the solutions of system* (3) *corresponding to* α_1 *and* $\alpha_2 \in \mathcal{U}$, *respectively.*

Proof. By a similar method as that in Theorem 1, it is easy to show that for any $\alpha \in \mathcal{U}$, system (3) has a unique nonnegative bounded solution $u \in L^\infty(Q)$ with μ replaced by $\mu + \alpha$. Next, we prove the inequality (11). For $i = 1, 2$, denote

$$E_i(t;t,x) = \exp\left\{ -\int\limits_\tau^t \bigl[\mu\bigl(\varphi(s;t,x),s\bigr) + g'\bigl(\varphi(s;t,x)\bigr) + \alpha_i\bigl(\varphi(s;t,x),s\bigr)\bigr] \,\mathrm{d}s \right\},$$

$$\varPi_i(t;t,x) = \exp\left\{ -\int\limits_0^t \bigl[\mu\bigl(\varphi(s;t,x),s\bigr) + g'\bigl(\varphi(s;t,x)\bigr) + \alpha_i\bigl(\varphi(s;t,x),s\bigr)\bigr] \,\mathrm{d}s \right\}.$$

Note that $u_1(x,t)$ and $u_2(x,t)$ are solutions of system (3) corresponding to $\alpha_1 \in \mathcal{U}$ and $\alpha_2 \in \mathcal{U}$, respectively. Then by (5) and $g(0)=1$, one has

$$\begin{aligned}
&\big\|u_1(\cdot,t)-u_2(\cdot,t)\big\|_{L^1[0,l)} \\
&= \int_0^l \big|u_1(x,t)-u_2(x,t)\big|\,\mathrm{d}x \\
&\leqslant \int_0^{z(t)} \big|b_1\big(\varphi^{-1}(0;t,x)\big)-b_2\big(\varphi^{-1}(0;t,x)\big)\big|\,\mathrm{d}x \\
&\quad + l\bar{u}(k_1\bar{\beta}+k_2)\mathrm{e}^{(k_1\bar{\beta}+k_2)T}\int_0^{z(t)}\int_0^t \big|\alpha_1\big(\varphi(s;t,x),s\big)-\alpha_2\big(\varphi(s;t,x),s\big)\big|\,\mathrm{d}s\,\mathrm{d}x \\
&\quad + \bar{u}\int_{z(t)}^l\int_0^t \big|\alpha_1\big(\varphi(s;t,x),s\big)-\alpha_2\big(\varphi(s;t,x),s\big)\big|\,\mathrm{d}s\,\mathrm{d}x \\
&:= \int_0^{z(t)} \big|b_1\big(\varphi^{-1}(0;t,x)\big)-b_2\big(\varphi^{-1}(0;t,x)\big)\big|\,\mathrm{d}x \\
&\quad + M_1\int_0^l\int_0^t \big|\alpha_1\big(\varphi(s;t,x),s\big)-\alpha_2\big(\varphi(s;t,x),s\big)\big|\,\mathrm{d}s\,\mathrm{d}x \\
&:= I_2+I_3, \qquad (12)
\end{aligned}$$

where $M_1=\max\{\bar{u}, l\bar{u}(k_1\bar{\beta}+k_2)\mathrm{e}^{(k_1\bar{\beta}+k_2)T}\}$.

For I_2, let $s=\varphi^{-1}(0;t,x)$. It follows from (10) and (A1) that

$$\begin{aligned}
I_2 &= \int_0^{z(t)} \big|b_1\big(\varphi^{-1}(0;t,x)\big)-b_2\big(\varphi^{-1}(0;t,x)\big)\big|\,\mathrm{d}x \\
&= -\int_t^0 \big|b_1(s)-b_2(s)\big|\exp\left\{\int_s^t g'\big(\varphi(r;s,0)\big)\,\mathrm{d}r\right\}\mathrm{d}s \\
&\leqslant \int_0^t \big|b_1(s)-b_2(s)\big|\,\mathrm{d}s. \qquad (13)
\end{aligned}$$

For I_3, let $w=\varphi(0;t,x)$. Then by Definition 1, $w=0$ when $x=0$, while $w=l$ when $x=l$. If $z(s)$ is the solution of the initial-valued problem

$$\frac{\mathrm{d}z}{\mathrm{d}s}=g'\big(\varphi(s;t,x)\big)z, \qquad z(t)=1, \qquad (14)$$

then $\mathrm{d}w/\mathrm{d}x = z(0)$. It is easy to get $z(s) = \exp\{\int_t^s g'(\varphi(r;t,x))\,\mathrm{d}r\}$. Thus, we can get $\mathrm{d}w/\mathrm{d}x = z(0) = \exp\{\int_t^0 g'(\varphi(r;t,x))\,\mathrm{d}r\}$, which means

$$\mathrm{d}x = \exp\left\{\int_0^t g'\big(\varphi(r;t,x)\big)\,\mathrm{d}r\right\}\mathrm{d}w. \tag{15}$$

Thus, by (A1), we have

$$\begin{aligned}
I_3 &= M_1\int_0^t\int_0^l \big|\alpha_1\big(\varphi(s;t,x),s\big) - \alpha_2\big(\varphi\big(s;t,x),s\big)\big|\,\mathrm{d}x\,\mathrm{d}s \\
&= M_1\int_0^t\int_0^l \big|\alpha_1(w,s) - \alpha_2(w,s)\big|\exp\left\{\int_0^t g'\big(\varphi(r;t,x)\big)\,\mathrm{d}r\right\}\mathrm{d}w\,\mathrm{d}s \\
&\leqslant M_1\int_0^t \big\|\alpha_1(\cdot,s) - \alpha_2(\cdot,s)\big\|_{L^1[0,l)}\,\mathrm{d}s.
\end{aligned} \tag{16}$$

It follows from (12), (13), and (16) that

$$\begin{aligned}
&\big\|u_1(\cdot,t) - u_2(\cdot,t)\big\|_{L^1[0,l)} \\
&\quad\leqslant \int_0^t \big|b_1(s) - b_2(s)\big|\,\mathrm{d}s + M_1\int_0^t \big\|\alpha_1(\cdot,s) - \alpha_2(\cdot,s)\big\|_{L^1[0,l)}\,\mathrm{d}s \\
&\quad= \int_0^t \big|\phi\big(P_1(s)\big) + F\big(I_1(s)\big) - \phi\big(P_2(s)\big) - F\big(I_2(s)\big)\big|\,\mathrm{d}s \\
&\qquad+ M_1\int_0^t \big\|\alpha_1(\cdot,s) - \alpha_2(\cdot,s)\big\|_{L^1[0,l)}\,\mathrm{d}s \\
&\quad\leqslant k_1\int_0^t \big|I_1(s) - I_2(s)\big|\,\mathrm{d}s + k_2\int_0^t \big|P_1(s) - P_2(s)\big|\,\mathrm{d}s \\
&\qquad+ M_1\int_0^t \big\|\alpha_1(\cdot,s) - \alpha_2(\cdot,s)\big\|_{L^1[0,l)}\,\mathrm{d}s \\
&\quad\leqslant M_1\int_0^t \big\|\alpha_1(\cdot,s) - \alpha_2(\cdot,s)\big\|_{L^1[0,l)}\,\mathrm{d}s \\
&\qquad+ (k_1\bar{\beta} + k_2)\int_0^t \big\|u_1(\cdot,s) - u_2(\cdot,s)\big\|_{L^1[0,l)}\,\mathrm{d}s.
\end{aligned} \tag{17}$$

It follows from Grönwall's inequality that

$$\left\| u_1(\cdot,t) - u_2(\cdot,t) \right\|_{L^1[0,l)} \leqslant M_1 \mathrm{e}^{(k_1\bar{\beta}+k_2)T} \int_0^t \left\| \alpha_1(\cdot,s) - \alpha_2(\cdot,s) \right\|_{L^1[0,l)} \mathrm{d}s.$$

The proof is complete. □

3 The adjoint system

In this section, we will derive the adjoint system of (3), which is necessary for optimality. First, we give the following lemma, which can be proved by means of the normal cone technique (see [4, Prop. 5.3]).

Lemma 4. *Suppose that $\vartheta(x,t) \in L^\infty(Q)$ satisfies*

$$\int_0^T \int_0^l \left[\vartheta(x,t)v(x,t) + \rho \left| v(x,t) \right| \right] \mathrm{d}x \, \mathrm{d}t \geqslant 0$$

for any $v \in \mathcal{T}_{\mathcal{U}}(\alpha)$. Then there exists $\theta \in L^\infty(Q)$ such that $\|\theta\|_\infty \leqslant 1$ and $\rho\theta - \vartheta \in \mathcal{N}_{\mathcal{U}}(\alpha)$.

Here and in the following, we denote by $\mathcal{T}_{\mathcal{U}}(\alpha)$ and $\mathcal{N}_{\mathcal{U}}(\alpha)$ the tangent cone and the normal cone of $\mathcal{U}$ at α, respectively.

Lemma 5. *Let $u^*(x,t)$ be the solution of system* (3) *corresponding to $\alpha^* \in \mathcal{U}$. For each $v \in T_{\mathcal{U}}(\alpha^*)$ such that $\alpha^* + \varepsilon v \in \mathcal{U}$ for sufficiently small $\varepsilon > 0$, we have $[u^\varepsilon - u^*]/\varepsilon \to z(x,t)$ as $\varepsilon \to 0$, where u^ε is the solution of* (3) *corresponding to $\alpha^* + \varepsilon v \in \mathcal{U}$, and $z(x,t)$ is the solution of the following system*:

$$\begin{aligned}
&D_\varphi z(x,t) = -\left[\mu(x,t) + g'(x) + \alpha^*(x,t)\right] z(x,t) - v(x,t)u^*(x,t), \\
&\quad (x,t) \in Q, \\
&g(0)z(0,t) = \phi'\left(P^*(t)\right)Q(t) + f'\left(I^*(t)\right)I^*(t)Z(t) + f\left(I^*(t)\right)Z(t), \\
&\quad t \in [0,T], \\
&z(x,0) = 0, \quad x \in [0,l), \\
&Z(t) = \int_0^l \beta(x,t)z(x,t)\,\mathrm{d}x, \quad Q(t) = \int_0^l z(x,t)\,\mathrm{d}x, \quad t \in [0,T],
\end{aligned} \tag{18}$$

in which $\phi'(P^*(t))$ and $f'(I^*(t))$ are the derivatives of ϕ and f with respect to P^* and I^*, respectively.

Proof. The existence and uniqueness of solution to (18) can be established by a similar way as that in the proofs of Theorems 1 and 2. According to Lemma 3.13 in [2],

$\lim_{\varepsilon\to 0}[u^\varepsilon - u^*]/\varepsilon$ makes sense. Note that u^ε and u^* are solutions of (3) corresponding to $\alpha^* + \varepsilon v$ and α^*, respectively. Thus, $[u^\varepsilon - u^*]/\varepsilon$ must be the solution of the following system:

$$\begin{aligned}
&D_\varphi \frac{1}{\varepsilon}\big[u^\varepsilon - u^*\big] = -[\mu + g']\frac{1}{\varepsilon}\big[u^\varepsilon - u^*\big] - \frac{1}{\varepsilon}\big[(\alpha^* + \varepsilon v)u^\varepsilon - \alpha^* u^*\big],\\
&g(0)\frac{1}{\varepsilon}\big[u^\varepsilon - u^*\big](0,t)\\
&\quad = \frac{1}{\varepsilon}\big[\phi\big(P^\varepsilon(t)\big) - \phi\big(P^*(t)\big)\big] + \frac{1}{\varepsilon}\big[f\big(I^\varepsilon(t)\big)I^\varepsilon(t) - f\big(I^*(t)\big)I^*(t)\big],\\
&\frac{1}{\varepsilon}\big[u^\varepsilon - u^*\big](x,0) = 0,\\
&\frac{1}{\varepsilon}\big[I^\varepsilon(t) - I^*(t)\big] = \int_0^l \beta(x,t)\frac{1}{\varepsilon}\big(u^\varepsilon - u^*\big)\,\mathrm{d}x,\\
&\frac{1}{\varepsilon}\big[P^\varepsilon(t) - P^*(t)\big] = \int_0^l \frac{1}{\varepsilon}\big(u^\varepsilon - u^*\big)\,\mathrm{d}x.
\end{aligned}\tag{19}$$

Letting $\varepsilon \to 0^+$, it follows from Theorem 2 and assumptions (A5)–(A6) that

$$\frac{1}{\varepsilon}\big[g\big(P^\varepsilon(t)\big) - g\big(P^*(t)\big)\big] \to g'\big(P^*(t)\big)Q(t), \tag{20}$$

$$\frac{1}{\varepsilon}\big[f\big(I^\varepsilon(t)\big)I^\varepsilon(t) - f\big(I^*(t)\big)I^*(t)\big] \to f'\big(I^*(t)\big)I^*(t)Z(t) + f\big(I^*(t)\big)Z(t), \tag{21}$$

Passing to the limit as $\varepsilon \to 0^+$ in (19) and using (20)–(21) produces the required result. The proof is complete. □

Next, we consider the following adjoint system of (3):

$$\begin{aligned}
&D_\varphi \xi(x,t) = \mu(x,t)\xi(x,t) + \big[\omega(x,t) + \xi(x,t)\big]\alpha^*(x,t)\\
&\qquad\qquad - \beta(x,t)\big[f'\big(I^*(t)\big)I^*(t) + f\big(I^*(t)\big)\big]\xi(0,t)\\
&\qquad\qquad - \xi(0,t)\phi'\big(P^*(t)\big) - r_1(t)\phi'\big(P^*(t)\big) - r_2(t)h(x,t),\\
&\quad (x,t) \in Q,\\
&\xi(l,t) = 0, \quad \xi(x,T) = 0, \quad (x,t) \in Q,\\
&I^*(t) = \int_0^l \beta(x,t)u^*(x,t)\,\mathrm{d}x, \quad P^*(t) = \int_0^l u^*(x,t)\,\mathrm{d}x, \quad t \in [0,T],
\end{aligned}\tag{22}$$

where $\xi(x,t)$ is the dual variable and u^* is the solution of system (3) corresponding to $\alpha^* \in \mathcal{U}$. Treating (22) in the same manner as that in Theorems 1 and 2, we can get the following result with the proof being omitted.

Theorem 3. *Let assumptions* (A1)–(A6) *hold. For each* $\alpha \in \mathcal{U}$*, the adjoint system* (22) *has a unique bounded solution* $\xi(x,t) \in L^\infty(Q)$*,* $|\xi(x,t)| \leqslant B_3$*. Moreover, there exists a positive constant* B_4 *such that*

$$\|\xi_1 - \xi_2\|_{L^1(Q)} \leqslant B_4 \|\alpha_1 - \alpha_2\|_\infty,$$

where ξ_1 *and* ξ_2 *are solutions of* (22) *corresponding to* α_1 *and* α_2*, respectively.*

4 Optimality conditions

In this section, we will derive the first-order necessary conditions of optimality in the form of an Euler–Lagrange system.

Theorem 4. *Let* $\alpha^*(x,t)$ *be an optimal policy for the control problem* (3)–(4)*. Under the conditions of Theorem* 2*, we have*

$$\alpha^*(x,t) = \mathcal{F}\left\{\frac{[\omega(x,t) + \xi(x,t)]u^*(x,t)}{\rho}\right\} \tag{23}$$

in which the truncated mapping $\mathcal{F} : L^1(Q) \to L^\infty(Q)$ *is given by*

$$(\mathcal{F}\eta)(x,t) = \begin{cases} \underline{\alpha}, & \eta(x,t) < \underline{\alpha}, \\ \eta(x,t), & \underline{\alpha} \leqslant \eta(x,t) \leqslant \overline{\alpha}, \\ \overline{\alpha}, & \eta(x,t) > \overline{\alpha}, \end{cases} \tag{24}$$

where $\xi(x,t)$ *is the solution of the adjoint system* (22)*.*

Proof. For any element of tangent cone $v \in \mathcal{T}_{\mathcal{U}}(\alpha^*)$, we have $\alpha^\varepsilon \doteq \alpha^* + \varepsilon v \in \mathcal{U}$ for sufficiently small $\varepsilon > 0$. Let $u^\varepsilon(x,t)$ be the solution of system (3) corresponding to α^ε. Since α^* is an optimal control, it follows that

$$\begin{aligned} &\int_0^T \int_0^l \omega(x,t)\alpha^\varepsilon(x,t)u^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T r_1(t)\phi\big(P^\varepsilon(t)\big)\,\mathrm{d}t \\ &\quad - \int_0^T \int_0^l r_2(t)h(x,t)u^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^T \int_0^l \rho(\alpha^* + \varepsilon v)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_0^T \int_0^l \omega(x,t)\alpha^*(x,t)u^*(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T r_1(t)\phi\big(P^*(t)\big)\,\mathrm{d}t \\ &\quad - \int_0^T \int_0^l r_2(t)h(x,t)u^*(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^T \int_0^l \rho\big(\alpha^*(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

It follows from Lemma 5 that

$$0 \geqslant \int_0^T \int_0^l \bigl[\omega(x,t)\alpha^*(x,t) - r_1(t)\phi'\bigl(P^*(t)\bigr) - r_2(t)h(x,t)\bigr] z(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_0^l \omega(x,t)v(x,t)u^*(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_0^l \rho\alpha^*(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t, \tag{25}$$

where $z(x,t)$ is the solution of (18). Multiplying the first equation of (18) by ξ and integrating on Q, we obtain

$$\begin{aligned}
&\int_0^T \int_0^l \bigl(D_\varphi \xi(x,t)\bigr) z(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_0^l \mu(x,t)\xi(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_0^l \alpha^*(x,t)\xi(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_0^l u^*(x,t)\xi(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_0^l \phi'\bigl(P^*(t)\bigr)\xi(0,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_0^l \beta(x,t)\bigl[f'\bigl(I^*(t)\bigr)I^*(t) + f\bigl(I^*(t)\bigr)\bigr]\xi(0,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{26}$$

Next, multiplying the first equation of (22) by z and integrating on Q, we have

$$\begin{aligned}
&\int_0^T \int_0^l \bigl(D_\varphi \xi(x,t)\bigr) z(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_0^l \mu(x,t)\xi(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_0^l \alpha^*(x,t)\xi(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_0^l \omega(x,t)\alpha^*(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_0^l \phi'\bigl(P^*(t)\bigr)\xi(0,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_0^l \beta(x,t)\bigl[f'\bigl(I^*(t)\bigr)I^*(t) + f\bigl(I^*(t)\bigr)\bigr]\xi(0,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_0^l \bigl[r_1(t)g'\bigl(P^*(t)\bigr) + r_2(t)h(x,t)\bigr] z(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{27}$$

It follows from (26) and (27) that

$$\int_0^T \int_0^l \left[\omega\alpha^* - r_1(t)\phi'\big(P^*(t)\big) - r_2(t)h(x,t)\right] z \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_0^l v u^* \xi \,\mathrm{d}x\,\mathrm{d}t. \tag{28}$$

Thus, from (25) and (28) we have

$$\int_0^T \int_0^l \left[\big(\omega(x,t) + \xi(x,t)\big)u^*(x,t) - \rho\alpha^*(x,t)\right] v(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0 \tag{29}$$

for each $v \in \mathcal{T}_{\mathcal{U}}(\alpha^*)$. Thus, $(\omega + \xi)u^* - \rho\alpha^* \in N_{\Omega}(\alpha^*)$, which implies the conclusion of this theorem. The proof is complete. □

5 Existence of a unique optimal control

In this section, we study the existence and uniqueness of the optimal control.

Definition 3. The embedding mapping $\tilde{J} : L^1(Q) \to [-\infty, +\infty)$ is given by

$$\tilde{J}(\alpha) = \begin{cases} J(\alpha), & \alpha \in \mathcal{U}, \\ +\infty, & \alpha \notin \mathcal{U}. \end{cases} \tag{30}$$

By a similar way as that in Lemma 7.1 in [21], we can get the following result with the proof being omitted.

Lemma 6. *The functional $\tilde{J}(\alpha)$ is upper semicontinuous.*

Theorem 5. *If $\rho^{-1}(\overline{\omega}B_2 + B_1B_4 + B_3B_2) < 1$, then control problem* (3)–(4) *has a unique solution.*

Proof. (i) We prove the uniqueness by contraction mapping theory. Define the mapping $\mathcal{B} : \mathcal{U} \subset L^\infty(Q) \to \mathcal{U}$ by

$$(\mathcal{B}\alpha)(x,t) = \mathcal{F}\left\{ \frac{[\omega(x,t) + \xi^\alpha(x,t)]u^\alpha(x,t)}{\rho} \right\}. \tag{31}$$

It is clear that $(L^\infty(Q), \|\cdot\|_\infty)$ is a Banach space and $\mathcal{U}$ is a closed subset of $L^\infty(Q)$. Thus, $(\mathcal{U}, d)$ is a complete metric space, where $d(u,v) = \|u - v\|_\infty$.

First, we show that $\mathcal{B}$ maps $\mathcal{U}$ into itself. For any $\alpha \in \mathcal{U}$, it follows from (24) that $\underline{\alpha} \leqslant (\mathcal{B}\alpha)(x,t) \leqslant \overline{\alpha}$ for any $(x,t) \in Q$, which means $(\mathcal{B}\alpha) \in \mathcal{U}$, that is, $\mathcal{B}$ is a mapping from $\mathcal{U}$ to $\mathcal{U}$.

Next, we discuss the compressibility of the mapping $\mathcal{B}$. For any $(x,t) \in Q$, it follows from (31) that

$$\begin{aligned} &\left|(\mathcal{B}\alpha_1)(x,t) - (\mathcal{B}\alpha_2)(x,t)\right| \\ &\quad \leqslant \rho^{-1}\left[\left|\omega(x,t)\right|\left|u^{\alpha_1}(x,t) - u^{\alpha_2}(x,t)\right| + \left|u^{\alpha_2}(x,t)\right|\left|\xi^{\alpha_1}(x,t) - \xi^{\alpha_2}(x,t)\right|\right] \\ &\quad\quad + \rho^{-1}\left[\left|\xi^{\alpha_2}(x,t)\right|\left|u^{\alpha_1}(x,t) - u^{\alpha_2}(x,t)\right|\right]. \end{aligned}$$

Combining Theorem 2 with Theorem 3, we have

$$\left\|\mathcal{B}\alpha_1 - \mathcal{B}\alpha_2\right\|_\infty \leqslant \rho^{-1}(\varpi B_2 + B_1 B_4 + B_3 B_2)\|\alpha_1 - \alpha_2\|_\infty.$$

This, together with the hypothesis of this theorem, implies that the mapping $\mathcal{B}$ is a contraction on the complete metric space $(\mathcal{U}, d)$.

Then by Banach fixed-point theorem, $\mathcal{B}$ owns a unique fixed point $\hat{\alpha} \in \mathcal{U}$. Theorem 4 implies that any optimal controller α^*, if exists, must be a fixed point of $\mathcal{B}$. Thus, the uniqueness of optimal control is proved.

(ii) We prove the existence of the optimal control, that is, the control $\hat{\alpha} \in \mathcal{U}$ is actually optimal. It follows from Lemma 6 and Ekeland's principle. We claim that, for each $\varepsilon > 0$, there exists $\alpha_\varepsilon \in \mathcal{U}$ such that

$$\tilde{J}(\alpha_\varepsilon) \geqslant \sup_{\alpha\in\mathcal{U}} \tilde{J}(\alpha) - \varepsilon, \tag{32}$$

$$\tilde{J}(\alpha_\varepsilon) \geqslant \sup_{\alpha\in\mathcal{U}} \bigl\{\tilde{J}(\alpha) - \sqrt{\varepsilon}\|\alpha_\varepsilon - \alpha\|_{L^1(Q)}\bigr\}. \tag{33}$$

Thus, the perturbed functional $\tilde{J}_\varepsilon(\alpha) = \tilde{J}(\alpha) - \sqrt{\varepsilon}\|\alpha_\varepsilon - \alpha\|_{L^1(Q)}$ attains its supremum at α_ε. Then, in the same manner as that in the previous section, for any $v \in \mathcal{T}_\mathcal{U}(\alpha_\varepsilon)$, we have

$$\int_0^T\int_0^l \bigl[\rho\alpha_\varepsilon(x,t) - \bigl(\omega(x,t) + \xi^\varepsilon(x,t)\bigr)u^\varepsilon(x,t)\bigr]v(x,t)\,\mathrm{d}x\,\mathrm{d}t + \sqrt{\varepsilon}\int_0^T\int_0^l |v(x,t)|\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.$$

Therefore, by Lemma 4, we claim that there exists $\theta \in L^\infty(Q)$, $\|\theta\|_\infty \leqslant 1$ such that $\sqrt{\varepsilon}\theta(x,t) + (\omega(x,t) + \xi^\varepsilon(x,t))u^\varepsilon(x,t) - \rho\alpha_\varepsilon(x,t) \in \mathcal{N}_\Omega(\alpha_\varepsilon)$. Consequently,

$$\alpha_\varepsilon(x,t) = \mathcal{F}\biggl\{\frac{[\omega(x,t) + \xi^\varepsilon(x,t)]u^\varepsilon(x,t)}{\rho} + \frac{\sqrt{\varepsilon}\theta(x,t)}{\rho}\biggr\}. \tag{34}$$

Now, we show that $\tilde{J}(\hat{\alpha}) = \sup\{\tilde{J}(\alpha)\colon \alpha \in \mathcal{U}\}$. From (31), (34) one has $\|\mathcal{B}\alpha_\varepsilon - \alpha_\varepsilon\|_\infty \leqslant \rho^{-1}\sqrt{\varepsilon}\|\theta(x,t)\|_\infty \leqslant \rho^{-1}\sqrt{\varepsilon}$. It is easy to derive that

$$\|\hat{\alpha} - \alpha_\varepsilon\|_\infty \leqslant \rho^{-1}(\varpi B_2 + B_1 B_4 + B_3 B_2)\left\|\hat{\alpha} - \alpha_\varepsilon\right\|_\infty + \rho^{-1}\sqrt{\varepsilon},$$

that is,

$$\|\hat{\alpha} - \alpha_\varepsilon\|_\infty \leqslant \bigl[1 - \rho^{-1}(\varpi B_2 + B_1 B_4 + B_3 B_2)\bigr]\rho^{-1}\sqrt{\varepsilon}.$$

Therefore, $\alpha_\varepsilon \to \hat{\alpha}$ in $L^\infty(Q)$ as $\varepsilon \to 0$. From Lemma 6 and inequality (32) we have $\tilde{J}(\hat{\alpha}) = \sup_{\alpha\in\mathcal{U}} \tilde{J}(\alpha)$, which implies that $\hat{\alpha} \in \mathcal{U}$ is the optimal policy. □

6 Numerical tests

In this section, we would like to examine a concrete example and try to obtain specifically the optimal control once the parameters in the control problem (3)–(4) have been chosen. In the following example, we do not consider the costs of controls and all the parameters are 2-periodic with respect to time t.

Example 1. Consider problem (3)–(4) with

$$\beta(x,t) = 20x^2(1-x)(1+\sin\pi t), \qquad \mu(x,t) = \mathrm{e}^{-4x}(1-x)^{-1.4}(2+\cos\pi t),$$
$$g(x) = 1-x, \qquad u_0(x) = 2(1-x), \qquad \omega(x,t) = \frac{1}{20}(2\pi x + \sin(\pi t) + 1),$$
$$h(x,t) = \frac{1}{40}\big(2\pi x + \sin(\pi t)\big), \qquad f\big(I(t)\big) = 0.8, \qquad \phi\big(P(t)\big) = 10 - 2P(t),$$
$$r_1(t) = 0.8\big(1+\sin(\pi t)\big), \qquad r_2(t) = 1 + \sin(\pi t),$$
$$\underline{\alpha} = 2, \qquad \overline{\alpha} = 6, \qquad l = 1, \qquad T = 10.$$

The following figures can be obtained from the results of the calculation.

From Fig. 2, if all the parameters in (3)–(4) are periodic with respect to the time t, then the solution of (3) tends to a periodic solution. Note that the harvest is equivalent

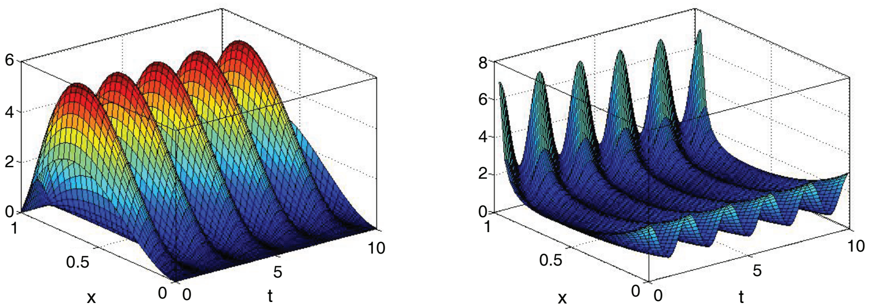

Figure 1. Egg-laying rate of the fish in Example 1 (left); mortality of the fish in Example 1 (right).

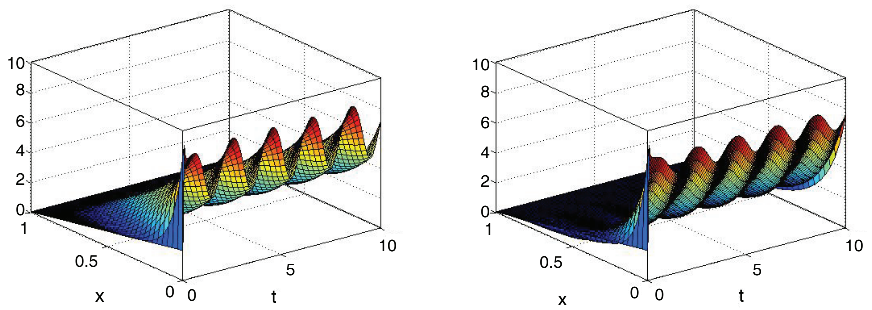

Figure 2. Population density in Example 1 with $\alpha = 0$ (left); with $\alpha = \alpha^*$ (right).

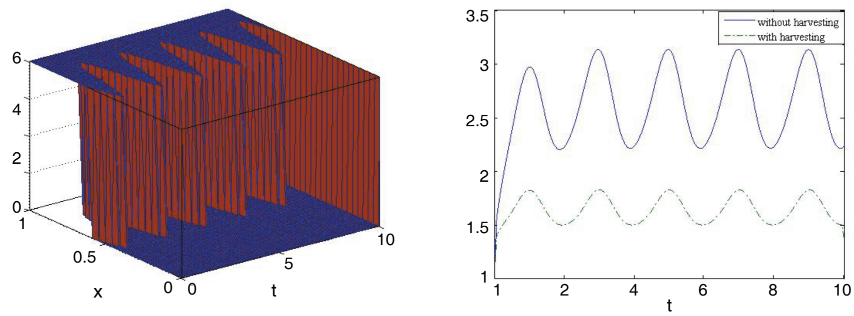

Figure 3. Optimal harvesting efforts $\alpha^*(x,t)$ in Example 1 (left); trend of the total population in the case of harvest and no harvest in Example 1 (right).

to death. It follows from Fig. 2 and the right of Fig. 3 that if $\mu_1(x,t) < \mu_2(x,t)$ for $(x,t) \in Q$, then $u_1(x,t) > u_2(x,t)$ for $(x,t) \in Q$. Here u_1 and u_2 are the solutions of system (3) corresponding to μ_1 and μ_2, respectively. It can be seen from the left of Fig. 3 that the optimal harvesting effort basically owns a bang-bang structure. We observe a significant similarity in the structure of optimal trajectories. It leads to the conclusion that the bang-bang structure of solutions is much more common in optimal population management.

7 Conclusion

In the previous sections, we have established the well-posedness of the state system by Banach fixed-point theorem. More important results are the existence and uniqueness of optimal policies, which supply us a solid theoretical ground for a practical application. As for the structure of the optimal policy, we have presented a feedback strategy in Theorem 4 by using an adjoint variable. The existence of a unique optimal policy is proved via Ekeland's variational principle and Banach fixed-point theorem. Moreover, some numerical results demonstrate the effectiveness of the theoretical results.

In addition, the optimal solution for small t is not substantially influenced by the choice of T if it is large enough. Assume that $0 < T_1 < T_2 < \infty$. We can prove that if (α^*, u^*) is the solution of the optimal control problem (3)–(4) on $[0,l) \times [0,T_2]$, then (α^*, u^*) is a solution of the following optimal control problem:

$$\max_{\alpha\in\mathcal{U}} J(\alpha) = \max_{\alpha\in\mathcal{U}} \Biggl\{ \int_0^{T_1}\int_0^{l} \omega(x,t)\alpha(x,t)u^\alpha(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^{T_1} r_1(t)\phi\bigl(P^\alpha(t)\bigr)\,\mathrm{d}t$$

$$- \int_0^{T_1}\int_0^{l} r_2(t)h(x,t)u^\alpha(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^{T_1}\int_0^{l} \rho\alpha^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \Biggr\}$$

under the restrictions

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} + \frac{\partial (g(x)u(x,t))}{\partial x} = -\mu(x,t)u(x,t) - \alpha(x,t)u(x,t),\\
&\quad (x,t) \in [0,l) \times [0,T_2],\\
&g(0)u(0,t) = \phi\bigl(P(t)\bigr) + f\bigl(I(t)\bigr)I(t), \quad t \in [0,T_2],\\
&u(x,0) = u_0(x), \quad x \in [0,l),\\
&u(x,T_1) = u^*(x,T_1), \quad x \in [0,l).
\end{aligned}$$

The proof is similar to that of Lemma 2 in [6].

In our paper, we only consider the optimal control problem of the fixed horizon $[0,T]$, where $T < \infty$. To our knowledge, most of optimal control problems for population systems are naturally formulated on an infinite time-horizon. However, infinite-horizon optimal control problems are still challenging even for systems of ordinary differential equations. For example, it is difficult to establish suitable transversality conditions, which allow one to choose the right solution of the adjoint system for which the Pontryagin maximum principle holds. In the infinite dimensional case (including age-structured systems and size-structured systems), this issue is open. For more details, see [17]. We leave these for our future work.

References

1. L.I. Aniţa, S. Aniţa, Note on some periodic optimal harvesting problems for age-structured population dynamics, *Appl. Math. Comput.*, **276**:21–30, 2016.
2. S. Aniţa, *Analysis and Control of Age-Dependent Population Dynamics*, Kluwer Academic, Dordrecht, 2000.
3. M.E. Araneda, J.M. Hernández, E. Gasca-Leyva, Optimal harvesting time of farmed aquatic populations with nonlinear size-heterogeneous growth, *Int. J. Biomath.*, **4**:201–218, 2014.
4. V. Barbu, M. Iannelli, Optimal control of population dynamics, *J. Optim. Theory Appl.*, **296**(1):1–14, 1999.
5. H. Caswell, *Matrix Population Models: Construction, Analysis and Interpretation,* 2nd ed., Sinauer Associates, Sunderland, MA, 2001.
6. W.L. Chan, B.Z. Guo, Optimal birth control of population dynamics, *J. Math. Anal. Appl.*, **144**:532–552, 1989.
7. G.L. Eucario, Optimal harvesting time in a size-heterogeneous population, *Ecol. Model.*, **210**(1–2):161–168, 2008.
8. G. Feichtinger, G. Tragler, V.M. Veliov, Optimality conditions for age-structured control systems, *J. Math. Anal. Appl.*, **288**(1):47–68, 2003.
9. W.M. Getz, Harvesting discrete nonlinear age and stage structured populations, *J. Optim. Theory Appl.*, **57**(1):69–83, 1988.
10. M. Ghergu, V.D. Rădulescu, *Nonlinear PDEs: Mathematical Models in Biology, Chemistry and Population Genetics*, Springer Monogr. Math., Springer, Berlin, Heidelberg, 2012.

11. Z.R. He, Y. Liu, An optimal birth control problem for a dynamical population model with size-structure, *Nonlinear Anal., Real World Appl.*, **13**(3):1369–1378, 2012.
12. N. Hritonenko, Yu. Yatsenko, R.-U. Goetz, A. Xabadia, Maximum principle for a size-structured model of forest and carbon sequestration management, *Appl. Math. Lett.*, **21**(10): 1090–1094, 2008.
13. N. Kato, Optimal harvesting for nonlinear size-structured population dynamics, *J. Math. Anal. Appl.*, **324**(2):1388–1398, 2008.
14. R. Liu, G.R. Liu, Optimal birth control problems for a nonlinear vermin population model with size-structure, *J. Math. Anal. Appl.*, **449**(1):265–291, 2017.
15. T. Malysheva, L. W. White, Sufficient conditions for hadamard well-posedness of a coupled thermo-chemo-poroelastic system, *Electron. J. Differ. Equ.*, **2016**:15, 2016.
16. C. Simon, B. Skritek, V.M. Veliov, Optimal immigration age-patterns in populations of fixed size, *J. Math. Anal. Appl.*, **405**(1):71–89, 2013.
17. B. Skritek, V.M. Veliov, On the infinite-horizon optimal control of age-structured systems, *J. Optim. Theory Appl.*, **167**(1):243–271, 2015.
18. V.M. Veliov, Optimal control of heterogeneous systems: Basic theory, *J. Math. Anal. Appl.*, **346**(1):227–242, 2008.
19. G.F. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*, Marcel Dekker, New York, 1985.
20. E.E. Werner, J.F. Gillian, The ontogenetic niche and species interactions in size-structured populations, *Annu. Rev. Ecol. Syst.*, **15**:393–425, 1984.
21. F.Q. Zhang, R. Liu, Y.M. Chen, Optimal harvesting in a periodic food chain model with size structures in predators, *Appl. Math. Optim.*, **75**(2):229–251, 2017.