Nonlinear Analysis: Modelling and Control, 2018, Vol. 23, No. 5, 664–690 ISSN 1392-5113
https://doi.org/10.15388/NA.2018.5.3

Coincidence and common fixed point theorems for four mappings satisfying (α_s, F)-contraction

Muhammad Nazam[a,b], **Muhammad Arshad**[a], **Mihai Postolache**[c,d,e]

[a]Department of Mathematics and Statistics,
International Islamic University,
H-10, Islamabad, Pakistan
nazim254.butt@gmail.com; marshadzia@iiu.edu.pk

[b]Department of Mathematics, Allama Iqbal Open University,
H-8, Islamabad, Pakistan

[c]Center for General Education, China Medical University,
Taichung 40402, Taiwan

[d]Gheorghe Mihoc–Caius Iacob Institute of Mathematical Statistics
and Applied Mathematics of the Romanian Academy,
Bucharest 050711, Romania

[e]Department of Mathematics and Informatics,
University Politehnica of Bucharest,
Bucharest 060042, Romania
emscolar@yahoo.com

Received: October 22, 2017 / **Revised:** May 31, 2018 / **Published online:** September 3, 2018

Abstract. In this paper, we manifest some coincidence and common fixed point theorems for four self-mappings satisfying Círíc-type and Hardy–Rogers-type (α_s, F)-contractions defined on an α_s-complete b-metric space. We apply these results to infer several new and old corresponding results in ordered b-metric spaces and graphic b-metric spaces. Our work generalizes several recent results existing in the literature. We present examples to validate our results. We discuss an application of main result to show the existence of common solution of the system of Volterra-type integral equations.

Keywords: fixed point, four mappings, α_s-complete b-metric space, (α_s, F)-contraction.

1 Introduction

The Banach contraction principle plays a fundamental role in metric fixed point theory, and a large number of researchers revealed many fruitful generalizations of this result in various directions. One of these generalizations is known as F-contraction presented by Wardowski [32]: every F-contraction defined on complete metric space has a unique fixed point. The concept of F-contraction proved to be a milestone in fixed point theory and numerous research papers on F-contraction have been published (see, for instance, [1, 3–5, 11, 13, 20, 23, 24, 27, 29–31]).

In 2012, Samet et al. [26] investigated the idea of (α, ψ)-contractive and α-admissible mappings and established some significant fixed point results for such kind of mappings defined on a complete metric space. Subsequently, Salimi et al. [25] and Hussain et al. [14–16] improved the concept of α-admissibility and proved some important (common) fixed point theorems.

In 1989, Bakhtin [7] investigated the concept of b-metric space. However, Czerwik [10] initiated the study of fixed point of self-mappings in a b-metric space and proved an analogue of Banach's fixed point theorem. Since then, numerous research articles have been published comprising fixed point theorems for various classes of single-valued and multi-valued operators in b-metric spaces (see, for example, [12, 20, 21, 28]).

Recently, Cosentino et al. [9] established a fixed point result for Hardy–Rogers-type F-contraction, and Minak et al. [22] presented a fixed point result for Círíc-type generalized F-contraction.

We bring into use the idea of Círíc-type and Hardy–Rogers-type (α_s, F)-contractions based on four self-mappings defined on a b-metric space. We present some common fixed point results for four self-mappings satisfying such kind of contractions on the α_s-complete b-metric space. We apply our results to infer several new and old results. We present ordered b-metric and graphic b-metric versions of these theorems as consequences. We discuss an application of main result to show the existence of common solution of the system of Volterra-type integral equations.

2 Preliminaries

We denote the set of natural numbers, rational numbers, $(-\infty, +\infty)$, $(0, +\infty)$, and $[0, +\infty)$ by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$, respectively. We bring back into reader's mind some definitions and properties of b-metric.

Definition 1. (See [10].) Let M be a nonempty set, and $s \geqslant 1$ be a real number. A mapping $d^* : M \times M \to \mathbb{R}_0^+$ is said to be a b-metric if, for all $\theta, \rho, \sigma \in M$, we have:

(d_1^*) $\theta = \rho$ if and only if $d^*(\theta, \rho) = 0$,
(d_2^*) $d^*(\theta, \rho) = d^*(\rho, \theta)$,
(d_3^*) $d^*(\theta, \rho) \leqslant s[d^*(\theta, \sigma) + d^*(\sigma, \rho)]$.

In this case, the triplet (M, d^*, s) is called a b-metric space (with coefficient s).

Remark 1. Definition 1 allows us to remark that b-metric space is effectually more general than metric space as a b-metric is a metric when $s = 1$. It is worth to mention that the b-metric structure produces some differences to the classical case of metric spaces: the b-metric on a nonempty set M need not be continuous, open balls in such spaces need not be open sets and so on. The following example describes the significance of a b-metric.

Example 1. Let (M, d) be a metric space and $d^*(\theta, \rho) = (d(\theta, \rho))^r$, $r > 1$, is a real number. Then d^* is b-metric with $s = 2^{r-1}$. Obviously, (d_1^*) and (d_2^*) of Definition 3 are satisfied. If $1 < r < \infty$, then the convexity of the function $f(\theta) = \theta^r$ $(\theta > 0)$ implies

$$\left(\frac{j+l}{2}\right)^r \leqslant \frac{1}{2}\bigl(j^r + l^r\bigr)$$

that gives $(j + l)^r \leqslant 2^{r-1}(j^r + l^r)$. Thus, for all $\theta, \rho, \sigma \in M$, we have

$$\begin{aligned} d^*(\theta, \sigma) &= \bigl(d(\theta, \sigma)\bigr)^r \leqslant \bigl(d(\theta, \rho) + d(\rho, \sigma)\bigr)^r \\ &\leqslant 2^{r-1}\bigl[\bigl(d(\theta, \rho)\bigr)^r + \bigl(d(\rho, \sigma)\bigr)^r\bigr] \\ &= 2^{r-1}\bigl[d^*(\theta, \rho) + d^*(\rho, \sigma)\bigr]. \end{aligned}$$

Therefore, $d^*(\theta, \rho) \leqslant s[d^*(\theta, \rho) + d^*(\rho, \sigma)]$, where $s = 2^{r-1}$, which shows that (M, d^*, s) is a b-metric space. Nevertheless, if (M, d) is a metric space, then (M, d^*, s) may not be a metric space. Indeed, if $M = \mathbb{R}$ and $d(\theta, \rho) = |\theta - \rho|$ (a usual metric), then $d^*(\theta, \rho) = [d(\theta, \rho)]^2$ does not define a metric on M.

For the notions like convergence, completeness, Cauchy sequence in the setting of b-metric spaces, the reader is referred to Aghajani et al. [2], Czerwik [10], Amini-Harandi [6], Huang et al. [12], Khamsi and Hussain [21].

In line with Wardowski [32], Cosentino et al. [8] investigated a nonlinear function $F : \mathbb{R}^+ \to \mathbb{R}$ complying with the following axioms:

- (F_1) F is strictly increasing.
- (F_2) For each sequence $\{r_n\}$ of positive numbers, $\lim_{n\to\infty} r_n = 0$ if and only if $\lim_{n\to\infty} F(r_n) = -\infty$.
- (F_3) For each sequence $\{r_n\}$ of positive numbers, $\lim_{n\to\infty} r_n = 0$, there exists $\theta \in (0, 1)$ such that $\lim_{r_n\to 0^+}(r_n)^\theta F(r_n) = 0$.
- (F_4) $\tau + F(sr_n) \leqslant F(r_{n-1})$ implies $\tau + F(s^n r_n) \leqslant F(s^{n-1} r_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$.

We denote the set of all functions satisfying the conditions (F_1)–(F_4) by $\mathcal{F}_s$.

Example 2. (See [8].) Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by

(a) $F(r) = \ln(r)$;
(b) $F(r) = r + \ln(r)$.

It is easy to check that functions given in (a) and (b) are members of $\mathcal{F}_s$.

Definition 2. Let (M, d^*, s) be a b-metric space, $S : M \to M$ and $\alpha_s : M \times M \to \mathbb{R}_0^+$ be two mappings. The mapping S is said to be an α_s-admissible if

$$\alpha_s(r_1, r_2) \geqslant s^2 \quad \Longrightarrow \quad \alpha_s\bigl(S(r_1), S(r_2)\bigr) \geqslant s^2 \quad \text{for all } r_1, r_2 \in M.$$

Definition 3. Let (M, d^*, s) be a b-metric space, $S : M \to M$ and $\alpha_s : M \times M \to \mathbb{R}_0^+$ be two mappings. The mapping S is said to be a triangular α_s-admissible mapping if:

(i) $\alpha_s(r_1, r_2) \geqslant s^2$ implies $\alpha_s(S(r_1), S(r_2)) \geqslant s^2$, $r_1, r_2 \in M$;
(ii) $\alpha_s(r_1, r_3) \geqslant s^2$, $\alpha_s(r_3, r_2) \geqslant s^2$ imply $\alpha_s(r_1, r_2) \geqslant s^2$ for all $r_1, r_2, r_3 \in M$.

Definition 4. Let (M, d^*, s) be a b-metric space and $f, g : M \to M$ and $\alpha_s : M \times M \to \mathbb{R}_0^+$ be three mappings. The pair (f, g) is said to be:

(i) weakly α_s-admissible pair if $\alpha_s(f(r), gf(r)) \geqslant s^2$ and $\alpha_s(g(r), fg(r)) \geqslant s^2$ for all $r \in M$;
(ii) partially weakly α_s-admissible pair if $\alpha_s(f(r), gf(r)) \geqslant s^2$ for all $r \in M$.

Let $f^{-1}(r) = \{m \in M\colon f(m) = r\}$.

Definition 5. Let (M, d^*, s) be a b-metric space and $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$ and α_s as defined in Definition 2. The pair (f, g) is said to be:

(i) weakly α_s-admissible pair with respect to h if and only if $\alpha_s(f(r_1), g(r_2)) \geqslant s^2$ for all $r_1 \in M$, for all $r_2 \in h^{-1}f(r_1)$ and $\alpha_s(g(r_1), f(r_2)) \geqslant s^2$ for all $r_2 \in h^{-1}g(r_1)$;
(ii) partially weakly α_s-admissible pair of mappings with respect to h if and only if $\alpha_s(f(r_1), g(r_2)) \geqslant s^2$ for all $r_1 \in M$, for all $r_2 \in h^{-1}f(r_1)$.

Definition 4 allows us to remark that:

(a) if $g = f$, then f is weakly α_s-admissible (partially weakly α_s-admissible) with respect to h;
(b) if $h = I_M$ (the identity mapping on M), then Definition 4 reduces to Definition 3.

Definition 6. Let (M, d^*, s) be a b-metric space and $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$ and α_s as defined in Definition 2. The pair (f, g) is said to be triangular weakly α_s-admissible pair of mappings with respect to h if:

(i) $\alpha_s(f(r_1), g(r_2)) \geqslant s^2$ for all $r_1 \in M$, $r_2 \in h^{-1}f(r_1)$ and $\alpha_s(g(r_1), f(r_2)) \geqslant s^2$ for all $r_2 \in h^{-1}g(r_1)$;
(ii) $\alpha_s(r_1, r_3) \geqslant s^2$, $\alpha_s(r_3, r_2) \geqslant s^2$ imply $\alpha_s(r_1, r_2) \geqslant s^2$ for all $r_1, r_2, r_3 \in M$.

Definition 7. Let $f, g, h : M \to M$ be three self-mappings defined on a b-metric space (M, d^*, s) such that $f(M) \cup g(M) \subseteq h(M)$ and α_s as defined in Definition 2. The pair (f, g) is said to be triangular partially weakly α_s-admissible pair of mappings with respect to h if:

(i) $\alpha_s(f(r_1), g(r_2)) \geqslant s^2$ for all $r_1 \in M$, for all $r_2 \in h^{-1}f(r_1)$;
(ii) $\alpha_s(r_1, r_3) \geqslant s^2$, $\alpha_s(r_3, r_2) \geqslant s^2$ imply $\alpha_s(r_1, r_2) \geqslant s^2$ for all $r_1, r_2, r_3 \in M$.

Example 3. Let $M = [0, \infty)$ and $d^*(r_1, r_2) = |r_1 - r_2|^2$ for all $r_1, r_2 \in M$ be a b-metric with $s = 2$,

$$f(r) = \begin{cases} r & \text{if } r \in [0,1); \\ 1 & \text{if } r \in [1,\infty), \end{cases} \qquad g(r) = \begin{cases} r^{1/3} & \text{if } r \in [0,1); \\ 1 & \text{if } r \in [1,\infty), \end{cases}$$

$$S(r) = \begin{cases} r^3 & \text{if } r \in [0,1); \\ 1 & \text{if } r \in [1,\infty), \end{cases} \qquad T(r) = \begin{cases} r^5 & \text{if } r \in [0,1); \\ 1 & \text{if } r \in [1,\infty). \end{cases}$$

Define $\alpha_s : M \times M \to \mathbb{R}_0^+$ by

$$\alpha_s(r_1, r_2) = \begin{cases} 4 + r_2 - r_1 & \text{if } r_1, r_2 \in [0,1); \\ 0 & \text{otherwise}. \end{cases}$$

Then the pair (f, g) is triangular weakly α_s-admissible pair of mappings with respect to T, and (g, f) is triangular weakly α_s-admissible pair with respect to S.

Indeed, if $\alpha_s(r_1, r_2) \geqslant s^2$ and $\alpha_s(r_2, r_3) \geqslant s^2$, then $r_1 - r_2 \leqslant 0$ and $r_2 - r_3 \leqslant 0$, which implies $r_1 - r_3 \leqslant 0$. Hence, $\alpha_s(r_1, r_3) = 4 + r_3 - r_1 \geqslant s^2$. To prove that (f, g) is partially weakly α_s-admissible pair of mappings with respect to T, let $r_1, r_2 \in M$ be such that $r_2 \in T^{-1}f(r_1)$, that is $T(r_2) = f(r_1)$, thus, we have $r_2^5 = r_1$ or $r_2 = r_1^{1/5}$. As $g(r_2) = r_1^{1/15} \geqslant r_1 = f(r_1)$ for all $r_1 \in [0,1)$. Thus, $\alpha_s(fr_1, gr_2) \geqslant s^2$. Hence, (f, g) is partially weakly α_s-admissible pair of mappings with respect to T. Similarly it can be proved that (g, f) is partially weakly α_s-admissible pair of mappings with respect to S.

In line with the concept of α-completeness for a metric space introduced by Hussain et al. [15], which is weaker than the concept of completeness, we introduce

Definition 8. Let (M, d^*, s) be a b-metric space and α_s as defined in Definition 2. The b-metric space M is said to be α_s-complete if and only if every Cauchy sequence $\{r_n\}$ in M such that $\alpha_s(r_n, r_{n+1}) \geqslant s^2$ for all $n \in \mathbb{N}$ converges in M.

If M is a complete metric space, then M is also an α_s-complete metric space, but the converse is not true. Following example explains this fact.

Example 4. Let $M = (0, \infty)$ and the b-metric $d^* : M \times M \to [0, \infty)$ given by $d^*(r_1, r_2) = |r_1 - r_2|^2$ for all $r_1, r_2 \in M$. Define $\alpha_s : M \times M \to [0, \infty)$:

$$\alpha_s(r_1, r_2) = \begin{cases} 4\mathrm{e}^{|r_1 - r_2|} & \text{if } r_1, r_2 \in [1,3]; \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to see that (M, d^*, s) is not a complete b-metric space, but (M, d^*, s) is an α_s-complete b-metric space. Indeed, if $\{r_n\}$ is a Cauchy sequence in M such that $\alpha_s(r_n, r_{n+1}) \geqslant 4$ for all $n \in \mathbb{N}$, then $r_n \in [1,3]$ for all $n \in \mathbb{N}$. Since $[1,3]$ is a closed subset of $\mathbb{R}$, we see that $([1,3], d^*, 2)$ is a complete b-metric space, and then there exists $r \in [1,3]$ such that $r_n \to r$ as $n \to \infty$.

Definition 9. Let (M, d^*, s) be a b-metric space and α_s as defined in Definition 2. We say the self-mapping T is an α_s-b-continuous mapping on (M, d^*, s) if, for given $r \in M$ and sequence $\{r_n\}$, $\lim_{n\to\infty} d^*(r_n, r) = 0$ and $\alpha_s(r_n, r_{n+1}) \geqslant s^2$ for all $n \in \mathbb{N}$ imply $\lim_{n\to\infty} d^*(T(r_n), T(r)) = 0$.

Example 5. Let $M = [0, \infty)$ and $d^* : M \times M \to [0, \infty)$, $d^*(r_1, r_2) = |r_1 - r_2|^2$ for all $r_1, r_2 \in M$. Define

$$T(r) = \begin{cases} \sin(\pi r), & r \in [0,1]; \\ \cos(\pi r) + 2, & r \in (1, \infty), \end{cases} \qquad \alpha_s(r_1, r_2) = \begin{cases} r_1^3 + r_2^3 + 4, & r_1, r_2 \in [0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

Then, clearly, T is not continuous on M, however, T is an α_s-continuous.

Definition 10. Let (M, d^*, s) be a b-metric space and α_s as defined in Definition 2. The pair of self-mappings (f, g) is said to be an α_s-compatible if $\lim_{n\to\infty} d^*(fg(r_n), gf(r_n)) = 0$ whenever $\{r_n\}$ is a sequence in M such that $\alpha_s(r_n, r_{n+1}) \geqslant s^2$ and $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = t$ for some $t \in M$.

Remark that if (f, g) is a compatible pair, then (f, g) is also an α_s-compatible pair, but the converse is not true. The following example illustrates this fact.

Example 6. Let $M = [1, \infty)$ and $d^* : M \times M \to [0, \infty)$, $d^*(r_1, r_2) = |r_1 - r_2|^2$ for all $r_1, r_2 \in M$, then $(M, d^*, 2)$ is a b-metric space. Define

$$f(r) = \begin{cases} 3, & r \in [1,3]; \\ 7, & r > 3, \end{cases} \qquad g(r) = \begin{cases} 12 - 3r, & r \in [1,3]; \\ 8, & r > 3, \end{cases}$$

and

$$\alpha_s(r_1, r_2) = \begin{cases} 5, & r_1, r_2 \in [1,3]; \\ 0, & \text{otherwise.} \end{cases}$$

Let us consider $\{r_n\}$ be a sequence such that $\alpha_s(r_n, r_{n+1}) \geqslant s^2$ and $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n)$, then $r_n = 3$. It is clear that $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = 3$, and thus, we obtain that $\lim_{n\to\infty} gf(r_n) = \lim_{n\to\infty} fg(r_n) = 3$. Hence, (f, g) is an α_s-compatible pair. Now if we consider $t_n = 3 - 1/n$, then $\lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} g(t_n) = 3$, but

$$\lim_{n\to\infty} gf(t_n) = 3 \neq \lim_{n\to\infty} fg(t_n) = \lim_{n\to\infty} f\left(3 + \frac{3}{n}\right) = 7.$$

Consequently, (f, g) is not compatible pair.

Definition 11. (See [18].) Let f and T be self-mappings defined on a nonempty set M. If $f(r) = T(r)$ for some $r \in M$, then r is called a coincidence point of f and T. Two self mappings f and T defined on M are said to be weakly compatible if they commute at their coincidence points, that is if $f(r) = T(r)$ for some $r \in M$, then $fT(r) = Tf(r)$.

Example 7. Let $M = \mathbb{R}$ and $T, f : M \to M$ be given by $T(r) = 6r - 5$ and $f(r) = 5r - 4$ for all $r \in M$. Then f, T are weakly compatible mappings for coincidence point $r = 1$.

Definition 12. Let (M, d^*, s) be a b-metric space and α_s as defined in Definition 2. The space (M, d^*, s) is said to be α_s-regular if, for any sequence $\{r_n\}$ in M, following condition holds: if $r_n \to r$ and $\alpha_s(r_n, r_{n+1}) \geqslant s^2$ for all $n \in \mathbb{N}$, then $\alpha_s(r_n, r) \geqslant s^2$ for all $n \in \mathbb{N}$.

Lemma 1. *Let (M, d^*, s) be a b-metric space. If there exist two sequences $\{r_n\}$, $\{s_n\}$ such that*

$$\lim_{n\to\infty} d^*(r_n, s_n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} r_n = t \quad \textit{for some } t \in M,$$

then $\lim_{n\to\infty} s_n = t$.

Proof. By triangle inequality, we have $d^*(s_n, t) \leqslant s[d^*(s_n, r_n) + d^*(r_n, t)]$, and the result follows from it by applying limit as $n \to \infty$. □

3 Fixed point theorems

Recently, Minak et al. [22] and Cosentino et al. [9] have employed Ćirić-type and Hardy–Rogers-type contractive conditions, respectively, on T in their definition of an F-contraction and found a unique fixed point of T in the context of a metric space. We introduce the notion of (α_s, F)-contraction by imposing some generalized type contractive conditions in terms of four self-mappings defined on a b-metric space and find their coincidence and common fixed points. We apply these results to obtain coincidence and common fixed points of four self-mappings defined on partially ordered b-metric space and graphic b-metric space.

Let (M, d^*, s) be a b-metric space and $f, g, S, T : M \to M$ be self-mappings and α_s as defined in Definition 2. We define the set γ_{f,g,α_s} by

$$\gamma_{f,g,\alpha_s} = \bigl\{(\alpha, \beta) \in M \times M\colon\ \alpha_s\bigl(S(\alpha), T(\beta)\bigr) \geqslant s^2 \text{ and } d^*\bigl(f(\alpha), g(\beta)\bigr) > 0\bigr\}.$$

Let

$$\mathcal{M}_1(\alpha, \beta) = \max\biggl\{d^*\bigl(S(\alpha), T(\beta)\bigr),\ d^*\bigl(f(\alpha), S(\alpha)\bigr),\ d^*\bigl(g(\beta), T(\beta)\bigr), \frac{d^*(S(\alpha), g(\beta)) + d^*(f(\alpha), T(\beta))}{2s}\biggr\}.$$

The following theorem is one of our main results.

Theorem 1. *Let M be a nonempty set and α_s as defined in Definition 2. Let f, g, S, T be α_s–b-continuous self-mappings defined on an α_s-complete b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$\tau + F\bigl(sd^*\bigl(f(r_1), g(r_2)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_1(r_1, r_2)\bigr). \tag{1}$$

Assume that the pairs (f, S), (g, T) are α_s-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α_s-admissible with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point (say) υ in M. Moreover, if $\alpha_s(S\upsilon, T\upsilon) \geqslant s^2$, then υ is a common fixed point of f, g, S, T.

Proof. Let $r_0 \in M$ be an arbitrary point. As $f(M) \subseteq T(M)$, there exists $r_1 \in M$ such that $f(r_0) = T(r_1)$. Since $g(r_1) \in S(M)$, we can choose $r_2 \in M$ such that $g(r_1) = S(r_2)$. In general, r_{2n+1} and r_{2n+2} are chosen in M such that $f(r_{2n}) = T(r_{2n+1})$ and $g(r_{2n+1}) = S(r_{2n+2})$. Define a sequence $\{j_n\}$ in M such that, for all $n \in \mathbb{N}$,

$$j_{2n+1} = f(r_{2n}) = T(r_{2n+1}) \quad \text{and} \quad j_{2n+2} = g(r_{2n+1}) = S(r_{2n+2}).$$

As $r_1 \in T^{-1}(fr_0)$, $r_2 \in S^{-1}(gr_1)$ and (f,g) and (g,f) are triangular partially weakly α_s-admissible pair of mappings with respect to T and S, respectively, we have

$$\alpha_s(Tr_1 = fr_0,\, gr_1 = Sr_2) \geqslant s^2 \quad \text{and} \quad \alpha_s(gr_1 = Sr_2,\, fr_2 = Tr_3) \geqslant s^2.$$

Continuing this way, we obtain $\alpha_s(Tr_{2n+1}, Sr_{2n+2}) = \alpha_s(j_{2n+1}, j_{2n+2}) \geqslant s^2$ and $\alpha_s(Sr_{2n+2}, Tr_{2n+3}) = \alpha_s(j_{2n+2}, j_{2n+3}) \geqslant s^2$ for all $n \in \mathbb{N}$. Thus, $\alpha_s(j_n, j_{n+1}) \geqslant s^2$ for all $n \in \mathbb{N}$.

We prove that $\lim_{l\to\infty} d^*(j_l, j_{l+1}) = 0$. Set $d_l = d^*(j_l, j_{l+1})$. Suppose that $d_{l_0} = 0$ for some l_0. Then $j_{l_0} = j_{l_0+1}$. If $l_0 = 2n$, then $j_{2n} = j_{2n+1}$ gives $j_{2n+1} = j_{2n+2}$. Indeed, by contractive condition (1), we get

$$F\bigl(sd^*(j_{2n+1}, j_{2n+2})\bigr) = F\bigl(sd^*\bigl(f(r_{2n}), g(r_{2n+1})\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_1(r_{2n}, r_{2n+1})\bigr) - \tau,$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}
&\mathcal{M}_1(r_{2n}, r_{2n+1})\\
&\quad= \max\biggl\{d^*\bigl(S(r_{2n}), T(r_{2n+1})\bigr),\, d^*\bigl(f(r_{2n}), S(r_{2n})\bigr),\, d^*\bigl(g(r_{2n+1}), T(r_{2n+1})\bigr),\\
&\qquad\qquad \frac{d^*(S(r_{2n}), g(r_{2n+1})) + d^*(f(r_{2n}), T(r_{2n+1}))}{2s}\biggr\}\\
&\quad= \max\biggl\{d^*(j_{2n}, j_{2n+1}),\, d^*(j_{2n+1}, j_{2n}),\, d^*(j_{2n+2}, j_{2n+1}),\\
&\qquad\qquad \frac{d^*(j_{2n}, j_{2n+2}) + d^*(j_{2n+1}, j_{2n+1})}{2s}\biggr\}\\
&\quad= \max\bigl\{d^*(j_{2n}, j_{2n+1}),\, d^*(j_{2n+1}, j_{2n+2})\bigr\}.
\end{aligned}$$

Since $d^*(j_{2n}, j_{2n+1}) = 0$, therefore, $\mathcal{M}(r_{2n}, r_{2n+1}) = d^*(j_{2n+1}, j_{2n+2})$, then

$$F\bigl(sd^*(j_{2n+1}, j_{2n+2})\bigr) \leqslant F\bigl(d^*(j_{2n+1}, j_{2n+2})\bigr) - \tau,$$

which is a contradiction to (F_1). Thus, $j_{2n+1} = j_{2n+2}$.

Similarly, if $l_0 = 2n+1$ then $j_{2n+1} = j_{2n+2}$ gives $j_{2n+2} = j_{2n+3}$.

Continuing this process, we find that j_l is a constant sequence for $l \geqslant l_0$. Hence, $\lim_{l\to\infty} d^*(j_l, j_{l+1}) = 0$ holds true.

Now suppose that $d_l = d^*(j_l, j_{l+1}) > 0$ for each l.

We claim that $\lim_{n\to\infty} F(d^*(j_n, j_{n+1})) = -\infty$.

Let $l = 2n$. As $\alpha_s(Sr_{2n}, Tr_{2n+1}) \geqslant s^2$, $d^*(f(r_{2n}), g(r_{2n-1})) > 0$, so, $(r_{2n}, r_{2n-1}) \in \gamma_{f,g,\alpha_s}$, by (1), we obtain

$$F\big(sd^*(j_{2n}, j_{2n+1})\big) \leqslant F\big(d^*(j_{2n-1}, j_{2n})\big) - \tau \tag{2}$$

for all $n \in \mathbb{N}$. Similarly, for $l = 2n - 1$,

$$F\big(sd^*(j_{2n-1}, j_{2n})\big) \leqslant F\big(d^*(j_{2n-2}, j_{2n-1})\big) - \tau \tag{3}$$

for all $n \in \mathbb{N}$. Hence, by (2) and (3), we have

$$F\big(sd^*(j_n, j_{n+1})\big) \leqslant F\big(d^*(j_{n-1}, j_n)\big) - \tau \tag{4}$$

for all $n \in \mathbb{N}$. Let $b_n = d^*(j_n, j_{n+1})$ for each $n \in \mathbb{N}$, by (4) and property (F_4), we have

$$\tau + F\big(s^n b_n\big) \leqslant F\big(s^{n-1} b_{n-1}\big), \quad n \in \mathbb{N}.$$

Repeating the process, we obtain

$$F\big(s^n b_n\big) \leqslant F(b_0) - n\tau, \quad n \in \mathbb{N}. \tag{5}$$

On taking limit $n \to \infty$ in (5), we have $\lim_{n\to\infty} F(s^n b_n)) = -\infty$. By property (F_2), we get $\lim_{n\to\infty} s^n b_n = 0$, and (F_3) implies that there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} \big(s^n b_n\big)^\kappa F\big(s^n b_n\big) = 0.$$

By (5), for all $n \in \mathbb{N}$, we obtain

$$\big(s^n b_n\big)^\kappa F\big(s^n b_n\big) - \big(s^n b_n\big)^\kappa F(b_0) \leqslant -\big(s^n b_n\big)^\kappa n\tau \leqslant 0. \tag{6}$$

On taking limit $n \to \infty$ in (6), we have

$$\lim_{n\to\infty} n\big(s^n b_n\big)^\kappa = 0.$$

This implies there exists $n_1 \in \mathbb{N}$ such that $n(s^n b_n)^\kappa \leqslant 1$ for all $n \geqslant n_1$, or

$$s^n b_n \leqslant \frac{1}{n^{1/\kappa}} \quad \text{for all } n \geqslant n_1. \tag{7}$$

To prove $\{j_n\}$ a Cauchy sequence, we use (7), and for $m > n \geqslant n_1$, we consider

$$d^*(j_n, j_m) \leqslant \sum_{i=n}^{m-1} s^i b_i \leqslant \sum_{i=n}^{\infty} s^i b_i \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.$$

The convergence of the series $\sum_{i=n}^{\infty} i^{-1/\kappa}$ entails $\lim_{n,m\to\infty} d^*(j_n, j_m) = 0$. Hence, $\{j_n\}$ is a Cauchy sequence in (M, d^*, s). Since $\{j_n\}$ is a Cauchy sequence in the α_s-complete b-metric space M and $\alpha_s(j_n, j_{n+1}) \geqslant s^2$, there exists $\upsilon \in M$ (say) such that

$$\lim_{n\to\infty} d^*(j_{2n+1}, \upsilon) = \lim_{n\to\infty} d^*(Tr_{2n+1}, \upsilon) = \lim_{n\to\infty} d^*(fr_{2n}, \upsilon) = 0$$

and

$$\lim_{n\to\infty} d^*(j_{2n}, \upsilon) = \lim_{n\to\infty} d^*(Sr_{2n}, \upsilon) = \lim_{n\to\infty} d^*(gr_{2n-1}, \upsilon) = 0.$$

Hence,

$$Sr_{2n} \to \upsilon, \quad fr_{2n} \to \upsilon \quad \text{as } n \to \infty.$$

Now, since (f, S) is α_s-compatible pair and $\alpha_s(j_{2n}, j_{2n+1}) \geqslant s^2$, therefore, we obtain $\lim_{n\to\infty} d^*(fSr_{2n}, Sfr_{2n}) = 0$. Moreover, from $\lim_{n\to\infty} d^*(fr_{2n}, \upsilon) = 0$, $\lim_{n\to\infty} d^*(Sr_{2n}, \upsilon) = 0$, and α_s-b-continuity of mappings f and S we obtain

$$\lim_{n\to\infty} d^*(fSr_{2n}, f\upsilon) = 0 = \lim_{n\to\infty} d^*(Sfr_{2n}, S\upsilon).$$

By the triangle inequality, we have

$$\begin{aligned} d^*(f\upsilon, S\upsilon) &\leqslant s\bigl[d^*(f\upsilon, Sfr_{2n}) + d^*(Sfr_{2n}, S\upsilon)\bigr] \\ &\leqslant sd^*(f\upsilon, fSr_{2n}) + s^2 d^*(fSr_{2n}, Sfr_{2n}) \\ &\quad + s^2 d^*(Sfr_{2n}, S\upsilon). \end{aligned} \tag{8}$$

Applying limit as $n \to \infty$ in (8), we obtain $d^*(f\upsilon, S\upsilon) \leqslant 0$, which yields that $f\upsilon = S\upsilon$. Thus, υ is a coincidence point of f and S. Arguing in a similar manner we can prove that $g\upsilon = T\upsilon$. Let $\alpha_s(T\upsilon, S\upsilon) \geqslant s^2$ and assume that $d^*(f\upsilon, g\upsilon) > 0$. As $\upsilon \in \gamma_{f,g,\alpha_s}$, using contractive condition (1), we have

$$F\bigl(sd^*\bigl(f(\upsilon), g(\upsilon)\bigr) \leqslant F\bigl(\mathcal{M}_1(\upsilon, \upsilon) - \tau, \tag{9}$$

where

$$\begin{aligned} \mathcal{M}_1(\upsilon, \upsilon) &= \max\biggl\{ d^*\bigl(S(\upsilon), T(\upsilon)\bigr),\ d^*\bigl(f(\upsilon), S(\upsilon)\bigr),\ d^*\bigl(g(\upsilon), T(\upsilon)\bigr), \\ &\qquad \frac{d^*(S(\upsilon), g(\upsilon)) + d^*(f(\upsilon), T(\upsilon))}{2s} \biggr\} \\ &= \max\biggl\{ d^*\bigl(f(\upsilon), g(\upsilon)\bigr),\ d^*\bigl(f(\upsilon), S(\upsilon)\bigr),\ d^*\bigl(g(\upsilon), T(\upsilon)\bigr), \\ &\qquad \frac{d^*(f(\upsilon), g(\upsilon)) + d^*(f(\upsilon), g(\upsilon))}{2s} \biggr\} \\ &= d^*\bigl(f(\upsilon), g(\upsilon)\bigr). \end{aligned}$$

Using (9), we deduce that $f\upsilon = g\upsilon$. Hence, $f\upsilon = g\upsilon = T\upsilon = S\upsilon$, that is υ is a coincidence point of f, g, S, T.

We show that υ is a common fixed point of f, g, S, and T. Since S is α_s-continuous and the pair (f, S) is α_s-compatible, therefore,

$$\lim_{n\to\infty} Sf(r_{2n}) = S(\upsilon) = \lim_{n\to\infty} S^2(r_{2n+2}),$$

$$\lim_{n\to\infty} d^*\big(fS(r_{2n}), Sf(r_{2n})\big) = 0,$$

and by Lemma 1,

$$\lim_{n\to\infty} fS(r_{2n}) = S(\upsilon).$$

Now put $r_1 = S(r_{2n})$ and $r_2 = r_{2n+1}$ in (1) and suppose on contrary that $d^*(S(\upsilon), \upsilon) > 0$, we obtain

$$F\big(sd^*\big(fS(r_{2n}), g(r_{2n+1})\big)\big) \leqslant F\big(\mathcal{M}_1\big(S(r_{2n}), r_{2n+1}\big)\big) - \tau, \tag{10}$$

where

$$\begin{aligned}\mathcal{M}_1(S(r_{2n}), r_{2n+1}) = \max\Big\{&d^*\big(S^2(r_{2n}), T(r_{2n+1})\big),\ d^*\big(fS(r_{2n}), S^2(r_{2n})\big),\\ &d^*\big(g(r_{2n+1}), T(r_{2n+1})\big)\\ &\frac{d^*(S^2(r_{2n}), g(r_{2n+1})) + d^*(fS(r_{2n}), T(r_{2n+1}))}{2s}\Big\}.\end{aligned}$$

Applying limit as $n \to \infty$ in (10) and using continuity of F, we have

$$F\big(sd^*\big(S(\upsilon), \upsilon\big) \leqslant F\big(d^*\big(S(\upsilon), \upsilon\big) - \tau < F\big(d^*\big(S(\upsilon), \upsilon\big),$$

a contradiction, therefore, $d^*(S(\upsilon), \upsilon) = 0$ implies $S(\upsilon) = \upsilon$. Hence, $f\upsilon = g\upsilon = T\upsilon = S\upsilon = \upsilon$, that is υ is a common fixed point of f, g, S, T. □

Remark 2. If we suppose that $\alpha_s(\upsilon, \omega) \geqslant s^2$ for each pair of common fixed point of f, g, S, T, then υ is unique. Indeed, if ω is another fixed point of f, g, S, T and assuming on contrary that $d^*(f\upsilon, g\omega) > 0$, then from (1) we have

$$F\big(sd^*(\upsilon, \omega)\big) = F\big(sd^*\big(S(\upsilon), T(\omega)\big)\big) \leqslant F\big(\mathcal{M}_1(\upsilon, \omega)\big) - \tau, \tag{11}$$

where

$$\begin{aligned}\mathcal{M}_1(\upsilon, \omega) = \max\Big\{&d^*\big(S(\upsilon), T(\omega)\big),\ d^*\big(f(\upsilon), S(\upsilon)\big),\ d^*\big(g(\omega), T(\omega)\big)\\ &\frac{d^*(S(\upsilon), g(\omega)) + d^*(f(\upsilon), T(\omega))}{2s}\Big\}.\end{aligned}$$

Thus, by (11), we have

$$F\big(sd^*(\upsilon, \omega)\big) < F\big(d^*(\upsilon, \omega)\big),$$

which is a contradiction. Hence, $\upsilon = \omega$ and υ is a unique common fixed point of self-mappings f, g, S, T.

The following example elucidates Theorem 1.

Example 8. Let $M = [0, \infty)$ and $d^* : M \times M \to \mathbb{R}_0^+$, $d^*(r_1, r_2) = |r_1 - r_2|^2$. Define $\alpha_s : M \times M \to [0, \infty)$ by the formula

$$\alpha_s(r_1, r_2) = \begin{cases} 4\mathrm{e}^{r_1 - r_2}, & r_1, r_2 \in M,\ r_1 \geqslant r_2; \\ 4\mathrm{e}^{r_2 - r_1}, & r_1, r_2 \in M,\ r_2 > r_1, \end{cases}$$

so, (M, d^*, s) is an α_s-complete b-metric space with $s = 2$. Define the mappings $f, g, S, T : M \to M$ for all $r \in M$ by

$$f(r) = \ln\left(1 + \frac{r}{6}\right), \qquad g(r) = \ln\left(1 + \frac{r}{7}\right),$$
$$S(r) = \mathrm{e}^{7r} - 1, \qquad T(r) = \mathrm{e}^{6r} - 1.$$

Clearly, f, g, S, T are α_s-continuous self mappings complying with $f(M) = T(M) = g(M) = S(M)$. We note that the pair (f, S) is an α_s-compatible. Indeed, let $\{r_n\}$ be a sequence in M satisfying $\alpha_s(r_n, r_{n+1}) \geqslant s^2$ and

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} S(r_n) = t \quad \text{for some } t \in M.$$

Then

$$\lim_{n\to\infty} \left|f(r_n) - t\right|^2 = \lim_{n\to\infty} \left|S(r_n) - t\right|^2 = 0,$$

equivalently,

$$\lim_{n\to\infty} \left|\ln\left(1 + \frac{r_n}{6}\right) - t\right|^2 = \lim_{n\to\infty} \left|\mathrm{e}^{7r_n} - 1 - t\right|^2 = 0$$

implies

$$\lim_{n\to\infty} \left|r_n - \left(6\mathrm{e}^t - 6\right)\right|^2 = \lim_{n\to\infty} \left|r_n - \frac{\ln(t+1)}{7}\right|^2 = 0.$$

Uniqueness of limit gives that $6\mathrm{e}^t - 6 = \ln(t+1)/7$, thus, $t = 0$ is only possible solution. Due to α_s-continuity of f, S, for $t = 0 \in M$, we have

$$\lim_{n\to\infty} d^*\left(fS(r_n), Sf(r_n)\right) = \lim_{n\to\infty} \left|fS(r_n) - Sf(r_n)\right|^2$$
$$= \left|f(t) - S(t)\right|^2 = |0 - 0|^2 = 0.$$

Similarly, the pair (g, T) is α_s-compatible. To prove that (f, g) is partially weakly α_s-admissible pair of mappings with respect to T, let $r_1, r_2 \in M$ be such that $r_2 \in T^{-1}(f(r_1))$, that is $T(r_2) = f(r_1)$. Thus, we have $\mathrm{e}^{6r_2} - 1 = \ln(1 + r_1/6)$ or $r_2 = \ln(1 + \ln(1 + r_1/6))/6$ as

$$f(r_1) = \ln\left(1 + \frac{r_1}{6}\right) \geqslant \ln\left(1 + \frac{\ln(1 + \ln(1 + \frac{r_1}{6}))}{42}\right)$$
$$= \ln\left(1 + \frac{r_2}{7}\right) = g(r_2).$$

Thus, $\alpha_s(fr_1, gr_2) = 4\mathrm{e}^{fr_1 - gr_2} \geqslant s^2$. Hence, (f, g) is partially weakly α_s-admissible pair of mappings with respect to T. To prove that (g, f) is partially weakly α_s-admissible pair of mappings with respect to S, let $r_1, r_2 \in M$ be such that $r_2 \in S^{-1}(g(r_1))$, that is $S(r_2) = g(r_1)$, thus, we have $\mathrm{e}^{7r_2} - 1 = \ln(1 + r_1/7)$ or $r_2 = \ln(1 + \ln(1 + r_1/7))/7$. Since

$$\begin{aligned} g(r_1) = \ln\left(1 + \frac{r_1}{7}\right) &\geqslant \ln\left(1 + \frac{\ln(1 + \ln(1 + \frac{r_1}{7}))}{42}\right) \\ &= \ln\left(1 + \frac{r_2}{6}\right) = f(r_2), \end{aligned}$$

thus, $\alpha_s(gr_1, fr_2) = 4\mathrm{e}^{gr_1 - fr_2} \geqslant s^2$. Hence, (g, f) is partially weakly α_s-admissible pair of mappings with respect to S. Now for each $r_1, r_2 \in M$, consider

$$\begin{aligned} d^*\bigl(f(r_1), g(r_2)\bigr) = \bigl|f(r_1) - g(r_2)\bigr|^2 &= \left|\ln\left(1 + \frac{r}{6}\right) - \ln\left(1 + \frac{r}{7}\right)\right|^2 \\ &\leqslant \left(\frac{r}{6} - \frac{r}{7}\right)^2 = \frac{1}{42^2}|7r - 6r|^2 \leqslant \frac{1}{1764}\bigl|\mathrm{e}^{7r} - \mathrm{e}^{6r}\bigr|^2 \\ &= \frac{1}{1764} d^*\bigl(T(r_1), S(r_2)\bigr) \leqslant \frac{1}{1764}\mathcal{M}_1(r_1, r_2). \end{aligned}$$

The above inequality can be written as

$$\ln 1764 + \ln\bigl(d^*\bigl(f(r_1), g(r_2)\bigr)\bigr) \leqslant \ln\bigl(\mathcal{M}_1(r_1, r_2)\bigr).$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ for all $r \in R^+ > 0$. Hence, for all $r_1, r_2 \in M$ such that $d^*(f(r_1), g(r_2)) > 0$, $\tau = \ln(1764)$, we obtain

$$\tau + F\bigl(d^*\bigl(f(r_1), g(r_2)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}(r_1, r_2)\bigr).$$

Thus, the contractive condition (1) is satisfied for all $r_1, r_2 \in M$. Hence, all the hypotheses of Theorem 1 are satisfied, note that f, g, S, T have a unique common fixed point $r = 0$.

The Corollary 1 is a generalization of [19, Thm. 3.1].

Corollary 1. *Let M be a nonempty set and $\alpha_s : M \times M \to [0, \infty)$ be a function. Let (M, d^*, s) be an α_s-complete metric space and f, g, S, T are α_s-continuous self-mappings on (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, the inequality*

$$sd^*\bigl(f(r_1), g(r_2)\bigr) \leqslant k\mathcal{M}_1(r_1, r_2), \tag{12}$$

holds. Assume that the pairs (f, S), (g, T) are α_s-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α_s-admissible pair of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point υ_1 in M. Moreover, if $\alpha_s(S\upsilon_1, T\upsilon_1) \geqslant s^2$, then υ_1 is a common point of f, g, S, T.

Proof. For all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, we have $sd^*(f(r_1), g(r_2)) \leqslant k\mathcal{M}_1(r_1, r_2)$. It follows that $\tau + \ln(d^*(f(r_1), g(r_2))) \leqslant \ln(\mathcal{M}_1(r_1, r_2))$, where $\tau = \ln(s/k) > 0$. Then the contraction condition (12) reduces to (1) with $F(r) = \ln(r)$, and the application of Theorem 1 ensures the existence of fixed point. □

In the following theorem, we omit the assumption of α_s-continuity of f, g, T, S and replace the α_s-compatibility of the pairs (f, S) and (g, T) by weak compatibility of the pairs.

Theorem 2. *Let f, g, S, T are self-mappings defined on an α_s-regular and α_s-complete metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$, and $T(M)$ and $S(M)$ are closed subsets of M. Suppose that for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$\tau + F\bigl(sd^*\bigl(f(r_1), g(r_2)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_1(r_1, r_2)\bigr). \tag{13}$$

Assume that the pairs (f, S), (g, T) are weakly compatible and the pairs (f, g) and (g, f) are triangular partially weakly α_s-admissible with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point υ in M. Moreover, if $\alpha_s(S\upsilon, T\upsilon) \geqslant s^2$, then υ is a coincidence point of f, g, S, T.

Proof. In line with the proof of Theorem 1, we know that there exists $\upsilon \in M$ such that

$$\lim_{l\to\infty} d^*(j_l, \upsilon) = 0.$$

Since $T(M)$ is closed subset of M and $\{j_{2n+1}\} \subseteq T(M)$, therefore, $\upsilon \in T(M)$. Thus, there exists $\omega_1 \in M$ such that $\upsilon = T(\omega_1)$ and

$$\lim_{n\to\infty} d^*\bigl(j_{2n+1}, T(\omega_1)\bigr) = \lim_{n\to\infty} d^*\bigl(Tr_{2n+1}, T(\omega_1)\bigr) = 0.$$

Similarly, there exists $\omega_2 \in M$ such that $\upsilon = T(\omega_1) = S(\omega_2)$ and

$$\lim_{n\to\infty} d^*\bigl(j_{2n}, S(\omega_2)\bigr) = \lim_{n\to\infty} d^*\bigl(Sr_{2n}, S(\omega_2)\bigr) = 0.$$

Now since, $\lim_{n\to\infty} d^*(Tr_{2n+1}, S(\omega_2)) = 0$, therefore, α_s-regularity of M implies that $\alpha_s(Tr_{2n+1}, S(\omega_2)) \geqslant s^2$, and from contractive condition (13) we have

$$F\bigl(sd^*\bigl(f(\omega_2), g(r_{2n+1})\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_1(\omega_2, r_{2n+1})\bigr) - \tau \tag{14}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}
&\mathcal{M}_1(\omega_2, r_{2n+1}) \\
&\quad = \max\biggl\{d^*\bigl(S(\omega_2), T(r_{2n+1})\bigr),\ d^*\bigl(f(\omega_2), S(\omega_2)\bigr),\ d^*\bigl(g(r_{2n+1}), T(r_{2n+1})\bigr), \\
&\qquad \frac{d^*(S(\omega_2), g(r_{2n+1})) + d^*(f(\omega_2), T(r_{2n+1}))}{2s}\biggr\} \\
&\quad = \max\biggl\{d^*(\upsilon, j_{2n+1}),\ d^*\bigl(f(\omega_2), \upsilon\bigr),\ d^*(j_{2n+2}, j_{2n+1}), \\
&\qquad \frac{d^*(\upsilon, j_{2n+2}) + d^*(f(\omega_2), j_{2n+1})}{2s}\biggr\}.
\end{aligned}$$

When $n \to \infty$ in (14), we obtain $f(\omega_2) = \upsilon = S(\omega_2)$, and weakly compatibility of f and S gives $f(\upsilon) = fS(\omega_2) = Sf(\omega_2) = S(\upsilon)$, which shows that υ is coincidence point of f and S. Similarly, it can be shown that υ is a coincidence point of the pair (g, T). The rest of the proof follows from similar arguments as in proof of Theorem 1. □

If we set $S = T$ in Theorem 1, we obtain the following result.

Corollary 2. *Let f, g, T be self-mappings defined on an α_s-complete metric space (M, d^*, s) such that $f(M) \cup g(M) \subseteq T(M)$ and $T(M)$ is α_s-continuous. Suppose that for all $r_1, r_2 \in M$ with $\alpha_s(Tr_1, Tr_2) \geqslant s^2$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\big(f(r_1), g(r_2)\big) > 0 \quad \Longrightarrow \quad \tau + F\big(sd^*\big(f(r_1), g(r_2)\big)\big) \leqslant F\big(\mathcal{M}_1(r_1, r_2)\big),$$

where

$$\mathcal{M}_1(r_1, r_2) = \max\bigg\{d^*\big(T(r_1), T(r_2)\big),\ d^*\big(f(r_1), T(r_1)\big),\ d^*\big(g(r_2), T(r_2)\big), \frac{d^*(T(r_1), g(r_2)) + d^*(f(r_1), T(r_2))}{2s}\bigg\}.$$

Assume that either the pair (f, T) is α_s-compatible and f is α_s-continuous or (g, T) is α_s-compatible and g is α_s-continuous. Then the pairs (f, T) and (g, T) have the coincidence point υ in M provided the pair (f, g) is triangular weakly α_s-admissible pair of mappings with respect to T. Moreover, if $\alpha_s(T\upsilon, T\upsilon) \geqslant s^2$, then υ is a common point of f, g, T.

If we set $S = T$ and $f = g$ in Theorem 1, we obtain the following result.

Corollary 3. *Let f, T be α_s-continuous self-mappings defined on an α_s-complete metric space (M, d^*, s) such that $f(M) \subseteq T(M)$. Suppose that for all $r_1, r_2 \in M$ with $\alpha_s(Tr_1, Tr_2) \geqslant s^2$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\big(f(r_1), f(r_2)\big) > 0 \quad \Longrightarrow \quad \tau + F\big(sd^*\big(f(r_1), f(r_2)\big)\big) \leqslant F\big(\mathcal{M}_1(r_1, r_2)\big),$$

where

$$\mathcal{M}_1(r_1, r_2) = \max\bigg\{d^*\big(T(r_1), T(r_2)\big),\ d^*\big(f(r_1), T(r_1)\big),\ d^*\big(f(r_2), T(r_2)\big), \frac{d^*(T(r_1), f(r_2)) + d^*(f(r_1), T(r_2))}{2s}\bigg\}.$$

Assume that the pair (f, T) is α_s-compatible. Then the mappings f, T have the coincidence point in M provided that the f is triangular weakly α_s-admissible mapping with respect to T. Moreover, if $\alpha_s(T\upsilon, T\upsilon) \geqslant s^2$, then f, T has a common point υ.

Corollary 4. *Let f, g, T are self-mappings defined on an α_s-regular and α_s-complete metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq T(M)$, and $T(M)$ is closed subset of M. Suppose that for all $r_1, r_2 \in M$ with $\alpha_s(Tr_1, Tr_2) \geqslant s^2$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\big(f(r_1), g(r_2)\big) > 0 \quad \Longrightarrow \quad \tau + F\big(sd^*\big(f(r_1), g(r_2)\big)\big) \leqslant F\big(\mathcal{M}_1(r_1, r_2)\big),$$

where

$$\mathcal{M}_1(r_1,r_2) = \max\biggl\{ d^*\bigl(T(r_1),T(r_2)\bigr),\, d^*\bigl(f(r_1),T(r_1)\bigr),\, d^*\bigl(g(r_2),T(r_2)\bigr), \frac{d^*(T(r_1),g(r_2)) + d^*(f(r_1),T(r_2))}{2s} \biggr\}.$$

Assume that the pairs (f,T), (g,T) are weakly compatible and the pair (f,g) is triangular weakly α_s-admissible pair of mapping with respect to T. Then the pairs (f,T), (g,T) have the coincidence point υ in M. Moreover, if $\alpha_s(T\upsilon,T\upsilon) \geqslant s^2$, then υ is a coincidence point of f, g, T.

Corollary 5. *Let f, T are self-mappings defined on an α_s-regular and α_s-complete metric space (M,d^*,s) such that $f(M) \subseteq T(M)$ and $T(M)$ is closed subset of M. Suppose that for all $r_1,r_2 \in M$ with $\alpha_s(Tr_1,Tr_2) \geqslant s^2$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\bigl(f(r_1),f(r_2)\bigr) > 0 \quad \Longrightarrow \quad \tau + F\bigl(sd^*\bigl(f(r_1),f(r_2)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_1(r_1,r_2)\bigr),$$

where

$$\mathcal{M}_1(r_1,r_2) = \max\biggl\{ d^*\bigl(T(r_1),T(r_2)\bigr),\, d^*\bigl(f(r_1),T(r_1)\bigr),\, d^*\bigl(f(r_2),T(r_2)\bigr), \frac{d^*(T(r_1),f(r_2)) + d^*(f(r_1),T(r_2))}{2s} \biggr\}.$$

Assume that the pair (f,T) is weakly compatible and f is triangular weakly α_s-admissible mapping with respect to T. Then the pair (f,T) has the coincidence point υ in M.

If we set $S = T = I_M$ (identity mapping) in Theorems 1 and 2, we obtain the following result.

Corollary 6. *Let f, g are self-mappings defined on an α_s-complete metric space (M,d^*,s). Suppose that for all $r_1,r_2 \in M$ with $\alpha_s(r_1,r_2) \geqslant s^2$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\bigl(f(r_1),f(r_2)\bigr) > 0 \quad \Longrightarrow \quad \tau + F\bigl(sd^*\bigl(f(r_1),g(r_2)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_1(r_1,r_2)\bigr),$$

where

$$\mathcal{M}_1(r_1,r_2) = \max\biggl\{ d^*(r_1,r_2),\, d^*\bigl(f(r_1),r_1\bigr),\, d^*\bigl(g(r_2),r_2\bigr), \frac{d^*(r_1,g(r_2)) + d^*(f(r_1),r_2)}{2s} \biggr\}.$$

Assume that the pair (f,g) is triangular weakly α_s-admissible pair of mappings. Then f, g has a common fixed point υ in M provided that either f or g is α_s-continuous, or M is α_s-regular.

Following theorem shows that the arguments given in the proof of Theorem 1 hold equally good if we replace $\mathcal{M}_1(r_1, r_2)$ with one of $\mathcal{M}_i(r_1, r_2)$, $i = 2, 3, 4, 5, 6$.

Theorem 3. *Let f, g, S, T are α_s-continuous self-mappings defined on an α_s-complete b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$\tau + F\bigl(sd^*\bigl(f(r_1), g(r_2)\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_i(r_1, r_2)\bigr) \tag{15}$$

holds for one of $i = 2, 3, 4, 5, 6$, where

$$\begin{aligned}\mathcal{M}_2(r_1, r_2) = {} & a_1 d^*\bigl(S(r_1), T(r_2)\bigr) + a_2 d^*\bigl(f(r_1), S(r_1)\bigr) + a_3 d^*\bigl(g(r_2), T(r_2)\bigr) \\ & + a_4\bigl[d^*\bigl(S(r_1), g(r_2)\bigr) + d^*\bigl(f(r_1), T(r_2)\bigr)\bigr]\end{aligned}$$

with $a_i \geqslant 0$, $i = 1, 2, 3, 4$, such that $a_1 + a_2 + a_3 + 2sa_4 = 1$;

$$\mathcal{M}_3(r_1, r_2) = a_1 d^*\bigl(S(r_1), T(r_2)\bigr) + a_2 d^*\bigl(f(r_1), S(r_1)\bigr) + a_3 d^*\bigl(g(r_2), T(r_2)\bigr),$$

with $a_1 + a_2 + a_3 = 1$;

$$\mathcal{M}_4(r_1, r_2) = k \max\bigl\{d^*\bigl(f(r_1), S(r_1)\bigr), d^*\bigl(g(r_2), T(r_2)\bigr)\bigr\} \quad \text{with } k \in [0, 1);$$

$$\begin{aligned}\mathcal{M}_5(r_1, r_2) = {} & a_1(r_1, r_2) d^*\bigl(S(r_1), T(r_2)\bigr) + a_2(r_1, r_2) d^*\bigl(f(r_1), S(r_1)\bigr) \\ & + a_3(r_1, r_2) d^*\bigl(g(r_2), T(r_2)\bigr) \\ & + a_4(r_1, r_2)\bigl[d^*\bigl(S(r_1), g(r_2)\bigr) + d^*\bigl(f(r_1), T(r_2)\bigr)\bigr]\end{aligned}$$

with $a_i(r_1, r_2)$, $i = 1, 2, 3, 4$, are non-negative functions such that

$$\sup_{r_1, r_2 \in M}\bigl\{a_1(r_1, r_2) + a_2(r_1, r_2) + a_3(r_1, r_2) + 2sa_4(r_1, r_2)\bigr\} = 1;$$

$$\begin{aligned}\mathcal{M}_6(r_1, r_2) = {} & a_1 d^*\bigl(S(r_1), T(r_2)\bigr) + \frac{a_2 + a_3}{2}\bigl[d^*\bigl(f(r_1), S(r_1)\bigr) + d^*\bigl(g(r_2), T(r_2)\bigr)\bigr] \\ & + \frac{a_4 + a_5}{2s}\bigl[d^*\bigl(S(r_1), g(r_2)\bigr) + d^*\bigl(f(r_1), T(r_2)\bigr)\bigr]\end{aligned}$$

with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$.

Assume that the pairs (f, S), (g, T) are α_s-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α_s-admissible pair of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point υ in M. Moreover, if $\alpha_s(S\upsilon, T\upsilon) \geqslant s^2$, then υ is a common point of f, g, S, T.

Proof. In line with the beginning part of proof of Theorem 1, for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, for some $F \in \mathcal{F}_s$ and $\tau > 0$, from contractive condition (15) we get

$$F\bigl(sd^*(j_{2n}, j_{2n+1})\bigr) = F\bigl(sd^*\bigl(f(r_{2n}), g(r_{2n+1})\bigr)\bigr) \leqslant F\bigl(\mathcal{M}_2(r_{2n}, r_{2n+1})\bigr) - \tau, \tag{16}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}
&\mathcal{M}_2(r_{2n}, r_{2n+1}) \\
&\quad = a_1 d^*\big(S(r_{2n}), T(r_{2n+1})\big) + a_2 d^*\big(f(r_{2n}), S(r_{2n})\big) + a_3 d^*\big(g(r_{2n+1}), T(r_{2n+1})\big) \\
&\qquad + a_4\big[d^*\big(S(r_{2n}), g(r_{2n+1})\big) + d^*\big(f(r_{2n}), T(r_{2n+1})\big)\big] \\
&\quad = a_1 d^*(j_{2n-1}, j_{2n}) + a_2 d^*(j_{2n}, j_{2n-1}) + a_3 d^*(j_{2n+1}, j_{2n}) \\
&\qquad + a_4\big[d^*(j_{2n-1}, j_{2n+1}) + d^*\big(j_{2n}, j_{2n}\big)\big] \\
&\quad = (a_1 + a_2 + s a_4) d^*(j_{2n-1}, j_{2n}) + (a_3 + s a_4) d^*(j_{2n}, j_{2n+1}).
\end{aligned}$$

Now from (16) we have

$$\begin{aligned}
F\big(sd^*(j_{2n}, j_{2n+1})\big) \leqslant{}& F\big((a_1 + a_2 + sa_4) d^*(j_{2n-1}, j_{2n}) \\
&+ (a_3 + sa_4) d^*(j_{2n}, j_{2n+1})\big) - \tau.
\end{aligned} \tag{17}$$

Since F is strictly increasing, (17) implies

$$\begin{aligned}
sd^*(j_{2n}, j_{2n+1}) &\leqslant (a_1 + a_2 + sa_4) d^*(j_{2n-1}, j_{2n}) \\
&\quad + (a_3 + sa_4) d^*(j_{2n}, j_{2n+1}), \\
(1 - a_3 - sa_4) d^*(j_{2n}, j_{2n+1}) &< (s - a_3 - sa_4) d^*(j_{2n}, j_{2n+1}) \\
&\leqslant (a_1 + a_2 + sa_4) d^*(j_{2n-1}, j_{2n}), \\
d^*(j_{2n}, j_{2n+1}) &\leqslant \frac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4} d^*(j_{2n-1}, j_{2n}).
\end{aligned}$$

Since $a_1 + a_2 + a_3 + 2sa_4 = 1$, therefore,

$$d^*(j_{2n}, j_{2n+1}) \leqslant \frac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4} d^*(j_{2n-1}, j_{2n}) = d^*(j_{2n-1}, j_{2n}).$$

Thus, from (17) we obtain

$$F\big(sd^*(j_{2n}, j_{2n+1})\big) \leqslant F\big(d^*(j_{2n-1}, j_{2n})\big) - \tau \tag{18}$$

for all $n \in \mathbb{N}$. Similarly,

$$F\big(sd^*(j_{2n-1}, j_{2n})\big) \leqslant F\big(d^*(j_{2n-2}, j_{2n-1})\big) - \tau \tag{19}$$

for all $n \in \mathbb{N}$. Hence, from (18) and (19) we have

$$F\big(sd^*(j_n, j_{n+1})\big) \leqslant F\big(d^*(j_{n-1}, j_n)\big) - \tau. \tag{20}$$

Inequality (20) leads us to remark that $\{j_n\}$ is a Cauchy sequence and remaining part of the proof can easily be followed from finishing part of the proof of Theorem 1.

For $\mathcal{M}_3(r_1, r_2)$: In line with beginning part of the proof of Theorem 1, for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, for some $F \in \mathcal{F}_s$ and $\tau > 0$, from contractive condition (15) we get

$$F\big(sd^*(j_{2n}, j_{2n+1})\big) = F\big(sd^*\big(f\big(r_{2n}\big), g(r_{2n+1})\big)\big) \leqslant F\big(\mathcal{M}_3(r_{2n}, r_{2n+1})\big) - \tau, \tag{21}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}
&\mathcal{M}_3(r_{2n}, r_{2n+1}) \\
&\quad = a_1 d^*\big(S(r_{2n}), T(r_{2n+1})\big) + a_2 d^*\big(f(r_{2n}), S(r_{2n})\big) + a_3 d^*\big(g(r_{2n+1}), T(r_{2n+1})\big) \\
&\quad = a_1 d^*(j_{2n-1}, j_{2n}) + a_2 d^*(j_{2n}, j_{2n-1}) + a_3 d^*(j_{2n+1}, j_{2n}) \\
&\quad = (a_1 + a_2) d^*(j_{2n-1}, j_{2n}) + a_3 d^*(j_{2n}, j_{2n+1}).
\end{aligned}$$

Now from (21) we have

$$F\big(sd^*(j_{2n}, j_{2n+1})\big) \leqslant F\big((a_1 + a_2) d^*(j_{2n-1}, j_{2n}) + a_3 d^*(j_{2n}, j_{2n+1})\big) - \tau. \tag{22}$$

Since F is strictly increasing, (22) implies

$$\begin{aligned}
sd^*(j_{2n}, j_{2n+1}) &\leqslant (a_1 + a_2) d^*(j_{2n-1}, j_{2n}) + a_3 d^*(j_{2n}, j_{2n+1}) \\
(1 - a_3) d^*(j_{2n}, j_{2n+1}) &< (s - a_3) d^*(j_{2n}, j_{2n+1}) \leqslant (a_1 + a_2) d^*(j_{2n-1}, j_{2n}) \\
d^*(j_{2n}, j_{2n+1}) &\leqslant \frac{a_1 + a_2}{1 - a_3} d^*(j_{2n-1}, j_{2n}).
\end{aligned}$$

Since $a_1 + a_2 + a_3 = 1$, therefore,

$$d^*(j_{2n}, j_{2n+1}) \leqslant \frac{a_1 + a_2}{1 - a_3} d^*(j_{2n-1}, j_{2n}) = d^*(j_{2n-1}, j_{2n}).$$

Thus, from (22) we obtain

$$F\big(sd^*(j_{2n}, j_{2n+1})\big) \leqslant F\big(d^*(j_{2n-1}, j_{2n})\big) - \tau \tag{23}$$

for all $n \in \mathbb{N}$. Similarly,

$$F\big(sd^*(j_{2n-1}, j_{2n})\big) \leqslant F\big(d^*(j_{2n-2}, j_{2n-1})\big) - \tau \tag{24}$$

for all $n \in \mathbb{N}$. Hence, from (23) and (24) we have

$$F\big(sd^*(j_n, j_{n+1})\big) \leqslant F\big(d^*(j_{n-1}, j_n)\big) - \tau. \tag{25}$$

Inequality (25) leads us to remark that $\{j_n\}$ is a Cauchy sequence and remaining part of the proof can easily be followed from finishing part of the proof of Theorem 1.

For $\mathcal{M}_4(r_1, r_2)$: In line with beginning part of the proof of Theorem 1, for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, for some $F \in \mathcal{F}_s$ and $\tau > 0$, from contractive condition (15) we get

$$F\big(sd^*(j_{2n}, j_{2n+1})\big) = F\big(sd^*\big(f(r_{2n}), g(r_{2n+1})\big)\big) \leqslant F\big(\mathcal{M}_4(r_{2n}, r_{2n+1})\big) - \tau$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}
\mathcal{M}_4(r_{2n}, r_{2n+1}) &= k \max\big\{d^*\big(f(r_{2n}), S(r_{2n})\big), d^*\big(g(r_{2n+1}), T(r_{2n+1})\big)\big\} \\
&= k \max\big\{d^*(j_{2n}, j_{2n-1}), d^*(j_{2n+1}, j_{2n})\big\}.
\end{aligned}$$

Remaining part of the proof can easily be followed from the proof of Theorem 1.

Similar arguments hold for $\mathcal{M}_5(r_1, r_2)$ and $\mathcal{M}_6(r_1, r_2)$. □

4 Results in ordered b-metric spaces

In this section, we present some common fixed point theorems on α_s complete b-metric spaces endowed with an arbitrary binary relation, specially a partial order relation, which can be regarded as consequences of the results presented in the previous section.

Let (M, d^*, s) be a b-metric space, and let $\prec$ be a binary relation over M.

Definition 13. Let f and g be two selfmappings on M and $\prec$ be a binary relation over M. A pair (f, g) is said to be:

(i) weakly $\prec$-increasing if $f(r) \prec gf(r)$ and $g(r) \prec fg(r)$ for all $r \in M$;
(ii) partially weakly $\prec$-increasing if $f(r) \prec gf(r)$ for all $r \in M$.

Definition 14. Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$. The pair (f, g) is said to be transitive weakly $\prec$-increasing pair of mappings with respect to h if:

(i) $f(r_1) \prec g(r_2)$ for all $r_1 \in M$, for all $r_2 \in h^{-1}f(r_1)$ and $g(r_1) \prec f(r_2))$ for all $r_2 \in h^{-1}g(r_1)$;
(ii) $r_1 \prec r_3, r_3 \prec r_2$ imply $r_1 \prec r_2$ for all $r_1, r_2, r_3 \in M$.

Definition 15. Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$. The pair (f, g) is said to be transitive partially weakly $\prec$-increasing pair of mappings with respect to h if:

(i) $f(r_1) \prec g(r_2)$ for all $r_1 \in M$, for all $r_2 \in h^{-1}f(r_1)$;
(ii) $r_1 \prec r_3, r_3 \prec r_2$ imply $r_1 \prec r_2$ for all $r_1, r_2, r_3 \in M$.

Let $\prec$ be a binary relation over M and

$$\alpha_s(r_1, r_2) = \begin{cases} s^2 & \text{if } r_1 \prec r_2; \\ 0 & \text{otherwise.} \end{cases}$$

By this assumption, we see that the above definitions are special cases from the definition of weak α_s-admissibility and partially weak α_s-admissibility.

Definition 16. Let (M, d^*, s) be a b-metric space. It is said to be $\prec$-complete if and only if every Cauchy sequence $\{r_n\}$ in M such that $r_n \prec r_{n+1})$ for all $n \in \mathbb{N}$ converges in M.

Definition 17. Let (M, d^*, s) be a b-metric space and $T : M \to M$ be a mapping. We say that T is an $\prec$-continuous mapping on (M, d^*, s) if, for given $r \in M$ and sequence $\{r_n\}$,

$$\lim_{n\to\infty} d^*(r_n, r) = 0, \quad r_n \prec r_{n+1},\ n \in \mathbb{N} \quad \Longrightarrow \quad \lim_{n\to\infty} d^*\bigl(T(r_n), T(r)\bigr) = 0.$$

Definition 18. Let (M, d^*, s) be a b-metric space. The pair (f, g) is said to be an $\prec$-compatible if and only if $\lim_{n\to\infty} d^*(fg(r_n), gf(r_n)) = 0$, whenever $\{r_n\}$ is a sequence in M such that $r_n \prec r_{n+1}$ and

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = t \quad \text{for some } t \in M.$$

Definition 19. The b-metric space (M, d^*, s) is said to be $\prec$-regular if, for any sequence $\{r_n\}$ in M, the following condition holds:

$$r_n \to r, \quad r_n \prec r_{n+1} \quad \Longrightarrow \quad r_n \prec r \quad \text{for all } n \in \mathbb{N}.$$

Now we are able to remodel Theorems 1 and 2 in the framework of ordered metric spaces.

Theorem 4. *Let f, g, S, T be $\prec$-continuous self-mappings define on a $\prec$-complete b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that for all $r_1, r_2 \in M$ with $S(r_1) \prec T(r_2)$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\big(f(r_1), g(r_2)\big) > 0 \quad \Longrightarrow \quad \tau + F\big(sd^*\big(f(r_1), g(r_2)\big)\big) \leqslant F\big(\mathcal{M}_1(r_1, r_2)\big).$$

Assume that the pairs (f, S), (g, T) are $\prec$-compatible and the pairs (f, g) and (g, f) are transitive partially weakly $\prec$-increasing pair of self-mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point υ in M. Moreover, if $S\upsilon \prec T\upsilon$, then υ is a common point of f, g, S, T.

Proof. Define

$$\alpha_s(r_1, r_2) = \begin{cases} s^2 & \text{if } r_1 \prec r_2; \\ 0 & \text{otherwise,} \end{cases}$$

and the proof follows from the proof of Theorem 1. □

Theorem 5. *Let f, g, S, T be $\prec$-continuous self-mappings defined on a $\prec$-regular and $\prec$-complete b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$, and $T(M)$ and $S(M)$ are closed subsets of M. Suppose that for all $r_1, r_2 \in M$ with $S(r_1) \prec T(r_2)$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\big(f(r_1), g(r_2)\big) > 0 \quad \Longrightarrow \quad \tau + F\big(sd^*\big(f(r_1), g(r_2)\big)\big) \leqslant F\big(\mathcal{M}_1(r_1, r_2)\big).$$

Assume that the pairs (f, S), (g, T) are weakly compatible and the pairs (f, g) and (g, f) are transitive partially weakly $\prec$-increasing pair of self-mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point υ in M. Moreover, if $S\upsilon \prec T\upsilon$, then υ is a coincidence point of f, g, S, T.

Proof. Define

$$\alpha_s(r_1, r_2) = \begin{cases} s^2 & \text{if } r_1 \prec r_2; \\ 0 & \text{otherwise,} \end{cases}$$

and the proof follows from proofs of Theorems 1 and 2. □

5 Results in b-metric spaces endowed with graph

Consistent with Jachymski [17], let (M, d^*, s) be a b-metric space and δ denotes the diagonal of the Cartesian product $M \times M$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with M and the set $E(G)$ of its edges contains all loops. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}0$ is a sequence $\{x_i\}_{i=1}^{N}$ of $N+1$ vertices such that $x_0 = x$ and $x_N = y$, and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, 3, \dots, N$.

Recently, some results have appeared in the setting of metric spaces, which are endowed with a graph. The first result in this direction was given by Jachymski [17].

Definition 20. Let f and g be two self-mappings on graphic b-metric space (M, d^*, s). A pair (f, g) is said to be:

(i) weakly G-increasing if $(f(r), gf(r)) \in E(G)$ and $(g(r), fg(r)) \in E(G)$ for all $r \in M$;
(ii) partially weakly G-increasing if $(f(r), gf(r)) \in E(G)$ for all $r \in M$.

Definition 21. Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$. The pair (f, g) is said to be transitive weakly G-increasing pair of mappings with respect to h if:

(i) $(f(r_1), g(r_2)) \in E(G)$ for all $r_1 \in M$, for all $r_2 \in h^{-1}f(r_1)$ and $(g(r_1), f(r_2)) \in E(G)$ for all $r_2 \in h^{-1}g(r_1)$;
(ii) $(r_1, r_3) \in E(G)$, $(r_3, r_2) \in E(G)$ imply $(r_1, r_2) \in E(G)$ for all $r_1, r_2, r_3 \in M$.

Definition 22. Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$. The pair (f, g) is said to be transitive partially weakly G-increasing pair of mappings with respect to h if:

(i) $(f(r_1), g(r_2)) \in E(G)$ for all $r_1 \in M$, for all $r_2 \in h^{-1}f(r_1)$;
(ii) $(r_1, r_3) \in E(G)$, $(r_3, r_2) \in E(G)$ imply $(r_1, r_2) \in E(G)$ for all $r_1, r_2, r_3 \in M$.

Let (M, d^*, s) be a graphic b-metric space, and let

$$\alpha_s(r_1, r_2) = \begin{cases} s^2 & \text{if } (r_1, r_2) \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

By this assumption, we see that the above definitions are special cases from the definition of weak α_s-admissibility and partially weak α_s-admissibility.

Definition 23. Let (M, d^*, s) be a graphic b-metric space. It is said to be G-complete if and only if every Cauchy sequence $\{r_n\}$ in M such that $(r_n, r_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ converges in M.

Definition 24. Let (M, d^*, s) be a graphic b-metric space and $T : M \to M$ be a mapping. We say that T is a G-continuous mapping on (M, d^*, s) if, for given $r \in M$ and sequence $\{r_n\}$, $\lim_{n\to\infty} d^*(r_n, r) = 0$ and $(r_n, r_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ implies $\lim_{n\to\infty} d^*(T(r_n), T(r)) = 0$.

Definition 25. Let (M, d^*, s) be a graphic b-metric space. The pair (f, g) is said to be an G-compatible if and only if $\lim_{n\to\infty} d^*(fg(r_n), gf(r_n)) = 0$, whenever $\{r_n\}$ is a sequence in M such that $(r_n, r_{n+1}) \in E(G)$ and

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = t \quad \text{for some } t \in M.$$

Definition 26. The graphic b-metric space (M, d^*, s) is said to be G-regular, if for any sequence $\{r_n\}$ in M, following condition holds:

$$r_n \to r, \quad (r_n, r_{n+1}) \in E(G) \quad \Longrightarrow \quad (r_n, r) \in E(G), \quad n \in \mathbb{N}.$$

Now we are able to remodel Theorems 1 and 2 in the framework of graphic metric spaces.

Theorem 6. *Let f, g, S, T be G-continuous self-mappings defined on a G-complete graphic b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$ and $g(M) \subseteq S(M)$. Suppose that for all $r_1, r_2 \in M$ with $(S(r_1), T(r_2)) \in E(G)$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\big(f(r_1), g(r_2)\big) > 0 \quad \Longrightarrow \quad \tau + F\big(sd^*\big(f(r_1), g(r_2)\big)\big) \leqslant F\big(\mathcal{M}_1(r_1, r_2)\big).$$

Assume that the pairs (f, S), (g, T) are G-compatible and the pairs (f, g) and (g, f) are transitive partially weakly G-increasing pair of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point (say) υ in M. Moreover, if $(S\upsilon, T\upsilon) \in E(G)$, then υ is a common point of f, g, S, T.

Proof. Define

$$\alpha_s(r_1, r_2) = \begin{cases} s^2 & \text{if } (r_1, r_2) \in E(G); \\ 0 & \text{otherwise,} \end{cases}$$

and the proof follows from the proof of Theorem 1. □

Theorem 7. *Let f, g, S, T be G-continuous self-mappings defined on a G-regular and G-complete graphic b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$, and $T(M)$ and $S(M)$ are closed subsets of M. Suppose that for all $r_1, r_2 \in M$ with $(S(r_1), T(r_2)) \in E(G)$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\big(f(r_1), g(r_2)\big) > 0 \quad \Longrightarrow \quad \tau + F\big(sd^*\big(f(r_1), g(r_2)\big)\big) \leqslant F\big(\mathcal{M}_1(r_1, r_2)\big).$$

Assume that the pairs (f, S), (g, T) are weakly compatible and the pairs (f, g) and (g, f) are transitive partially weakly G-increasing pair of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point (say) υ in M. Moreover, if $(S\upsilon, T\upsilon) \in E(G)$, then υ is a coincidence point of f, g, S, T.

Proof. Define

$$\alpha_s(r_1, r_2) = \begin{cases} s^2 & \text{if } (r_1, r_2) \in E(G); \\ 0 & \text{otherwise,} \end{cases}$$

and the proof follows from proofs of Theorems 1 and 2. □

Corollaries 2–6 given above hold equally good in ordered b-metric space and graphic b-metric space.

We extend Theorem 1 for all $r_1, r_2 \in M$ as it follows.

Theorem 8. *Let f, g, S, T be self-mappings defined on a complete b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. If there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$d^*\big(f(r_1), g(r_2)\big) > 0 \quad \Longrightarrow \quad \tau + F\big(sd^*\big(f(r_1), g(r_2)\big)\big) \leqslant F\big(\mathcal{M}_1(r_1, r_2)\big)$$

for all $r_1, r_2 \in M$. Then f, g, S, T have a unique common fixed point in M provided that S, T are continuous and pairs $\{f, S\}$, $\{g, T\}$ are compatible.

Proof. The arguments follow the same lines as in proof of Theorem 1. □

6 Application to a system of integral equations

Let $M = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a, b]$. Let the function $d^* : M \times M \to [0, \infty)$ be defined by

$$d^*(u, v) = \Big(\sup_{t \in [a,b]} \big|u(t) - v(t)\big|\Big)^2$$

for all $u, v \in C([a, b], \mathbb{R})$ and define $\alpha_s : M \times M \to [0, \infty)$ by the rule

$$\alpha_s(u, v) = s^2 \quad \text{for all } u, v \in M.$$

Obviously, $(M, d^*, 2)$ is an α_s-complete b-metric space.

We will apply Theorem 1 to show the existence of common solution of the system of Volterra-type integral equations given by

$$u(t) = p(t) + \int_a^t K\big(t, r, S\big(u(t)\big)\big)\, \mathrm{d}r, \tag{26}$$

$$w(t) = p(t) + \int_a^t J\big(t, r, T\big(v(t)\big)\big)\, \mathrm{d}r \tag{27}$$

for all $t \in [a, b]$ and $a > 0$, where $f : M \to \mathbb{R}$ is continuous function and $K, J : [a, b] \times [a, b] \times M \to \mathbb{R}$ are lower semi continuous operators. Now we prove the following theorem to ensure the existence of solution of system of integral equations (26) and (27).

Theorem 9. *Let M=$C([a,b],\mathbb{R})$ and define the mappings $f, g : M \to M$ by*

$$fu(t) = p(t) + \int_a^t K\bigl(t, r, S\bigl(u(t)\bigr)\bigr)\,\mathrm{d}r,$$

$$gu(t) = p(t) + \int_a^t J\bigl(t, r, T\bigl(v(t)\bigr)\bigr)\,\mathrm{d}r$$

for all $t \in [a,b]$ and $a > 0$, where $f : M \to \mathbb{R}$ is continuous function and $K, J : [a,b] \times [a,b] \times M \to \mathbb{R}$ are lower semi continuous operators. Assume the following conditions are satisfied:

(i) *there exists a continuous function $H : M \to [0,\infty)$ such that*

$$\bigl|K(t,r,S) - J(t,r,T)\bigr| \leqslant H(r)\bigl|S\bigl(u(t)\bigr) - T\bigl(v(t)\bigr)\bigr|$$

for each $t, r \in [a,b]$ and $S, T \in M$;

(ii) *there exists $\tau > 0$, and for each $r \in M$, we have*

$$\int_a^t H(r)\,\mathrm{d}r \leqslant \sqrt{\frac{\mathrm{e}^{-\tau}}{s}}, \quad t \in [a,b].$$

Then the system of integral equations given by (26) *and* (27) *has a solution.*

Proof. By assumptions (i) and (ii), we have

$$\begin{aligned}
d^*\bigl(fu(t), gv(t)\bigr) &= \Bigl(\sup_{t\in[a,b]} \bigl|fu(t) - gv(t)\bigr|\Bigr)^2 \\
&= \left(\sup_{t\in[a,b]} \int_a^t \bigl|K\bigl(t,r,S\bigl(u(t)\bigr) - J\bigl(t,r,T\bigl(v(t)\bigr)\bigr)\bigr)\bigr|\,\mathrm{d}r\right)^2 \\
&\leqslant \left(\sup_{t\in[a,b]} \int_a^t H(r)\bigl|S\bigl(u(t)\bigr) - T\bigl(v(t)\bigr)\bigr|\,\mathrm{d}r\right)^2 \\
&\leqslant \left(\sqrt{\sup_{t\in[a,b]} \bigl|S\bigl(u(t)\bigr) - T\bigl(v(t)\bigr)\bigr|^2} \int_a^t H(r)\,\mathrm{d}r\right)^2 \\
&= d^*\bigl(S\bigl(u(t)\bigr), T\bigl(v(t)\bigr)\bigr)\left(\int_a^t H(r)\,\mathrm{d}r\right)^2 \\
&\leqslant d^*\bigl(S\bigl(u(t)\bigr), T\bigl(v(t)\bigr)\bigr)\frac{\mathrm{e}^{-\tau}}{s} \leqslant \mathcal{M}_1\bigl(u(t), v(t)\bigr)\frac{\mathrm{e}^{-\tau}}{s}.
\end{aligned}$$

Consequently, we obtain $sd^*(fu(t), gv(t)) \leqslant e^{-\tau}\mathcal{M}_1(u(t), v(t))$, which implies that $\tau + \ln(sd^*(fu(t), gv(t))) \leqslant \ln(\mathcal{M}_1(u(t), v(t)))$. For $F(r) = \ln(r)$, all hypotheses of Theorem 8 are satisfied. Hence, the system of integral equations given in (26) and (27) has a unique common solution. □

7 Conclusion

As we know, the concepts of α_s-complete b-metric space, α_s-continuity of a mapping, and α_s-compatibility of a pair of mappings are weaker than the concepts of complete metric space, continuity of a mapping, and compatibility of a pair of mappings, respectively. Therefore, Theorems 1, 2 and corresponding corollaries enrich the fixed point theory on F-contraction under weaker conditions.

References

1. O. Acar, I. Altun, Multivalued F-contractive mappings with a graph and some fixed point results, *Publ. Math.*, **88**:305–317, 2016.
2. A. Aghajani, M. Abbas, J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Math. Slovaca*, **4**:941–960, 2014.
3. M.U. Ali, T. Kamran, Multivalued F-contractions and related fixed point theorems with an application, *Filomat*, **30**:3779–3793, 2016.
4. M.U. Ali, T. Kamran, M. Postolache, Solution of volterra integral inclusion in b-metric spaces via new fixed point theorem, *Nonlinear Anal. Model. Control*, **22**(1):17–30, 2017.
5. I. Altun, G. Minak, M. Olgun, Fixed points of multivalued nonlinear F-contractions on complete metric spaces, *Nonlinear Anal. Model. Control*, **21**(2):201–210, 2016.
6. A. Amini-Harandi, Fixed point theory for quasi-contraction mappings in b-metric spaces, *Fixed Point Theory*, **15**(1):351–358, 2014.
7. I.A. Bakhtin, The contraction principle in quasi metric spaces, *Funkts. Anal.*, **30**:26–37, 1989.
8. M. Cosentino, M. Jleli, B. Samet, C. Vetro, Solvability of integrodifferential problem via fixed point theory in b-metric spaces, *Fixed Point Theory Appl.*, **2015**:70, 2015.
9. M. Cosentino, P. Vetro, Fixed point results for F-contractive mappings of Hardy–Rogers type, *Filomat*, **28**:715–722, 2014.
10. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostrav.*, **1**: 5–11, 1993.
11. G. Durmaz, G. Minak, I. Altun, Fixed points of ordered F-contractions, *Hacettepe J. Math. Stat.*, **45**:15–21, 2016.
12. H. Huang, J. Vujakovic, S. Radenovic, A note on common fixed point theorems for isotone increasing mappings in ordered b-metric spaces, *J. Nonlinear Sci. Appl.*, **8**:808–815, 2015.
13. A. Hussain, M. Nazam, M. Arshad, Connection of Ciric type F-contraction involving fixed point on closed ball, *Gazi Univ. J. Sci.*, **30**:283–291, 2017.
14. N. Hussain, M. Arshad, A. Shoaib, Fahimuddin, Common fixed point results for (α, ψ)-contractions on a metric space endowed with graph, *J. Inequal. Appl.*, **2014**:136, 2014.

15. N. Hussain, M.A. Kutbi, P. Salimi, Fixed point theory in α-complete metric spaces with applications, *Abstr. Appl. Anal.*, **2014**:280817, 2014.
16. N. Hussain, C. Vetro, F. Vetro, Fixed point results for α-implicit contractions with application to integral equations, *Nonlinear Anal. Model. Control*, **21**:362–378, 2016.
17. J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Am. Math. Soc.*, **136**:1359–1373, 2008.
18. G. Jungck, Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.*, **9**:771–779, 1986.
19. G. Jungck, Common fixed points for commuting and compatible mappings on compacta, *Proc. Am. Math. Soc.*, **103**, 1988.
20. T. Kamran, M. Postolache, M.U. Ali, Q. Kiran, Feng and Liu type F-contraction in b-metric spaces with application to integral equations, *J. Math. Anal.*, **7**(5):18–27, 2016.
21. M.A. Khamsi, N. Hussain, KKM mappings in metric type spaces, *Nonlinear Anal., Theory Methods Appl.*, **73**(9):3123–3129, 2010.
22. G. Minak, A. Helvaci, I. Altun, Ćirić type generalized F-contractions on complete metric spaces and fixed point results, *Filomat*, **28**:1143–1151, 2014.
23. M. Nazam, M. Arshad, M. Abbas, Existence of common fixed points of improved F-contraction on partial metric spaces, *Appl. Gen. Topol.*, **18**:277–287, 2017.
24. H. Piri, P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, *Fixed Point Theory Appl.*, **2014**:210, 2014.
25. P. Salimi, A. Latif, N. Hussain, Modified (α, ψ)-contractive mappings with applications, *Fixed Point Theory Appl.*, **2013**:151, 2013.
26. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha - \psi$-contractive type mappings, *Nonlinear Anal., Theory Methods Appl.*, **75**(4):2154–2165, 2012.
27. N.A. Secelean, Iterated function systems consisting of F-contractions, *Fixed Point Theory Appl.*, **2013**:277, 2013.
28. W. Shatanawi, A. Pitea, R. Lazovic, Contraction conditions using comparison functions on b-metric spaces, *Fixed Point Theory Appl.*, **2014**:135, 2014.
29. D. Singh, V. Chauhan, I. Altun, Common fixed point of a power graphic (F, ψ)-contraction pair on partial b-metric spaces with application, *Nonlinear Anal. Model. Control*, **22**(5):662–678, 2017.
30. C. Vetro, F. Vetro, The class of F-contraction mappings with a measure of noncompactness, in J. Banaś, M. Jleli, M. Mursaleen, B. Samet, C. Vetro (Eds.), *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, Springer, Singapore, 2017, pp. 297–331.
31. F. Vetro, F-contractions of Hardy–Rogers type and application to multistage decision processes, *Nonlinear Anal. Model. Control*, **21**(4):531–546, 2016.
32. D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012**:94, 2012.