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On the center-stable manifolds for some fractional differential equations of Caputo type*

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Abstract. This paper is devoted to study the existence of center-stable manifolds for some planar fractional differential equations of Caputo type with relaxation factor. After giving some necessary estimation for Mittag–Leffler functions, some existence results for center-stable manifolds are established under the mild conditions by virtue of a suitable Lyapunov–Perron operator. Moreover, an explicit example is given to illustrate the above result. Finally, high-dimensional case is considered.

Keywords: fractional differential equations, Mittag–Leffler functions, center-stable manifolds.

1 Introduction

Fractional calculus (FC) has a long history almost as well as the one of standard integer calculus. Thereafter, fractional differential equations (FDEs) have been recognized as one of the best tools to be applied in interdisciplinary field such as viscoelastic materials and electromagnetic problems; see [3, 4, 7, 12, 13, 16, 18, 19, 25–29] and references therein. In particular, existence and stability results for some FDEs involving two different Caputo derivatives have been studied extensively, one can refer to [1, 2, 5, 8–10, 14, 15, 17, 20–22, 24]. Very recently, an interesting local stable manifold theorem near a hyperbolic equilibrium point for planar fractional differential equations is given in [6], where the fixed point of Lyapunov–Perron operator describes the set of all solutions near the fixed

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point tend to zero is called stable manifold of hyperbolic fixed point. However, stable manifolds results for FDEs of different type are still not enough.

Let $X_\infty(\mathbb{R}_+, \mathbb{R})$ be the Banach space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ with the norm $\|v\|_\infty = \sup_{t\in\mathbb{R}_+} |v(t)|$. Motivated by [6,21,22], Wang et al. [23] consider local stable manifold of the following planar fractional damped equations:

$$\begin{aligned} &{}^cD_t^\alpha x(t) + A\,{}^cD_t^\beta x(t) = f\bigl(x(t),t\bigr), \quad \alpha \in (1,2),\ \beta \in (0,1),\ t \geqslant 0, \\ &x(0) = x = (x_1,x_2)^{\mathrm{T}}, \qquad x'(0) = \overline{x} = (x_3,x_4)^{\mathrm{T}}, \end{aligned} \tag{1}$$

where ${}^cD_t^\gamma$ denotes the Caputo fractional derivative of order γ with the lower limit zero (see [12]), $A = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ ($\lambda_1 > 0$, $\lambda_2 < 0$), $f(x,t) = (f_1(x,t), f_2(x,t))^{\mathrm{T}}$. However, the related issues for the case of $\beta \in \{0,1\}$ has not been discussed. In the present paper, we consider local stable manifold of the following nonautonomous fractional Cauchy problems:

$$\begin{aligned} &{}^cD_t^\alpha x(t) - Ax(t) = f\bigl(x(t),t\bigr), \quad \alpha \in (1,2),\ t \geqslant 0, \\ &x(0) = x = (x_1,x_2)^{\mathrm{T}}, \qquad x'(0) = \overline{x} = (x_3,x_4)^{\mathrm{T}}, \end{aligned} \tag{2}$$

where $f \in X(\mathbb{R}^2 \times \mathbb{R}_+, \mathbb{R}^2)$ is a local Lipschitz function:

$$\bigl\|f(x,t) - f(y,t)\bigr\| \leqslant l_f(r)\varpi(t)\|x-y\|$$

for $\|x\|, \|y\| \leqslant r$, where set $\|x\| = \|(x_1,x_2)\| = \max\{|x_1|, |x_2|\}$, $f(0,t) = 0$, $\lim_{r\to 0} l_f(r) = 0$, and $\varpi \in X_\infty(\mathbb{R}_+, (0,\infty))$ satisfies $b_\varpi := \sup_{t\geqslant 1} \int_0^{t-1} (t-s)^{-1} \times \varpi(s)\,\mathrm{d}s < \infty$.

Here we would like to emphasize that the methods used to deal with (2) are much different from (1). Concerning on (2), we cannot apply a known formula on Mittag–Leffler functions to simplify the form of solution to (2). In order to obtain the existence of the stability of the solution, we simply our problem and set $x(0) = x = (0,x_2)^{\mathrm{T}}$. Even in this special case, we also have to overcome expatiatory computation from the estimation on the possible integral of Mittag–Leffler functions, more precisely, we have to study the asymptotic behavior of Mittag–Leffler functions E_α, $E_{\alpha,2}$, and $E_{\alpha,\alpha}$ for $\alpha \in (1,2)$.

By [21, Lemma 2.1], taking Laplace transform of Caputo fractional derivative and inverse Laplace transforms of the functions, the solution $\phi(\cdot,x,\overline{x})$ of (2) is given by

$$\begin{aligned} \phi\bigl(t,x,\overline{x}\bigr) &= E_\alpha\bigl(t^\alpha A\bigr)x + E_{\alpha,2}\bigl(t^\alpha A\bigr)t\overline{x} \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl((t-s)^\alpha A\bigr) f\bigl(\phi(s,x,\overline{x}),s\bigr)\,\mathrm{d}s. \end{aligned} \tag{3}$$

By reviewing (3) we can know that when $t \to \infty$, the solution $\phi(\cdot,x,\overline{x})$ of (2) is stable for $\lambda_2 < 0$. However, when $t \to \infty$, the solution $\phi(\cdot,x,\overline{x})$ of (2) is not stable for $\lambda_1 > 0$.

By a center-stable manifold of (2) we mean the set of all small x and $\bar{x}$ for which the solution of (2) is bounded on $\mathbb{R}_+$ when the time variable tends infinite.

To achieve our aim, we adopt the same idea in [6] and construct a suitable Lyapunov–Perron operator

$$F = (F_1, F_2)\colon\ X_\infty\big(\mathbb{R}_+, \mathbb{R}^2\big) \to X_\infty\big(\mathbb{R}_+, \mathbb{R}^2\big) \tag{4}$$

as follows:

$$\begin{aligned}
F_1(\eta)(t) &= -\lambda_1^{(2-\alpha)/\alpha} t E_{\alpha,2}\big(t^\alpha \lambda_1\big) \int\limits_0^\infty \exp\big(-\lambda_1^{1/\alpha} s\big) f_1\big(\eta(s), s\big)\,\mathrm{d}s \\
&\quad + \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big((t-s)^\alpha \lambda_1\big) f_1\big(\eta(s), s\big)\,\mathrm{d}s, \\
F_2(\eta)(t) &= E_\alpha\big(t^\alpha \lambda_2\big) x_2 + E_{\alpha,2}\big(t^\alpha \lambda_2\big) t x_4 \\
&\quad + \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big((t-s)^\alpha \lambda_2\big) f_2\big(\eta(s), s\big)\,\mathrm{d}s.
\end{aligned}$$

Then we need show that the center-stable manifold of (2) can be characterized as a fixed point of the above Lyapunov–Perron operator F and the fixed point is bounded.

This paper is organized as follows. In Section 2, we give some fundamental estimation related to Mittag–Leffler functions. In Section 3, we give the main result of this paper about center-stable manifolds. An example is given to demonstrate the application of our main result. In the final section, we extend the previous stable manifold result for planar fractional order relaxation differential equations involving the order $\alpha \in (0,1)$ to high-dimensional case.

2 A sequence of integral estimation related to Mittag–Leffler functions

The following explicit estimation of Mittag–Leffler functions is useful in the sequel, which has been reported in one of our submitted paper. To give some results for the asymptotic behavior of Mittag–Leffler functions E_α, $E_{\alpha,2}$, and $E_{\alpha,\alpha}$ for $\alpha \in (1,2)$, we recall the following.

Lemma 1. (*See* [11].) *Let $\varrho \in (0,2)$ and $\upsilon \in \mathbb{R}$ be arbitrary. Then, for $\bar{p} = [\upsilon/\varrho]$, the following asymptotic expansions hold*:

(i) $E_{\sigma,\upsilon}(z) = \frac{1}{\sigma} z^{(1-\upsilon)/\sigma} \exp\big(z^{1/\sigma}\big) - \sum_{k=1}^{\bar{p}} \frac{z^{-k}}{\Gamma(\upsilon-\sigma k)} + O\big(z^{-1-\bar{p}}\big)$ *as* $z \to \infty$.

(ii) $E_{\sigma,\upsilon}(z) = -\sum_{k=1}^{\bar{p}} \frac{z^{-k}}{\Gamma(\upsilon-\sigma k)} + O\big(|z|^{-1-\bar{p}}\big)$ *as* $z \to -\infty$.

By inserting $\bar{\alpha} = \alpha$, $\bar{\beta} = 1$, and $z = t^\alpha \lambda$ we give the first asymptotic expansions for Mittag–Leffler functions E_α, which extend to our case.

Lemma 2. *For any $\lambda > 0$, $\alpha \in (1,2)$, and $\bar{p} = [1/\alpha]$, the following asymptotic expansions hold*:

(i) $E_\alpha(\lambda t^\alpha) = \frac{1}{\alpha}\exp(\lambda^{1/\alpha} t) - \sum_{k=1}^{\bar{p}} \frac{t^{-k\alpha}}{\lambda^k \Gamma(1-\alpha k)} + O(t^{-\alpha(1+\bar{p})})$ *as* $t \to \infty$.

(ii) $E_\alpha(-\lambda t^\alpha) = -\sum_{k=1}^{\bar{p}} \frac{t^{-k\alpha}}{(-\lambda)^k \Gamma(1-\alpha k)} + O(t^{-\alpha(1+\bar{p})})$ *as* $t \to \infty$.

Remark 1. Lemma 2(i) presents the relationship between Mittag–Leffler functions and exponential function when the parameter $\lambda > 0$. Lemma 2(ii) shows that Mittag–Leffler functions can be formulated by power series when the parameter $\lambda < 0$.

By inserting $\sigma = \alpha$, $\upsilon = 2$ or $\upsilon = \alpha$, and $z = t^\alpha \lambda$ we give asymptotic expansions for Mittag–Leffler functions $E_{\alpha,2}$ and $E_{\alpha,\alpha}$.

Lemma 3. *For any $\lambda > 0$, $\alpha \in (1,2)$ with $\bar{p} = 1$, the following asymptotic expansions hold*:

(i) $tE_{\alpha,2}(\lambda t^\alpha) = \frac{1}{\alpha}\lambda^{-1/\alpha}\exp(\lambda^{1/\alpha} t) - \frac{t^{1-\alpha}}{\lambda\Gamma(2-\alpha)} + O(t^{1-2\alpha})$ *as* $t \to \infty$.

(ii) $tE_{\alpha,2}(-\lambda t^\alpha) = \frac{t^{1-\alpha}}{\lambda\Gamma(2-\alpha)} + O(t^{1-2\alpha})$ *as* $t \to \infty$.

Remark 2. Lemma 3(i) presents explicit formula to compute $tE_{\alpha,2}(\lambda t^\alpha)$ by virtue of exponential function and power series when the parameter $\lambda > 0$. Lemma 3(ii) shows that $tE_{\alpha,2}(-\lambda t^\alpha)$ can be formulated by power series when the parameter $\lambda < 0$.

Lemma 4. *For any $\lambda > 0$, $\alpha \in (1,2)$ with $\bar{p} = 1$, the following asymptotic expansions hold*:

(i) $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) = \frac{1}{\alpha}\lambda^{(1-\alpha)/\alpha}\exp(\lambda^{1/\alpha} t) - \frac{1}{t\lambda} + O(t^{-1-\alpha})$ *as* $t \to \infty$.

(ii) $t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) = \frac{1}{t\lambda} + O(t^{-1-\alpha})$ *as* $t \to \infty$.

Remark 3. Lemma 4(i) presents explicit formula to compute $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ by virtue of exponential function and power series when the parameter $\lambda > 0$. Lemma 4(ii) shows that $t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$ can be formulated by power series when the parameter $\lambda < 0$.

We give the estimation for Mittag–Leffler functions.

Lemma 5. *For $\lambda > 0$, we define*

$$M(\alpha,\lambda) = \max\left\{E_{\alpha,\alpha+1}(\lambda),\ \lambda^{(1-\alpha)/\alpha}E_{\alpha,2}(\lambda),\ \frac{E_{\alpha,\alpha}(\lambda)}{\alpha} + \lambda^{(1-\alpha)/\alpha}E_{\alpha,2}(\lambda)\right\}.$$

Then, for any function $g \in X_\infty(\mathbb{R}_+,\mathbb{R})$, the following statements hold for all $t \in [0,1]$:

(i) $\left|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\lambda)g(s)\,\mathrm{d}s\right| \leqslant M(\alpha,\lambda)\|g\|_\infty$.

(ii) $\left|\lambda^{(2-\alpha)/\alpha}E_{\alpha,2}(t^\alpha\lambda)t\int_t^\infty \exp(-\lambda^{1/\alpha}s)g(s)\,\mathrm{d}s\right| \leqslant M(\alpha,\lambda)\|g\|_\infty$.

(iii) $\left|\int_0^t \left[(t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha\lambda) - \lambda^{(2-\alpha)/\alpha}E_{\alpha,2}(t^\alpha\lambda)t\exp(-\lambda^{1/\alpha}s)\right]g(s)\,\mathrm{d}s\right| \leqslant M(\alpha,\lambda)\|g\|_\infty$.

Proof. (i) Using $\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha\lambda)\,\mathrm{d}s = t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha)$ and the fact that Mittag–Leffler functions are increasing functions on $[0,\infty)$, we have

$$\begin{aligned}&\left|\int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-(t-s)^\alpha\lambda\bigr) g(s)\,\mathrm{d}s\right| \\ &\quad\leqslant t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha)\|g\|_\infty \leqslant E_{\alpha,\alpha+1}(\lambda)\|g\|_\infty \leqslant M(\alpha,\lambda)\|g\|_\infty.\end{aligned}$$

(ii) Like above, we have

$$\begin{aligned}&\left|\lambda^{(2-\alpha)/\alpha} E_{\alpha,2}\bigl(t^\alpha\lambda\bigr) t \int\limits_t^\infty \exp\bigl(-\lambda^{1/\alpha}s\bigr) g(s)\,\mathrm{d}s\right| \\ &\quad\leqslant \lambda^{(2-\alpha)/\alpha} E_{\alpha,2}(\lambda) \int\limits_0^\infty \exp\bigl(-\lambda^{1/\alpha}s\bigr)\,\mathrm{d}s\|g\|_\infty \\ &\quad\leqslant \lambda^{(1-\alpha)/\alpha} E_{\alpha,2}(\lambda)\|g\|_\infty \leqslant M(\alpha,\lambda)\|g\|_\infty.\end{aligned}$$

(iii) Similarly, we derive

$$\begin{aligned}&\left|\int\limits_0^t \bigl[(t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl((t-s)^\alpha\lambda\bigr) - \lambda^{(2-\alpha)/\alpha} E_{\alpha,2}\bigl(t^\alpha\lambda\bigr) t \exp\bigl(-\lambda^{1/\alpha}s\bigr)\bigr] g(s)\,\mathrm{d}s\right| \\ &\quad\leqslant \left[\frac{1}{\alpha} t^\alpha E_{\alpha,\alpha}(\lambda) + \lambda^{(2-\alpha)/\alpha} E_{\alpha,2}(\lambda)\bigl(-\lambda^{-1/\alpha}\exp\bigl(-\lambda^{1/\alpha}t\bigr) + \lambda^{-1/\alpha}\bigr)\right]\|g\|_\infty \\ &\quad\leqslant \left[\frac{E_{\alpha,\alpha}(\lambda)}{\alpha} + \lambda^{(1-\alpha)/\alpha} E_{\alpha,2}(\lambda)\right]\|g\|_\infty \leqslant M(\alpha,\lambda)\|g\|_\infty.\end{aligned}$$

The proof is complete. □

Remark 4. Lemma 5 presents some explicit integral bound estimation by using the asymptotic behavior of Mittag–Leffler functions, which will be used to derive the stable manifold for the proposed fractional systems.

By Lemmas 3 and 4 we can define finite constants:

$$\begin{aligned}m_1(\alpha,\lambda) &= \sup_{t\geqslant 1} t^{2\alpha-1}\left| tE_{\alpha,2}(\lambda t^\alpha) - \frac{\exp(\lambda^{1/\alpha}t)}{\lambda^{1/\alpha}\alpha} + \frac{t^{1-\alpha}}{\lambda\Gamma(2-\alpha)}\right|, \\ m_2(\alpha,\lambda) &= \sup_{t\geqslant 1} t^{2\alpha-1}\left| tE_{\alpha,2}(-\lambda t^\alpha) - \frac{t^{1-\alpha}}{\lambda\Gamma(2-\alpha)}\right|, \\ m_3(\alpha,\lambda) &= \sup_{t\geqslant 1} t^{\alpha+1}\left| t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) - \frac{\exp(\lambda^{1/\alpha}t)}{\lambda^{(\alpha-1)/\alpha}\alpha} + \frac{1}{\lambda t}\right|, \\ m_4(\alpha,\lambda) &= \sup_{t\geqslant 1} t^{\alpha+1}\left| t^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda t^\alpha\bigr) - \frac{1}{\lambda t}\right|.\end{aligned}$$

Now we can prove the following lemma.

Lemma 6. *For $\lambda > 0$, we define*

$$N(\alpha,\lambda,b_\varpi) = \max\biggl\{E_{\alpha,\alpha+1}(\lambda) + \frac{\max\{m_3(\alpha,\lambda), m_4(\alpha,\lambda)\}}{\alpha} + \frac{b_\varpi}{\lambda} + \frac{1}{\alpha\lambda}\exp\bigl(\lambda^{1/\alpha}\bigr) + \frac{2\lambda^{(1-2\alpha)/\alpha}}{\Gamma(2-\alpha)} + 2m_1(\alpha,\lambda)\lambda^{(1-\alpha)/\alpha}\biggr\}.$$

Then, for any function $g \in X_\infty(\mathbb{R}_+,\mathbb{R})$ with $\|g\|_\varpi := \sup_{t\geqslant 0}|g(t)/\varpi(t)| < \infty$, the following statements hold for all $t > 1$:

(i) $|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\lambda)g(s)\,\mathrm{d}s| \leqslant N(\alpha,\lambda,b_\varpi)(\|g\|_\infty + \|g\|_\varpi)$.

(ii) $|\lambda^{(2-\alpha)/\alpha}E_{\alpha,2}(t^\alpha\lambda)t\int_t^\infty \exp(-\lambda^{1/\alpha}s)g(s)\,\mathrm{d}s| \leqslant N(\alpha,\lambda,b_\varpi)\|g\|_\infty$.

(iii) $|\int_0^t [(t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha\lambda) - \lambda^{(2-\alpha)/\alpha}E_{\alpha,2}(t^\alpha\lambda)t\exp(-\lambda^{1/\alpha}s)]g(s)\,\mathrm{d}s| \leqslant N(\alpha,\lambda,b_\varpi)(\|g\|_\infty + \|g\|_\varpi)$.

Proof. (i) Note that $\int_{t-1}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha\lambda)\,\mathrm{d}s = E_{\alpha,\alpha+1}(\lambda)$. So we have

$$\biggl|\int_{t-1}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-(t-s)^\alpha\lambda\bigr)g(s)\,\mathrm{d}s\biggr| \leqslant \int_{t-1}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((t-s)^\alpha\lambda\bigr)\bigl|g(s)\bigr|\,\mathrm{d}s \leqslant E_{\alpha,\alpha+1}(\lambda)\|g\|_\infty.$$

On the other hand, we get

$$\begin{aligned}\biggl|\int_0^{t-1}(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-(t-s)^\alpha\lambda\bigr)g(s)\,\mathrm{d}s\biggr| &\leqslant m_4(\alpha,\lambda)\int_0^{t-1}\frac{|g(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s + \frac{1}{\lambda}\int_0^{t-1}\frac{|g(s)|}{t-s}\,\mathrm{d}s\\ &\leqslant \frac{m_4(\alpha,\lambda)\|g\|_\infty}{\alpha} + \frac{1}{\lambda}\int_0^{t-1}\frac{\varpi(s)}{t-s}\frac{|g(s)|}{\varpi(s)}\,\mathrm{d}s \leqslant \frac{m_4(\alpha,\lambda)\|g\|_\infty}{\alpha} + \frac{b_\varpi\|g\|_\varpi}{\lambda}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\biggl|\int_0^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-(t-s)^\alpha\lambda\bigr)g(s)\,\mathrm{d}s\biggr| &\leqslant \biggl(E_{\alpha,\alpha+1}(\lambda) + \frac{m_4(\alpha,\lambda)}{\alpha}\biggr)\|g\|_\infty + \frac{b_\varpi}{\lambda}\|g\|_\varpi\\ &\leqslant N(\alpha,\lambda,b_\varpi)\bigl(\|g\|_\infty + \|g\|_\varpi\bigr).\end{aligned}$$

(ii) Similarly, we get

$$\begin{aligned}
&\left|\lambda^{(2-\alpha)/\alpha} E_{\alpha,2}\bigl(t^\alpha\lambda\bigr) t \int_t^\infty \exp\bigl(-\lambda^{1/\alpha}s\bigr) g(s)\,\mathrm{d}s\right| \\
&\quad\leqslant \frac{1}{\alpha}\lambda^{(1-\alpha)/\alpha} \int_t^\infty \exp\bigl(\lambda^{1/\alpha}(t-s)\bigr)\bigl|g(s)\bigr|\,\mathrm{d}s \\
&\qquad + \lambda^{(2-\alpha)/\alpha}\left(m_1(\alpha,\lambda) + \frac{1}{\lambda\Gamma(2-\alpha)}\right) \int_t^\infty \exp\bigl(-\lambda^{1/\alpha}s\bigr) g(s)\,\mathrm{d}s \\
&\quad\leqslant \left(\frac{1}{\alpha\lambda} + \frac{\lambda^{(1-2\alpha)/\alpha}}{\Gamma(2-\alpha)} + m_1(\alpha,\lambda)\lambda^{(1-\alpha)/\alpha}\right)\|g\|_\infty \leqslant N(\alpha,\lambda,b_\varpi)\|g\|_\infty.
\end{aligned}$$

(iii) Like above, we get

$$\begin{aligned}
&\bigl|(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((t-s)^\alpha\lambda\bigr) - \lambda^{(2-\alpha)/\alpha}E_{\alpha,2}\bigl(t^\alpha\lambda\bigr) t \exp\bigl(-\lambda^{1/\alpha}s\bigr)\bigr| \\
&\quad\leqslant \frac{m_3(\alpha,\lambda)}{(t-s)^{\alpha+1}} + \frac{1}{\lambda(t-s)} + m_1(\alpha,\lambda)\frac{\lambda^{(2-\alpha)/\alpha}\exp(-\lambda^{1/\alpha}s)}{t^{2\alpha-1}} \\
&\qquad + \frac{t^{1-\alpha}}{\lambda\Gamma(2-\alpha)}\lambda^{(2-\alpha)/\alpha}\exp\bigl(-\lambda^{1/\alpha}s\bigr)
\end{aligned}$$

for $t-s\geqslant 1$ and $t\geqslant 1$, but since $s\geqslant 0$, it is enough to suppose $t-s\geqslant 1$ because then $t\geqslant 1+s\geqslant 1$. So we derive

$$\begin{aligned}
&\left|\int_0^{t-1}\bigl[(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((t-s)^\alpha\lambda\bigr) - \lambda^{(2-\alpha)/\alpha}E_{\alpha,2}\bigl(t^\alpha\lambda\bigr) t\exp\bigl(-\lambda^{1/\alpha}s\bigr)\bigr] g(s)\,\mathrm{d}s\right| \\
&\quad\leqslant \int_0^{t-1}\frac{m_3(\alpha,\lambda)|g(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s + \int_0^{t-1}\frac{|g(s)|}{\lambda(t-s)}\,\mathrm{d}s \\
&\qquad + m_1(\alpha,\lambda)\lambda^{(2-\alpha)/\alpha}\int_0^{t-1}\frac{\exp(-\lambda^{1/\alpha}s)}{t^{2\alpha-1}}\,\mathrm{d}s\|g\|_\infty \\
&\qquad + \frac{1}{\lambda\Gamma(2-\alpha)}\int_0^{t-1} t^{1-\alpha}\lambda^{(2-\alpha)/\alpha}\exp\bigl(-\lambda^{1/\alpha}s\bigr)\,\mathrm{d}s\|g\|_\infty \\
&\quad\leqslant \left[\frac{m_3(\alpha,\lambda)}{\alpha} + m_1(\alpha,\lambda)\lambda^{(1-\alpha)/\alpha} + \frac{\lambda^{(1-2\alpha)/\alpha}}{\Gamma(2-\alpha)}\right]\|g\|_\infty + \frac{b_\varpi}{\lambda}\|g\|_\varpi.
\end{aligned}$$

On the other hand,

$$\begin{aligned}&\left|\int_{t-1}^{t}\left[(t-s)^{\alpha-1}E_{\alpha,\alpha}\big((t-s)^{\alpha}\lambda\big)-\lambda^{(2-\alpha)/\alpha}E_{\alpha,2}\big(t^{\alpha}\lambda\big)t\exp\big(-\lambda^{1/\alpha}s\big)\right]g(s)\,\mathrm{d}s\right|\\&\quad\leqslant\left[E_{\alpha,\alpha+1}(\lambda)+\int_{t-1}^{t}\lambda^{(2-\alpha)/\alpha}E_{\alpha,2}\big(t^{\alpha}\lambda\big)t\exp\big(-\lambda^{1/\alpha}s\big)\,\mathrm{d}s\right]\|g\|_{\infty}.\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}&\left|\int_{t-1}^{t}\lambda^{(2-\alpha)/\alpha}E_{\alpha,2}\big(t^{\alpha}\lambda\big)t\exp\big(-\lambda^{1/\alpha}s\big)\,\mathrm{d}s\right|\\&\quad\leqslant\frac{1}{\alpha}\lambda^{(1-\alpha)/\alpha}\int_{t-1}^{t}\exp\big(\lambda^{1/\alpha}(t-s)\big)\,\mathrm{d}s+m_1(\alpha,\lambda)\lambda^{(2-\alpha)/\alpha}\int_{t-1}^{t}\exp(-\lambda^{1/\alpha}s)\,\mathrm{d}s\\&\qquad+\frac{1}{\lambda\Gamma(2-\alpha)}\int_{t-1}^{t}\lambda^{(2-\alpha)/\alpha}\exp\big(-\lambda^{1/\alpha}s\big)\,\mathrm{d}s\\&\quad\leqslant\frac{1}{\alpha\lambda}\exp\big(\lambda^{1/\alpha}\big)+m_1(\alpha,\lambda)\lambda^{(1-\alpha)/\alpha}+\frac{\lambda^{(1-2\alpha)/\alpha}}{\Gamma(2-\alpha)}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}&\left|\int_{0}^{t}\left[(t-s)^{\alpha-1}E_{\alpha,\alpha}\big((t-s)^{\alpha}\lambda\big)-\lambda^{(2-\alpha)/\alpha}E_{\alpha,2}\big(t^{\alpha}\lambda\big)t\exp\big(-\lambda^{1/\alpha}s\big)\right]g(s)\,\mathrm{d}s\right|\\&\quad\leqslant\left[\frac{m_3(\alpha,\lambda)}{\alpha}+E_{\alpha,\alpha+1}(\lambda)+\frac{1}{\alpha\lambda}\exp\big(\lambda^{1/\alpha}\big)+2m_1(\alpha,\lambda)\lambda^{(1-\alpha)/\alpha}\right.\\&\qquad\left.+\frac{2\lambda^{(1-2\alpha)/\alpha}}{\Gamma(2-\alpha)}\right]\|g\|_{\infty}+\frac{b_{\varpi}}{\lambda}\|g\|_{\varpi}\\&\quad\leqslant N(\alpha,\lambda,b_{\varpi})\big(\|g\|_{\infty}+\|g\|_{\varpi}\big).\end{aligned}$$

The proof is finished. □

3 Existence of stable manifolds theorem

By Lemmas 5 and 6 the operator F in (4) is well defined. In what follows, we state and prove some fundamental properties of F, which are used later to prove the existence of stable manifolds.

Define $\Omega(\alpha,\lambda_1,\lambda_2,\varpi) = (1+\|\varpi\|_\infty)\varrho$ and $\varrho := \max\{2M(\alpha,\lambda_1), 2N(\alpha,\lambda_1,b_\varpi), M(\alpha,-\lambda_2), N(\alpha,-\lambda_2,b_\varpi)\}$, where M and N are the functions defined as in Lemmas 5 and 6.

Proposition 1. *For any $\eta,\hat\eta \in X_\infty(\mathbb{R}_+,\mathbb{R}^2)$, it holds*

$$\big\|F(\eta)-F(\hat\eta)\big\|_\infty \leqslant \Omega(\alpha,\lambda_1,\lambda_2,\varpi)l_f\big(\max\big(\|\eta\|_\infty,\|\hat\eta\|_\infty\big)\big)\|\eta-\hat\eta\|_\infty \tag{5}$$

and

$$\begin{aligned}\big\|F(\eta)\big\|_\infty \leqslant \|x\| + \left(E_{\alpha,2}(-\lambda_2)+m_2(\alpha,-\lambda_2)+\frac{1}{\lambda\Gamma(2-\alpha)}\right)|x_4| \\ + \Omega(\alpha,\lambda_1,\lambda_2,\varpi)l_f\big(\|\eta\|_\infty\big)\|\eta\|_\infty,\end{aligned} \tag{6}$$

where $\|\eta\|_\infty = \max(\|\eta_1\|_\infty,\|\eta_2\|_\infty)$.

Proof. Note that

$$\begin{aligned}&\big|F_1(\eta)(t)-F_1(\hat\eta)(t)\big| \\ &\quad= \Bigg|\int_0^t \big((t-s)^{\alpha-1}E_{\alpha,\alpha}\big((t-s)^\alpha\lambda_1\big) \\ &\qquad\quad - \lambda_1^{(2-\alpha)/\alpha}E_{\alpha,2}(\lambda_1 t^\alpha)t\exp\big(-\lambda_1^{1/\alpha}s\big)\big)\big(f_1\big(\eta(s),s\big)-f_1\big(\hat\eta(s),s\big)\big)\,\mathrm{d}s \\ &\qquad - \lambda_1^{(2-\alpha)/\alpha}E_{\alpha,2}(\lambda_1 t^\alpha)t\int_t^\infty \exp\big(-\lambda_1^{1/\alpha}s\big)\big(f_1\big(\eta(s),s\big)-f_1\big(\hat\eta(s),s\big)\big)\,\mathrm{d}s\Bigg|.\end{aligned}$$

By Lemmas 5 and 6 we have

$$\begin{aligned}&\sup_{t\in[0,1]}\big|F_1(\eta)(t)-F_1(\hat\eta)(t)\big| \\ &\quad\leqslant 2M(\alpha,\lambda_1)l_f\big(\max\big(\|\eta\|_\infty,\|\hat\eta\|_\infty\big)\big)\|\varpi\|_\infty\|\eta-\hat\eta\|_\infty, \\ &\sup_{t>1}\big|F_1(\eta)(t)-F_1(\hat\eta)(t)\big| \\ &\quad\leqslant 2N(\alpha,\lambda_1,b_\varpi)l_f\big(\max\big(\|\eta\|_\infty,\|\hat\eta\|_\infty\big)\big)\big(1+\|\varpi\|_\infty\big)\|\eta-\hat\eta\|_\infty.\end{aligned}$$

So

$$\sup_{t\geqslant 0}\big|F_1(\eta)(t)-F_1(\hat\eta)(t)\big| \leqslant \Omega(\alpha,\lambda_1,\lambda_2,\varpi)l_f\big(\max(\|\eta\|_\infty,\|\hat\eta\|_\infty)\big)\|\eta-\hat\eta\|_\infty.$$

On the other hand, note that

$$\big|F_2(\eta)-F_2(\hat\eta)\big| \leqslant \Bigg|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big((t-s)^\alpha\lambda_2\big)\big(f_2(\eta(s),s)-f_2(\hat\eta(s),s)\big)\,\mathrm{d}s\Bigg|.$$

By using Lemmas 5 and 6 we have

$$\begin{aligned}
&\sup_{t\in[0,1]}\bigl|F_2(\eta)(t)-F_2(\hat\eta)(t)\bigr| \\
&\quad\leqslant M(\alpha,-\lambda_2)l_f\bigl(\max(\|\eta\|_\infty,\|\hat\eta\|_\infty)\bigr)\|\varpi\|_\infty\|\eta-\hat\eta\|_\infty, \\
&\sup_{t>1}\bigl|F_2(\eta)(t)-F_2(\hat\eta)(t)\bigr| \\
&\quad\leqslant N(\alpha,-\lambda_2,b_\varpi)l_f\bigl(\max(\|\eta\|_\infty,\|\hat\eta\|_\infty)\bigr)\bigl(1+\|\varpi\|_\infty\bigr)\|\eta-\hat\eta\|_\infty.
\end{aligned}$$

So

$$\sup_{t\geqslant 0}\bigl|F_2(\eta)(t)-F_2(\hat\eta)(t)\bigr|\leqslant \Omega(\alpha,\lambda_1,\lambda_2,\varpi)l_f\bigl(\max(\|\eta\|_\infty,\|\hat\eta\|_\infty)\bigr)\|\eta-\hat\eta\|_\infty.$$

Consequently, we can get conclusion (5). Next, it is obvious that $F_1(0)(t)=0$, we derive

$$\begin{aligned}
\bigl|F_2(0)(t)\bigr| &\leqslant |x_2|+\bigl|\,tE_{\alpha,2}\bigl(t^\alpha\lambda_2\bigr)\bigr||x_4| \\
&\leqslant \|x\|+\left(E_{\alpha,2}(-\lambda_2)+m_2(\alpha,-\lambda_2)+\frac{1}{\lambda\Gamma(2-\alpha)}\right)|x_4|.
\end{aligned}$$

Hence, we get conclusion (6). The proof is completed. □

Before stating and proving the stable invariant manifold result, we show the following technical lemma.

Lemma 7. *For any function $g\in X_\infty(\mathbb{R}_+,\mathbb{R})$ and $\lambda>0$, it holds*

$$\begin{aligned}
&\lim_{p\to\infty}\frac{\int_0^p(p-s)^{\alpha-1}E_{\alpha,\alpha}((p-s)^\alpha\lambda)g(s)\,\mathrm{d}s}{pE_{\alpha,2}(\lambda p^\alpha)} \\
&\quad=\lambda^{(2-\alpha)/\alpha}\int_0^\infty\exp\bigl(-\lambda^{1/\alpha}s\bigr)g(s)\,\mathrm{d}s.
\end{aligned}$$

Proof. According to Lemma 2(i), we obtain $\lim_{p\to\infty}\exp(\lambda^{1/\alpha}p)/(pE_{\alpha,2}(\lambda p^\alpha))=\alpha\lambda^{1/\alpha}$. Next, since for $p>1$,

$$\left|\int_{p-1}^{p}(p-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((p-s)^\alpha\lambda\bigr)g(s)\,\mathrm{d}s\right|\leqslant E_{\alpha,\alpha+1}(\lambda)\|g\|_\infty,$$

then

$$\lim_{p\to\infty}\int_{p-1}^{p}\frac{(p-s)^{\alpha-1}E_{\alpha,\alpha}((p-s)^\alpha\lambda)}{\exp(\lambda^{1/\alpha}p)}g(s)\,\mathrm{d}s=0.$$

Next, we have

$$\lim_{p\to\infty}\left|\int_0^{p-1}\frac{(p-s)^{\alpha-1}E_{\alpha,\alpha}\big((p-s)^\alpha\lambda\big)-\frac{1}{\alpha}\lambda^{(1-\alpha)/\alpha}\exp(\lambda^{1/\alpha}(p-s))}{\exp(\lambda^{1/\alpha}p)}g(s)\,\mathrm{d}s\right|$$
$$\leqslant \lim_{p\to\infty}\int_0^{p-1}\frac{1}{\exp(\lambda^{1/\alpha}p)}\left(\frac{|g(s)|}{(t-s)\lambda}+\frac{m_3(\alpha,\lambda)|g(s)|}{(p-s)^{\alpha+1}}\right)\mathrm{d}s=0.$$

So we get

$$\lim_{p\to\infty}\int_0^{p-1}\frac{(p-s)^{\alpha-1}E_{\alpha,\alpha}((p-s)^\alpha\lambda)}{\exp(\lambda^{1/\alpha}p)}g(s)\,\mathrm{d}s=\frac{\lambda^{(1-\alpha)/\alpha}}{\alpha}\int_0^\infty\exp\big(-\lambda^{1/\alpha}s\big)g(s)\,\mathrm{d}s.$$

Finally,

$$\lim_{p\to\infty}\frac{\int_0^p(p-s)^{\alpha-1}E_{\alpha,\alpha}((p-s)^\alpha\lambda)g(s)\,\mathrm{d}s}{pE_{\alpha,2}(\lambda p^\alpha)}=\lambda^{(2-\alpha)/\alpha}\int_0^\infty\exp\big(-\lambda^{1/\alpha}s\big)g(s)\,\mathrm{d}s.$$

The proof is completed. □

Let $V\subset U\subset\mathbb{R}^2$ and $W\subset\mathbb{R}^2$ be open neighborhoods of zero. Define a center-stable manifold

$$W^{cs}(V\times W,U)=\big\{x\in V,\ \overline{x}\in W\colon\ \phi(t,x,\overline{x})\in U\ \ \forall t\geqslant 0\big\}.$$

Proposition 2. $(x,\overline{x})\in W^{cs}(V\times W,U)$ *if and only if* $\phi(\cdot,x,\overline{x})$ *is a fixed point of* F *along with* $\phi(t,x,\overline{x})\in U$ *for all* $t\geqslant 0$.

Proof. If $(x,\overline{x})\in W^{cs}(V\times W,U)$, then by (3) we get

$$\begin{aligned}\phi_1(t,x,\overline{x})&=E_{\alpha,2}\big(t^\alpha\lambda_1\big)tx_3\\&\quad+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big((t-s)^\alpha\lambda_1\big)f_1\big(\phi(s,x,\overline{x}),s\big)\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\phi_2(t,x,\overline{x})&=E_\alpha\big(t^\alpha\lambda_2\big)x_2+E_{\alpha,2}\big(t^\alpha\lambda_2\big)tx_4\\&\quad+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big((t-s)^\alpha\lambda_2\big)f_2\big(\phi(s,x,\overline{x}),s\big)\,\mathrm{d}s.\end{aligned}$$

By the above results we know that $\phi_2(\cdot,x,\overline{x})=F_2(\phi_2(\cdot,x,\overline{x}))$.

Furthermore, we arrive at

$$x_3 = -\lim_{t\to\infty} \frac{\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha \lambda_1) f_1(\phi(s,x,\overline{x}),s)\,\mathrm{d}s}{E_{\alpha,2}(t^\alpha \lambda_1)t}$$
$$= -\lambda_1^{(2-\alpha)/\alpha} \int_0^\infty \exp\bigl(-\lambda_1^{1/\alpha}s\bigr) f_1\bigl(\phi(s,x,\overline{x}),s\bigr)\,\mathrm{d}s$$

due to Lemma 7. Hence,

$$\begin{aligned}\phi_1(t,x,\overline{x}) &= -\lambda_1^{(2-\alpha)/\alpha} t E_{\alpha,2}\bigl(t^\alpha\lambda_1\bigr) \int_0^\infty \exp\bigl(-\lambda_1^{1/\alpha}s\bigr) f_1\bigl(\phi(s,x,\overline{x}),s\bigr)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl((t-s)^\alpha\lambda_1\bigr) f_1\bigl(\phi(s,x,\overline{x}),s\bigr)\,\mathrm{d}s \\ &= F_1\bigl(\phi_1(\cdot,x,\overline{x})\bigr).\end{aligned}$$

So $\phi(t,x,\overline{x})$ is a fixed point of F. Clearly, $\phi(t,x,\overline{x}) \in U$ for all $t \geqslant 0$.

On the other hand, let $\eta \in X_\infty(\mathbb{R}_+,\mathbb{R}^2) \cap X^1(\mathbb{R}_+,\mathbb{R}^2)$, $(\eta(0),\eta'(0)) \in V \times W$ be a fixed point of F such that $\eta(t) \in U$ for all $t \geqslant 0$. Then

$$\begin{aligned}\eta_1(t) &= -\lambda_1^{(2-\alpha)/\alpha} t E_{\alpha,2}\bigl(t^\alpha\lambda_1\bigr) \int_0^\infty \exp\bigl(-\lambda_1^{1/\alpha}s\bigr) f_1\bigl(\eta(s),s\bigr)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl((t-s)^\alpha\lambda_1\bigr) f_1\bigl(\eta(s),s\bigr)\,\mathrm{d}s, \\ \eta_2(t) &= E_\alpha\bigl(t^\alpha\lambda_2\bigr)x_2 + E_{\alpha,2}\bigl(t^\alpha\lambda_2\bigr)tx_4 \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl((t-s)^\alpha\lambda_2\bigr) f_2\bigl(\eta(s),s\bigr)\,\mathrm{d}s.\end{aligned}$$

Defining

$$x_3 = -\lambda_1^{(2-\alpha)/\alpha} \int_0^\infty \exp\bigl(-\lambda_1^{1/\alpha}s\bigr) f_1\bigl(\eta(s),s\bigr)\,\mathrm{d}s,$$

we get

$$\begin{aligned}\eta(t) &= E_\alpha\bigl(t^\alpha A\bigr)x + E_{\alpha,2}\bigl(t^\alpha A\bigr)t\overline{x} \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl((t-s)^\alpha A\bigr) f\bigl(\phi(s,x,\overline{x}),s\bigr)\,\mathrm{d}s\end{aligned}$$

with $\eta(0) = (0,x_2)^{\mathrm{T}}$ and $\eta'(0) = (x_3,x_4)^{\mathrm{T}}$, which is a bounded solution of (2). □

Now we are ready to state and prove the main result on stable manifolds.

Theorem 1. *Take* $R^* > 0$ *such that* $\sigma = l_f(R^*)\Omega(\alpha, \lambda_1, \lambda_2, \varpi) < 1$ *and set*

$$r = \frac{(1-\sigma)R^*}{1 + E_{\alpha,2}(-\lambda_2) + m_2(\alpha, -\lambda_2) + \frac{1}{\lambda\Gamma(2-\alpha)}},$$

$$R^{**} = \lambda_1^{(2-\alpha)/\alpha} l_f(R^*)R^* \int_0^\infty \exp\bigl(-\lambda_1^{1/\alpha}s\bigr)\varpi(s)\,\mathrm{d}s.$$

Then, for any $\tilde{x} = (x_1, x_2, x_4) \in (-r, r)^3$*, there exists a unique* $h(\tilde{x}) \in (-R^{**}, R^{**})$ *such that* $(\tilde{x}, h(\tilde{x})) \in W^{cs}(V \times W, U)$ *with* $V = (-r, r)^2$, $W = (-r, r) \times (-R^{**}, R^{**})$, *and* $U = (-R^*, R^*)^2$. *Furthermore,* $h : (-r, r)^3 \to (-R^{**}, R^{**})$ *satisfies the following properties*:

(i) $h(0) = 0$.

(ii) h *is Lipschitz continuous*:

$$\bigl|h(\tilde{x}) - h(\tilde{y})\bigr| \leqslant \frac{R^{**}}{R^*(1-\sigma)}\biggl[\|x - y\|_\infty + \biggl(E_{\alpha,2}(-\lambda_2) + m_2(\alpha, -\lambda_2) + \frac{1}{\lambda\Gamma(2-\alpha)}\biggr)|x_4 - y_4|\biggr]$$

for any $\tilde{x} = (x, x_4), \tilde{y} = (y, y_4) \in (-r, r)^3$.

Proof. Define $B_{R^*}(0) = \{\eta \in X_\infty(\mathbb{R}_+, \mathbb{R}^2)\colon \|\eta\|_\infty \leqslant R^*\}$. By Proposition 1 we have

$$\bigl\|F(\eta) - F(\hat{\eta})\bigr\|_\infty \leqslant \Omega(\alpha, \lambda_1, \lambda_2, \varpi) l_f(r^*)\|\eta - \hat{\eta}\|_\infty = \sigma\|\eta - \hat{\eta}\|_\infty,$$

and

$$\bigl\|F(\eta)\bigr\|_\infty < \biggl(1 + E_{\alpha,2}(-\lambda_2) + m_2(\alpha, -\lambda_2) + \frac{1}{\lambda\Gamma(2-\alpha)}\biggr)r + \Omega(\alpha, \lambda_1, \lambda_2, \varpi) l_f(R^*)R^* = R^*$$

for any $\tilde{x} \in (-r, r)^3$ and $\eta, \hat{\eta} \in B_{R^*}(0)$. We find $\|F(\eta)\|_\infty \leqslant r^*$, i.e., $F : B_{R^*}(0) \to B_{R^*}(0)$. The Banach fixed-point theorem uniquely determines $\eta_{\tilde{x}}$ by $\eta_{\tilde{x}} = F(\eta_{\tilde{x}})$ with $\|\eta_{\tilde{x}}\|_\infty < R^*$. We set

$$h(\tilde{x}) = -\lambda_1^{(2-\alpha)/\alpha} \int_0^\infty \exp\bigl(-\lambda_1^{1/\alpha}s\bigr) f_1\bigl(\eta_{\tilde{x}}(s), s\bigr)\,\mathrm{d}s. \tag{7}$$

Then

$$\bigl|h(\tilde{x})\bigr| \leqslant \lambda_1^{(2-\alpha)/\alpha} l_f(R^*)R^* \int_0^\infty \exp\bigl(-\lambda_1^{1/\alpha}s\bigr)\varpi(s)\,\mathrm{d}s = R^{**}.$$

Furthermore, Proposition 2 implies $(\tilde{x}, h(\tilde{x})) \in W^{cs}(V \times W, U)$. Clearly, (i) holds.

Next, we have $F = F_{\tilde{x}}$, in particular, $\eta_{\tilde{x}} = F_{\tilde{x}}(\eta_{\tilde{x}})$. Then, for any $\tilde{x} = (x, x_4), \tilde{y} = (y, y_4) \in (-r, r)^3$, it follows from Proposition 1 and definition of $F_{\tilde{x}}$ that

$$\begin{aligned}\|\eta_{\tilde{x}} - \eta_{\tilde{y}}\|_\infty &\leqslant \sigma\|\eta_{\tilde{x}} - \eta_{\tilde{y}}\|_\infty + \|x - y\|_\infty \\ &\quad + \left(E_{\alpha,2}(-\lambda_2) + m_2(\alpha, -\lambda_2) + \frac{1}{\lambda\Gamma(2-\alpha)}\right)|x_4 - y_4|\end{aligned}$$

since $F_{\tilde{x}}(\eta) - F_{\tilde{y}}(\eta) = F_{\tilde{x}-\tilde{y}}(0)$ for any $\eta \in X_\infty(\mathbb{R}_+, \mathbb{R}^2)$. This yields that

$$\begin{aligned}\|\eta_{\tilde{x}} - \eta_{\tilde{y}}\|_\infty \leqslant \frac{1}{1-\sigma}\bigg[\|x - y\|_\infty + \bigg(E_{\alpha,2}(-\lambda_2) + m_2(\alpha, -\lambda_2) \\ + \frac{1}{\lambda\Gamma(2-\alpha)}\bigg)|x_4 - y_4|\bigg].\end{aligned}$$

Hence,

$$\begin{aligned}\big|h(\tilde{x}) - h(\tilde{y})\big| \leqslant \frac{r^{**}}{r^*(1-\sigma)}\bigg[\|x - y\|_\infty + \bigg(E_{\alpha,2}(-\lambda_2) + m_2(\alpha, -\lambda_2) \\ + \frac{1}{\lambda\Gamma(2-\alpha)}\bigg)|x_4 - y_4|\bigg],\end{aligned}$$

which gives (ii). The proof is completed. □

To end this section, we give an example for which we can compute explicitly its stable manifold.

Example 1. Consider the following fractional damped equations motivated by

$$\begin{aligned}&{}^{c}D_t^{3/2}x_1(t) - x_1(t) = \frac{x_2(t)}{(1+t)^{2/5}}, \\ &{}^{c}D_t^{3/2}x_2(t) + x_2(t) = 0, \\ &x(0) = x = (0, x_2)^{\mathrm{T}}, \qquad x'(0) = \overline{x} = (x_3, x_4)^{\mathrm{T}}.\end{aligned}$$

Set $\alpha = 3/2$ and $\varpi(t) = 1/(1+t)^{2/5}$. Next, we have

$$\int_0^{t-1} \frac{1}{t-s}\varpi(s)\,\mathrm{d}s \leqslant \int_0^t (t-s)^{-7/10}s^{-2/5}\,\mathrm{d}s \leqslant B\left(\frac{3}{10}, \frac{3}{5}\right) = 4.16891 := b_\varpi$$

for $t \geqslant 1$, where B is the beta function. By (3) the solution $\phi(t, x, \overline{x})$ is given by (see Fig. 1)

$$\begin{aligned}\phi_1(t, x, \overline{x}) &= E_{3/2,2}(t^{3/2})tx_3 + \int_0^t (t-s)^{1/2}E_{3/2,3/2}(t-s)^{3/2}\frac{\phi_2(s, x, \overline{x})}{(1+s)^{2/5}}\,\mathrm{d}s, \\ \phi_2(t, x, \overline{x}) &= E_{3/2}\big(-t^{3/2}\big)x_2 + E_{3/2,2}\big(-t^{3/2}\big)tx_4.\end{aligned}$$

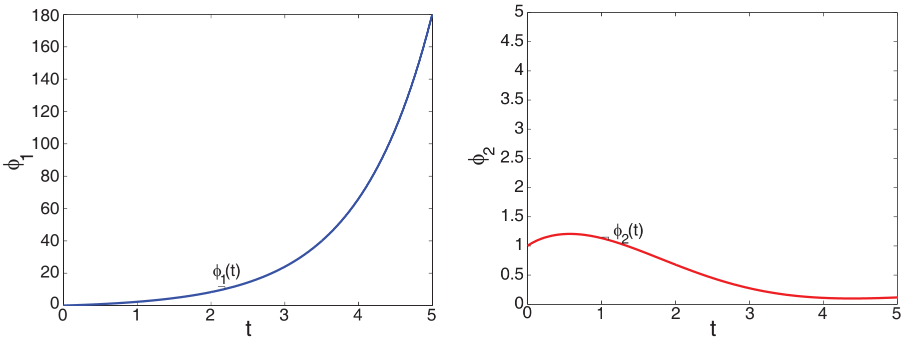

Figure 1. The solution ϕ_1 and ϕ_2.

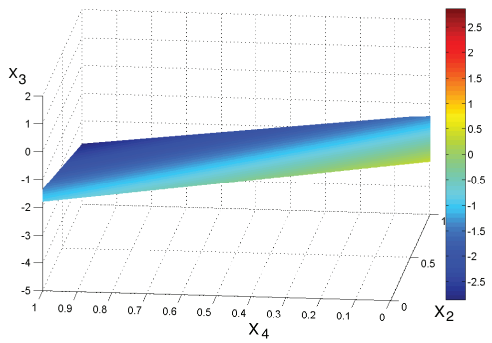

Figure 2. $x_2, x_4 \in [0, 1]$, $x_3 = -1.4722x_2 - 1.3737x_4$.

Hence, the Lyapunov–Perron operator F has the form

$$F_1(\xi)(t) = -E_{3/2,2}\left(t^{3/2}\right)t\int\limits_0^\infty \exp(-s)\frac{\phi_2(s,x,\overline{x})}{(1+s)^{2/5}}\,\mathrm{d}s + \int\limits_0^t (t-s)^{1/2}E_{3/2,3/2}(t-s)^{3/2}\frac{\phi_2(s,x,\overline{x})}{(1+s)^{2/5}}\,\mathrm{d}s,$$

$$F_2(\xi)(t) = E_{3/2}\left(-t^{3/2}\right)x_2 + E_{32,2}\left(-t^{3/2}\right)tx_4.$$

By (7) we derive $h(\tilde{x}) = -1.4722x_2 - 1.3737x_4$.

Consequently, Proposition 2 and Theorem 1 imply that the local center-stable manifold around the origin is given by $\{(0, x_2, -1.4722x_2 - 1.3737x_4, x_4)\}$, where x_2, x_4 are small enough (see Fig. 2).

4 High-dimensional case

In this section, we extend the result for planar fractional differential equations in [6] to high-dimensional case:

$$\begin{aligned} &{}^cD_{0+}^{\alpha}x(t) = Ax(t) + f\bigl(x(t)\bigr), \quad \alpha \in (0,1),\\ &x(0) = x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}, \end{aligned} \tag{8}$$

where $A = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_1, \lambda_2, \dots, \lambda_r > 0$ and $\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_n < 0$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function: $\|f(x) - f(y)\| \leqslant l_f(r)\|x-y\|$, $\|x\|, \|y\| \leqslant r$ with $f(0) = 0$, $\lim_{r\to 0} l_f(r) = 0$. The solution of (8) is formulated by

$$\varphi(t,x) = E_{\alpha}\bigl(t^{\alpha}A\bigr)x + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((t-s)^{\alpha}A\bigr)f\bigl(\varphi(s,x)\bigr)\,\mathrm{d}s.$$

For $i = 1, 2, \dots, r$, set

$$\varphi_i(t,x) = E_{\alpha}\bigl(t^{\alpha}\lambda_i\bigr)x_i + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((t-s)^{\alpha}\lambda_i\bigr)f_i\bigl(\varphi(s,x)\bigr)\,\mathrm{d}s.$$

For $j = r+1, r+2, \dots, n$, $i + j = n$, set

$$\varphi_j(t,x) = E_{\alpha}\bigl(t^{\alpha}\lambda_j\bigr)x_j + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((t-s)^{\alpha}\lambda_j\bigr)f_j\bigl(\varphi(s,x)\bigr)\,\mathrm{d}s.$$

Consider the Lyapunov–Perron operator $T : X_{\infty}(\mathbb{R}_+, \mathbb{R}^n) \to X_{\infty}(\mathbb{R}_+, \mathbb{R}^n)$.
For $i = 1, 2, \dots, r$,

$$\begin{aligned} (T\xi)_i(t) = {}&\Biggl(-\lambda_i^{1/\alpha-1}\int_0^{\infty} \mathrm{e}^{-\lambda_i^{1/\alpha}s} f_i\bigl(\xi(s)\bigr)\,\mathrm{d}s\Biggr)E_{\alpha}\bigl(t^{\alpha}\lambda_i\bigr)\\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(\lambda_i(t-s)^{\alpha}\bigr)f_i\bigl(\xi(s)\bigr)\,\mathrm{d}s. \end{aligned}$$

For $j = r+1, r+2, \dots, n$,

$$(T\xi)_j(t) = E_{\alpha}\bigl(t^{\alpha}\lambda_j\bigr)x_j + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((t-s)^{\alpha}\lambda_j\bigr)f_j\bigl(\xi(s)\bigr)\,\mathrm{d}s.$$

Denote $C(\alpha, \lambda_1, \dots, \lambda_n) = \max\{2K(\alpha, \lambda_i), 2M(\alpha, \lambda_i), K(\alpha, -\lambda_j), M(\alpha, -\lambda_j)\}$, where K, M are defined in [6, pp. 161–162].

Theorem 2. *For $x_j, \hat{x}_j \in \mathbb{R}$, $\xi, \hat{\xi} \in X_\infty(\mathbb{R}_+, \mathbb{R}^n)$, there exists the following corresponding result*:

$$\begin{aligned}&\|T\xi - T\hat{\xi}\|_\infty \\ &\quad \leqslant |x_k - \hat{x}_k| + C(\alpha, \lambda_1, \ldots, \lambda_n) l_f\bigl(\max\bigl(\|\xi\|_\infty, \|\hat{\xi}\|_\infty\bigr)\bigr)\|\xi - \hat{\xi}\|_\infty. \end{aligned} \tag{9}$$

In particular,

$$\|T\xi\|_\infty \leqslant |x_k| + C(\alpha, \lambda_1, \ldots, \lambda_n) l_f\bigl(\|\xi\|_\infty\bigr)\|\xi\|_\infty,$$

where $|x_k - \hat{x}_k| = \max\{|x_j - \hat{x}_|\}$, $k = r+1, r+2, \ldots, n$.

Proof. Recall $\|\xi\|_\infty = \sup_{t\in\mathbb{R}} \|\xi(t)\| = \max(\|\xi_1\|, \|\xi_2\|, \ldots, \|\xi_n\|)$. One can derive

$$\begin{aligned}\bigl|(T\xi)_i - (T\hat{\xi})_i\bigr| \leqslant{}& \Biggl|\int_0^t \bigl(f_i\bigl(\varphi(s)\bigr) - f_i\bigl(\hat{\varphi}(s)\bigr)\bigr)\bigl((t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(\lambda_i(t-s)^\alpha\bigr) \\ &- \lambda_i^{1/\alpha-1}E_\alpha\bigl(\lambda_i t^\alpha\bigr)\mathrm{e}^{-\lambda_i^{1/\alpha}s}\bigr)\,\mathrm{d}s\Biggr| \\ &+ \Biggl|\lambda_i^{1/\alpha-1}E_\alpha\bigl(\lambda_i t^\alpha\bigr)\int_0^\infty \mathrm{e}^{-\lambda_i^{1/\alpha}s}\bigl(f_i\bigl(\xi(s)\bigr) - f_i\bigl(\hat{\xi}(s)\bigr)\bigr)\,\mathrm{d}s\Biggr|.\end{aligned}$$

Next, we can use [6, Lemmas 5, 6] to estimate the following formulas:

$$\Biggl|\int_0^t \bigl((t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(\lambda_i(t-s)^\alpha\bigr) - \lambda_i^{1/\alpha-1}E_\alpha\bigl(\lambda_i t^\alpha\bigr)\mathrm{e}^{-\lambda_i^{1/\alpha}s}\bigr)g(s)\,\mathrm{d}s\Biggr|$$

and

$$\lambda_i^{1/\alpha-1}E_\alpha\bigl(\lambda_i t^\alpha\bigr)\Biggl|\int_0^\infty \mathrm{e}^{-\lambda_i^{1/\alpha}s}g(s)\,\mathrm{d}s\Biggr|.$$

Thus, using the above facts, we derive that

$$\begin{aligned}\sup_{t\in[0,1]}\bigl|(T\xi)_i(t) - (T\hat{\xi})_i(t)\bigr| &\leqslant 2K(\alpha,\lambda_i) l_f\bigl(\max\bigl(\|\xi\|_\infty, \|\hat{\xi}\|_\infty\bigr)\bigr)\|\xi - \hat{\xi}\|_\infty, \\ \sup_{t>1}\bigl|(T\xi)_i(t) - (T\hat{\xi})_i(t)\bigr| &\leqslant 2M(\alpha,\lambda_i) l_f\bigl(\max\bigl(\|\xi\|_\infty, \|\hat{\xi}\|_\infty\bigr)\bigr)\|\xi - \hat{\xi}\|_\infty.\end{aligned}$$

This yields that

$$\sup_{t\geqslant 0}\bigl|(T\xi)_i(t) - (T\hat{\xi})_i(t)\bigr| \leqslant C(\alpha,\lambda_1,\ldots,\lambda_n) l_f\bigl(\max\bigl(\|\xi\|_\infty, \|\hat{\xi}\|_\infty\bigr)\bigr)\|\xi - \hat{\xi}\|_\infty.$$

On the other hand,

$$\begin{aligned}&(T\xi)_j(t) - (T\hat{\xi})_j(t) \\ &\quad = E_\alpha(t^\alpha\lambda_j)(x_j - \hat{x}_j) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl((t-s)^\alpha\lambda_j\bigr)\bigl(f_j\bigl(\xi(s)\bigr) - f_j\bigl(\hat{\xi}(s)\bigr)\bigr)\,\mathrm{d}s.\end{aligned}$$

Noticing the fact $\sup_{t\geqslant 0} E_\alpha(-\lambda t^\alpha) = 1$ ($\lambda > 0$), we have

$$\sup_{t\geqslant 0} \bigl|(T\xi)_j(t) - (T\hat{\xi})_j(t)\bigr| \\ \leqslant |x_j - \hat{x}_j| + C(\alpha, \lambda_1, \ldots, \lambda_n) l_f\bigl(\max\bigl(\|\xi\|_\infty, \|\hat{\xi}\|_\infty\bigr)\bigr)\|\xi - \hat{\xi}\|_\infty.$$

This reduces to

$$\|T\xi - T\hat{\xi}\|_\infty \leqslant |x_k - \hat{x}_k| + C(\alpha, \lambda_1, \ldots, \lambda_n) l_f\bigl(\max\bigl(\|\xi\|_\infty, \|\hat{\xi}\|_\infty\bigr)\bigr)\|\xi - \hat{\xi}\|_\infty.$$

Note that $T(0) = 0$, $(T\xi)_j(0) = E_\alpha(0)x_j = 0$, and due to (9), we can get $x_j = 0$.

Moreover, $\|T\xi\|_\infty \leqslant |x_k| + C(\alpha, \lambda_1, \ldots, \lambda_n) l_f(\|\xi\|_\infty)\|\xi\|_\infty$. The proof is completed. □

Theorem 3. *Denote* $W^s(U) = \{x \in U\colon \varphi(t, x) \in U,\ \lim_{t\to\infty} \varphi(t, x) = 0\}$.

(i) *If* $x \in W^s(U)$*, then* φ *is the fixed point of* T.

(ii) *If* $\xi(t)$ *is a fixed point of* T*, then* $\xi(t)$ *is a solution of (8) with* $\xi_j(0) = x_j$*. Furthermore, suppose additionally that* $\|\xi\|_\infty < r^*$*, where satisfied that* $l_f(r^*)C(\alpha, \lambda_1, \lambda_2, \ldots, \lambda_n) < 1$.

Then $\lim_{t\to\infty} \xi(t) = 0$.

Proof. (i) Since $x \in W^s(U)$, $\lim_{t\to\infty} \varphi(t, x) = 0$. That is,

$$\lim_{t\to\infty} \varphi_i(t, x) \\ = \lim_{t\to\infty} \Biggl[E_\alpha(t^\alpha \lambda_i)\Biggl(x_i + \int_0^t (t - s)^{\alpha-1} \frac{E_{\alpha,\alpha}((t - s)^\alpha \lambda_i)}{E_\alpha(t^\alpha \lambda_i)} f_i\bigl(\varphi(s, x)\bigr)\,\mathrm{d}s\Biggr)\Biggr] = 0.$$

According to $\lambda > 0$, $\lim_{t\to\infty} E_\alpha(t^\alpha \lambda) = \infty$, so we can get

$$x_1 = -\lim_{u\to\infty} \int_0^t (t - s)^{\alpha-1} \frac{E_{\alpha,\alpha}(\lambda_i(t - s)^\alpha f_i(\varphi(s, x)))}{E_\alpha(t^\alpha \lambda_i)}\,\mathrm{d}s,$$

Linking with [6, Lemma 8], when $\lambda > 0$,

$$\lim_{u\to\infty} \int_0^u (u - s)^{\alpha-1} \frac{E_{\alpha,\alpha}(\lambda(u - s)^\alpha)}{E_\alpha(u^\alpha \lambda)} g(s)\,\mathrm{d}s = \lambda^{1/\alpha-1} \int_0^\infty \mathrm{e}^{-\lambda^{1/\alpha} s} g(s)\,\mathrm{d}s.$$

Then $(T\varphi)_i(t) = \varphi_i(t, x)$, and it is clearly $(T\varphi)_j(t) = \varphi_j(t, x)$. So the proof of (i) is completed.

(ii) Since $\xi(t)$ is a fixed point of T, we can get

$$\begin{aligned}(T\xi_i)(t) &= \xi_i(t) \\ &= \left(-\lambda_i^{1/\alpha-1}\int_0^\infty \mathrm{e}^{-\lambda_i^{1/\alpha}s} f_i\big(\xi(s)\big)\,\mathrm{d}s\right) E_\alpha\big(t^\alpha\lambda_i\big) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(\lambda_i(t-s)^\alpha\big) f_i\big(\xi(s)\big)\,\mathrm{d}s.\end{aligned}$$

Now we set $x_i = -\lambda_i^{1/\alpha-1}\int_0^\infty \mathrm{e}^{-\lambda_i^{1/\alpha}s} f_i(\xi(s))\,\mathrm{d}s$. On the other hand,

$$(T\xi_j)(t) = \xi_j(t) = E_\alpha(t^\alpha\lambda_j)x_j + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big((t-s)^\alpha\lambda_j\big) f_j(\xi(s))\,\mathrm{d}s.$$

It is easy to get $\xi_j(0) = x_j$, and thus we get

$$\xi(t) = E_\alpha(t^\alpha A)x + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big((t-s)^\alpha A\big) f\big(\xi(s)\big)\,\mathrm{d}s,$$

which is a solution of (8).

We can define $\lim_{t\to\infty}\sup\|\xi(t)\| = a \in [0, r^*]$, there exists a sequence t_n, such that $\lim_{n\to\infty}\sup\|\xi(t_n)\| = a$ as $\lim_{n\to\infty} t_n = \infty$. In other words, for all $\epsilon > 0$, there exists $T(\epsilon) > 0$ when $t > T$, $\|\xi(t)\|_\infty < a + \epsilon$.

We note that $l_f(a+\epsilon)C(\alpha, \lambda_1, \lambda_2, \dots, \lambda_n) = q < 1$. According to [6, Lemmas 3, 6, 7],

$$\left|\int_{T(\epsilon)}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big((t-s)^\alpha\lambda_j\big) f_j\big(\xi(s)\big)\,\mathrm{d}s\right| \leqslant C(\alpha, \lambda_1, \lambda_2, \dots, \lambda_n),$$

we can get

$$\begin{aligned}\lim_{t\to\infty}\sup\big|\xi_j(t)\big| &\leqslant \max\big|f_j\big(\xi(t)\big)\big| \lim_{t\to\infty}\sup\left|\int_0^{T(\epsilon)} \frac{m(\alpha,\lambda_j)}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right. \\ &\quad \left. + \int_{T(\epsilon)}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big((t-s)^\alpha\lambda_j\big) f_j\big(\xi(s)\big)\,\mathrm{d}s\right| \\ &\leqslant (a+\epsilon) l_f(a+\epsilon) C(\alpha, \lambda_1, \lambda_2, \dots, \lambda_n) = q(a+\epsilon).\end{aligned}$$

For any function $\hat{\xi} \in C_\infty(\mathbb{R}^n)$, we can get

$$\lim_{t\to\infty}\sup\left|\int_0^{T(\epsilon)} \big((t-s)^{\alpha-1} E_{\alpha,\alpha}\big((t-s)^\alpha\lambda_i\big) - \lambda_i^{1/\alpha-1} E_\alpha\big(t^\alpha\lambda_i\big)\mathrm{e}^{-\lambda_i^{1/\alpha}s}\big) f_i\big(\hat{\xi}(s)\big)\,\mathrm{d}s\right|$$

$$\leqslant \lim_{t\to\infty}\sup\int\limits_0^{T(\epsilon)}\left(\frac{m(\alpha,\lambda)}{(t-s)^{1+\alpha}}+\frac{1}{\alpha}\lambda_i^{1/\alpha-1}\mathrm{e}^{\lambda_i^{1/\alpha}(t-s)}-\frac{1}{\alpha}\lambda_i^{1/\alpha-1}\mathrm{e}^{\lambda_i^{1/\alpha}t}\mathrm{e}^{-\lambda_i^{1/\alpha}s}\right.$$
$$\left.+\frac{m(\alpha,\lambda)}{t^\alpha}\mathrm{e}^{-\lambda_i^{1/\alpha}s}\lambda_i^{1/\alpha-1}\right)\left|f_i\big(\hat{\xi}(s)\big)\,\mathrm{d}s\right|=0.$$

Note the fact $\lim_{t\to\infty}\sup|(T\xi)_i(t)| = \lim_{t\to\infty}\sup|(T\hat{\xi})_i(t)|$, where $\hat{\xi}(t) = \xi(t)$ $(t > T(\epsilon))$, and $\hat{\xi}(t) = (t/T(\epsilon))\xi(T(\epsilon))$ $(t \in (0, T(\epsilon))$.

In view of $\lim_{t\to\infty}\sup|\xi_i(t)| = \lim_{t\to\infty}\sup|(T\hat{\xi})_i(t)|$. Let $\hat{\xi}$ be corresponding to $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ and set $\hat{x}_k = 0$. By using

$$\|T\hat{\xi}\|_\infty \leqslant C(\alpha,\lambda_1,\dots,\lambda_n)l_f\big(\|\hat{\xi}\|_\infty\big)\|\hat{\xi}\|_\infty = (a+\epsilon)l_f(a+\epsilon)C(\alpha,\lambda_1,\dots,\lambda_n)$$

we derive that $\lim\limits_{t\to\infty}\sup|\xi_i(t)| \leqslant (a+\epsilon)q$ and

$$\lim_{t\to\infty}\sup\|\xi_i(t)\| = \max\Big(\lim_{t\to\infty}\sup\big|\xi_1(t)\big|, \lim_{t\to\infty}\sup\big|\xi_2(t)\big|\Big) = a.$$

Finally, we can get $a \leqslant q(a+\epsilon)$, which implies that $(1-q)a \leqslant 0$ as $\epsilon \to 0$. Due to the condition $q < 1$, we derive that $a = 0$. So the proof is completed. □

Define $B_{r^*}(0) = \{\xi \in X_\infty(\mathbb{R}^+, \mathbb{R}^n)\colon \|\xi\|_\infty \leqslant r^*\}$.

Theorem 4. *Let $r^* > 0$, which satisfies $l_f(r^*)C(\alpha,\lambda_1,\lambda_2,\dots,\lambda_n) < 1$, and set $r = (1-q)r^*$. Then, for for all $x_k \in [-r, r]$, there exists $s_1(x_k), s_2(x_k), \dots, s_r(x_k) \in [-r^*, r^*]$ such that $(s_1(x_k), s_2(x_k), \dots, s_r(x_k), x_{r+1}, \dots, x_n) \in W^s(B_{r^*}(0))$, where $s : [-r, r] \to [-r^*, r^*]$ and satisfies the following properties*:

(i) $s_i(0) = 0$.

(ii) *s is a Lipschitz continuous function, it means that, for $x_k, \hat{x}_k \in [-r, r]$, we have*

$$\big|s_i(x_k) - s_i(\hat{x}_k)\big| \leqslant \frac{1}{1 - l_f(r^*)C(\alpha,\lambda_1,\lambda_2,\dots,\lambda_n)}|x_k - \hat{x}_k|.$$

Proof. (i) Let $\xi \in W^s(B_{r^*}(0))$ be the solution. Then ξ is a fixed point of T. Now, we show that $T : B_{r^*}(0) \to B_{r^*}(0)$, let $\xi, \hat{\xi} \in B_{r^*}(0)$, $x_k = \hat{x}_k$. By using elementary computation we have

$$\|T\xi - T\hat{\xi}\|_\infty \leqslant C(\alpha,\lambda_1,\dots,\lambda_n)l_f\big(\max(\|\xi\|_\infty, \|\hat{\xi}\|_\infty)\big)\|\xi - \hat{\xi}\|_\infty = q\|\xi - \hat{\xi}\|_\infty,$$

and $\|T\xi\|_\infty \leqslant |x_k| + C(\alpha,\lambda_1,\dots,\lambda_n)l_f(r^*)r^* \leqslant r + qr^*$. Due to $r^*(1-q) = r$, $\|T\xi\|_\infty \leqslant r^*$. By using the contraction mapping theorem there exists a unique fixed point $\xi \in B_{r^*}(0)$. According to the definition of T, we can know $T(0) = 0$,$(0, 0, \dots, 0) \in W^s(B_{r^*}(0))$, $s_i(0) = 0$, $i = 1, 2, \dots r$.

(ii) It is easy to see that $\|T\xi - T\hat{\xi}\|_\infty = \|\xi - \hat{\xi}\|_\infty \leqslant |x_k - \hat{x}_k| + q\|\xi - \hat{\xi}\|_\infty$. Thus, $\|\xi - \hat{\xi}\|_\infty \leqslant 1/(1-q)|x_k - \hat{x}_k|$, which yields that $|s_i(x_k) - s_i(\hat{x}_k)| \leqslant \|\xi - \hat{\xi}\|_\infty \leqslant 1/(1-q)|x_k - \hat{x}_k|$. The proof is completed. □

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