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Global exponential stability of positive periodic solutions for a cholera model with saturated treatment*

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Abstract. In this paper, we consider a cholera model with periodic incidence rate and saturated treatment function. Under certain conditions, we establish a criterion on the global exponential stability of positive periodic solutions for this model by using a novel method. We illustrate our theoretical results with numerical simulations by using Matlab.

Keywords: cholera model, global exponential stability, periodic incidence rate, positive periodic solution.

1 Introduction

Cholera is an acute intestinal infectious disease caused by infection of the bacterium Vibrio cholerae, such as Vibrio cholerae serogroups O1 and O139, which is the major public health problem and affect primarily developing world populations with no proper access to adequate water and sanitation resources. Once they colonise the intestinal gut, then produce enterotoxin (which stimulates water and electrolyte secretion by the endothelial cells of the small intestine) that leads to copious, painless, and watery diarrhoea that can quickly lead to severe dehydration and death if treatment isn't promptly given [4]. Up to now, the control of deadly outbreaks remains a challenge. In recent years, the number of cholera cases reported to World Health Organization (WHO) is on the increase. In 2015, 172454 cases and 1304 deaths of cholera were reported to WHO worldwide [25]. Outbreaks continued to affect several countries [25]. Overall, 41% of cases were reported from Africa, 37% from Asia, and 21% from the Americas [25]. So, cholera is also a global threat to public health, and it is one of the important indicators of social development.

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Mathematical models have been proven to be central importance for understanding dynamical behavior of the epidemic spreading in the infectious diseases [12, 21, 27]. The mathematical model of cholera epidemics pandemic was first proposed by Capasso et al in 1979 [2]. Some researchers considered a cholera model with imperfect vaccination, which studied the stability of a disease-free equilibrium and an endemic equilibrium [4, 14, 22, 23, 26, 29, 30]. Also, the literature [22] analysed control strategies of cholera. Mwasa et al. formulated a mathematical model that captures some essential dynamics of cholera transmission to study the impact of some control strategies, such as public health educational campaigns, vaccination and treatment in reducing the incidence of disease [15, 20]. In [16], Safi presented a new two-strain model, for assessing the impact of basic control measures, treatment and dose-structured mass vaccination on cholera transmission dynamics in a population. In [10], Khan et al. studied the dynamical behavior of cholera epidemic model with nonlinear incidence rate. To the best of our knowledge, these is no paper to consider a cholera model with both periodic incidence rate and saturated treatment function.

As it is well known, many infectious diseases exhibit seasonal fluctuations, and there is a saturation phenomenon during the treatment process. Therefore, the coefficients in the differential equations of ecology, epidemics, and population problems are usually time-varying. Usually, we use the periodic coefficients. So, to describe the dynamics of the cholera, we consider the following model:

$$\begin{aligned} S'(t) &= A(t) - \mu_1(t)S(t) - \frac{\beta(t)S(t)B(t)}{K(t)+B(t)}, \\ I'(t) &= \frac{\beta(t)S(t)B(t)}{K(t)+B(t)} - \big(\gamma(t)+\mu_1(t)+\delta(t)\big)I(t) - \frac{\vartheta(t)I(t)}{1+\alpha(t)I(t)}, \\ R'(t) &= \gamma(t)I(t) + \frac{\vartheta(t)I(t)}{1+\alpha(t)I(t)} - \mu_1(t)R(t), \\ B'(t) &= \eta(t)I(t) - \mu_2(t)B(t) \end{aligned} \tag{1}$$

Here S represents the number of individuals susceptible to the disease, I represents the number of infected individuals infectious and able to spread the disease by contacting with the susceptibles, R is the number of the infectives removed or recovered, and B is the number of the pathogen population. In this paper, $A, K, \mu_1, \mu_2, \gamma, \delta, \vartheta, \alpha, \eta : \mathbb{R} \to (0, \infty)$ and $\beta : \mathbb{R} \to \mathbb{R}_+$ are continuous T-periodic functions with $T > 0$. Moreover,the natural human birth rate $A(t)$ is $a + b\,\mathrm{term}(t)$ and $\mathrm{term}(t) = \sin(2\pi t/365)$ is a periodic function, and a and b are positive. The contact rate $\beta(t)$ is $c + d\,\mathrm{term}(t)$ and $\mathrm{term}(t) = \cos(2\pi t/365)$ is a periodic function, and c and d are positive.

The notation $\mathbb{R}$ and $\mathbb{R}_+$ refers to the space of real number and nonnegative real number, respectively. The notation $\mathbb{R}^n$ and $\mathbb{R}^n_+$ refers to the space of n-dimensional real column vector and n-dimensional nonnegative real column vector, respectively. The notation $\mathbb{R}^{n\times n}$ refers to the $n\times n$ nonnegative real matrix space. For any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, let $|x|$ denotes the absolute-value vector given by $|x| = (|x_1|, |x_2|, \dots, |x_n|)$, "T" denotes the transpose ($x^{\mathrm{T}} = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$), and we define $\|x\| = \max_{i\in\{1,2,\dots,n\}} |x_i|$. If $A \in \mathbb{R}^{m\times n}$, A^{T} refers to the transpose of A.

The initial conditions associated with (1) are as follows:

$$S(t_0) > 0, \qquad I(t_0) > 0, \qquad B(t_0) \geqslant 0, \qquad R(t_0) \geqslant 0. \tag{2}$$

For simplicity, we first assume that a bounded continuous function g defined on $\mathbb{R}$ given by

$$g^+ = \sup_{t\in\mathbb{R}} \bigl|g(t)\bigr| \quad \text{and} \quad g^- = \inf_{t\in\mathbb{R}} \bigl|g(t)\bigr|.$$

In the following, we will always assume that

$$\frac{\eta(t)}{K(t)\mu_2(t)} \leqslant \alpha(t) \quad \text{for all } t \in \mathbb{R}. \tag{3}$$

2 Preliminaries and lemmas

Firstly, we show that the existence of the disease free periodic solution of (1). To find the disease free periodic solution of (1), we consider the following equation:

$$S'(t) = A(t) - \mu_1(t)S(t), \tag{4}$$

with initial condition $S(0) = S^0 \in \mathbb{R}_+$. (4) admits a unique positive ω-periodic solution $S^*(t) > 0$, which is globally attractive in $\mathbb{R}_+$ and hence, (1) has a unique disease free periodic solution $(S^*(t), 0, 0, 0)$.

Let us define the basic reproduction number of (1), by applying the theory in Wang and Zhang [28] with

$$\begin{gathered}
\mathcal{F}(x) = \begin{pmatrix} \frac{\beta(t)S(t)B(t)}{K(t)+B(t)} \\ \eta(t)I \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{V}^+(x) = \begin{pmatrix} 0 \\ 0 \\ A(t) \\ \gamma(t)I + \frac{\vartheta(t)I}{1+\alpha(t)I} \end{pmatrix}, \\
\mathcal{V}^-(x) = \begin{pmatrix} (\gamma(t) + \mu_1(t) + \delta(t))I + \frac{\vartheta(t)I}{1+\alpha(t)I} \\ \mu_2 B(t) \\ \frac{\beta(t)SB}{K(t)+B} + \mu_1(t)S \\ \mu_1 R \end{pmatrix},
\end{gathered} \tag{5}$$

where $x = (I, B, S, R)^{\mathrm{T}}$. For our purpose, we check conditions (A1)–(A7) in Section 1 of [28]. (1) is equivalent to the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \mathcal{F}\bigl(t, x(t)\bigr) - \mathcal{V}\bigl(t, x(t)\bigr). \tag{6}$$

where $\mathcal{V}(t, x(t)) = \mathcal{V}^-(t, x(t)) - \mathcal{V}^+(t, x(t))$. It is easy to see that conditions (A1)–(A5) are satisfied.

We know that (6) has the disease free periodic solution $x^*(t) = (S^*(t), 0, 0, 0)$. Now, we define $f(t, x(t)) = \mathcal{F}(t, x(t)) - \mathcal{V}(t, x(t))$ and $M(t) = (\partial f_i(t, x^*(t))/\partial x_j)_{3\leqslant i,j\leqslant 4}$, where $f_i(t, x(t))$ and x_i is the ith component of $f(t, x(t))$ and x, respectively. From (5) we can get

$$M(t) = \begin{pmatrix} -\mu_1(t) & 0 \\ 0 & -\mu_1(t) \end{pmatrix},$$

and hence, $r(\Phi_M(\omega)) < 1$, which implies that $x^*(t)$ is linearly asymptotically stable in the disease free subspace $X_s = (0, 0, S, R) \in \mathbb{R}^4_+$. Thus, the condition (A6) also holds.

Next, we set $F(t)$ and $V(t)$ are 2×2 matrices defined by $F(t) = (\partial \mathcal{F}_i(t, x^*(t))/\partial x_j)_{1 \leqslant i,j \leqslant 2}$ and $V(t) = (\partial \mathcal{V}_i(t, x^*(t))/\partial x_j)_{1 \leqslant i,j \leqslant 2}$, where $\mathcal{F}_i(t, x)$ and $\mathcal{V}_i(t, x)$ is the ith component of $\mathcal{F}(t, x)$ and $\mathcal{V}(t, x)$, respectively. Then from (2.4) it follows that

$$F = \begin{pmatrix} 0 & \frac{\beta(t)}{K(t)} S^*(t) \\ \eta(t) & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} \gamma(t) + \mu_1(t) + \delta(t) + \vartheta(t) & 0 \\ 0 & \mu_2(t) \end{pmatrix}.$$

Let $Y(t, s)$ is a 2×2 matrix solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t, s) = -V(t) Y(t, s)$$

for any $t \geqslant s, Y(s, s) = I$, where I is 2×2 identity matrix. Therefore, condition (A7) holds.

Let C_ω be the ordered Banach space of all ω-periodic function from $\mathbb{R} \to \mathbb{R}^2$, which is equipped with maximum norm $\|\cdot\|$ and the positive cone $C^\omega_+ = \{\phi \in C_\omega \colon \phi(t) \geqslant 0$ for any $t \in \mathbb{R}\}$. Consider the following linear operator $L : C_\omega \to C_\omega$ by

$$(L\phi)(t) = \int_0^{+\infty} Y(t,\, t-a) F(t,\, t-a) \phi(t, t-a)\, \mathrm{d}a \quad \text{for any } t \in \mathbb{R},\ \phi \in C_\omega.$$

Finally, we can define the basic reproduction number $\Re_0$ of (1) as follows:

$$\Re_0 = r(L).$$

From the above discussion, we obtain the following result for the local asymptotic stability of the disease free periodic solution $(S^*(t), 0, 0, 0)$ for (1).

Theorem 1. (*See* [28, *Thm.* 2.2].) *The following statements are valid*:

(i) $\Re_0 = 1$ *if and only if* $r(\phi_{F-V}(\omega)) = 1$.
(ii) $\Re_0 > 1$ *if and only if* $r(\phi_{F-V}(\omega)) > 1$.
(iii) $\Re_0 < 1$ *if and only if* $r(\phi_{F-V}(\omega)) < 1$.

Thus, $(S^*(t), 0, 0, 0)$ *of* (1) *is asymptotically stable if* $\Re_0 < 1$*, and unstable if* $\Re_0 > 1$.

Lemma 1. *Every solution of system* (1) *with initial value conditions* (2) *is positive and bounded on* (t_0, ∞).

Proof. From Theorem 1.3.1 in [6] we can deduce that there exists a unique solution $(S(t,t_0,x_0), I(t,t_0,x_0), R(t,t_0,x_0), B(t,t_0,x_0))$ of (1) passing through (t_0,x_0) with initial value $x_0 = (S(t_0), I(t_0), R(t_0), B(t_0)$ satisfying (2). Let $[t_0,T^*)$ be the maximal right-interval of existence of

$$\bigl(S(t), I(t), R(t), B(t)\bigr) = \bigl(S(t,t_0,x_0), I(t,t_0,x_0), R(t,t_0,x_0), B(t,t_0,x_0)\bigr).$$

We first prove that

$$S(t) > 0 \quad \text{for all } t \in [t_0,T^*). \tag{7}$$

Assume, by way of contradiction, that (7) doesn't hold. Then there must exist $T_1 \in [t_0,T^*)$ such that

$$S(T_1) = 0, S(s) > 0 \quad \text{for all } s \in [t_0,T_1),\ S'(T_1) \leqslant 0.$$

But from the first equation of (1), we have

$$S'(T_1) = A(T_1) - \mu_1(T_1)S(T_1) - \frac{\beta(T_1)S(T_1)B(T_1)}{K(T_1)+B(T_1)} = A(T_1) > 0,$$

a contradiction. Hence, (7) holds.

Next, we claim that $I(t) > 0$ for $t \in [t_0,T^*)$. Otherwise, there must exist $T_2 \in [t_0,T^*)$ such that

$$I(T_2) = 0, \quad I(s) > 0 \quad \text{for all } s \in [t_0,T_2).$$

From the second equation of (1) we obtain

$$\begin{aligned} I'(v) &= \frac{\beta(v)S(v)B(v)}{K(v)+B(v)} - \bigl(\gamma(v)+\mu_1(v)+\delta(v)\bigr)I(v) - \frac{\vartheta(v)I(v)}{1+\alpha(v)I(v)} \\ &\geqslant \frac{\beta(v)S(v)B(v)}{K(v)+B(v)} - \bigl(\gamma(v)+\mu_1(v)+\delta(v)\bigr)I(v) \\ &\geqslant \frac{\beta(v)S(v)B(v)}{K(v)+B(v)} - \bigl(\gamma^+ + \mu_1^+ + \delta^+\bigr)I(v) \quad \text{for all } v \in [t_0,T_2], \end{aligned}$$

and hence,

$$\begin{aligned} I'(T_2) &\geqslant \mathrm{e}^{-(T_2-t_0)(\gamma^+ + \mu_1^+ + \delta^+)} I(t_0) \\ &\quad + \int_{t_0}^{T_2} \mathrm{e}^{-(T_2-v)(\gamma^+ + \mu_1^+ + \delta^+)} \frac{\beta(v)S(v)B(v)}{K(v)+B(v)}\,\mathrm{d}v > 0. \end{aligned}$$

This contradicts $I(T_2) = 0$ and the claim is proved.

Now, we prove that $R(t) > 0$ for all $t \in [t_0,T^*)$. If $R(t) > 0$, then by continuity we can choose a small positive constant ρ^* such that

$$R(t) > 0 \quad \text{for all } t \in (t_0,\ t_0+\rho^*] \subset (t_0,T^*). \tag{8}$$

If $R(t_0) = 0$, then

$$\begin{aligned} R'(t_0) &= \gamma(t_0)I(t_0) + \frac{\vartheta(t_0)I(t_0)}{1+\alpha(t_0)I(t_0)} - \mu_1(t_0)R(t_0) \\ &= \gamma(t_0)I(t_0) + \frac{\vartheta(t_0)I(t_0)}{1+\alpha(t_0)I(t_0)} > 0, \end{aligned}$$

which implies that (8) also holds. Now, we claim that

$$R(t) > 0 \quad \text{for all } t \in (t_0+\rho^*,\, T^*). \tag{9}$$

Otherwise, there must exist $T_3 \in [t_0+\rho^*,\, T^*)$ such that

$$R(T_3) = 0, \qquad R(s) > 0 \quad \text{for all } s \in [t_0+\rho^*,\, T_3). \tag{10}$$

From (1) and (10) we have

$$0 \geqslant R'(T_3) = \gamma(T_3)I(T_3) + \frac{\vartheta(T_3)I(T_3)}{1+\alpha(T_3)I(T_3)} > 0,$$

which is a contradiction, and hence, (9) holds.

Finally, we prove that

$$B(t) > 0 \quad \text{for all } t \in [t_0, T^*). \tag{11}$$

We will prove it by the way of contradiction. Assume that (11) doesn't hold. Then there must exist $T_4 \in [t_0, T^*)$ such that

$$B(T_4) = 0, \qquad B(s) > 0 \quad \text{for all } s \in [t_0, T_4), \qquad B'(T_4) \leqslant 0.$$

But in view of the first equation of (1), we have

$$B'(T_4) = \eta(T_4)I(T_4) - \mu_2(T_4)B(T_4) = \eta(T_4)I(T_4) > 0,$$

a contradiction. Hence, (11) holds.

From the above discussion, we find that

$$S(t) > 0, \quad I(t) > 0, \quad R(t) > 0, \quad B(t) > 0 \quad \text{for all } t \in (t_0, T^*),$$

which, together with (1), yields

$$\begin{aligned} S'(t) &= A(t) - \mu_1(t)S(t) - \frac{\beta(t)S(t)B(t)}{K(t)+B(t)} \leqslant A^+ - \mu_1^- S(t), \\ I'(t) &= \frac{\beta(t)S(t)B(t)}{K(t)+B(t)} - \big(\gamma(t)+\mu_1(t)+\delta(t)\big)I(t) - \frac{\vartheta(t)I(t)}{1+\alpha(t)I(t)}, \\ &\leqslant \beta^+ S(t) - (\gamma^- + \mu_1^- + \delta^-)I(t) \\ R'(t) &= \gamma(t)I(t) + \frac{\vartheta(t)I(t)}{1+\alpha(t)I(t)} - \mu_1(t)R(t) \leqslant (\gamma^+ + \vartheta^+)I(t) - \mu_1^- R(t), \\ B'(t) &= \eta(t)I(t) - \mu_2(t)B(t) \leqslant \eta^+ I(t) - \mu_2^- B(t). \end{aligned}$$

Therefore,

$$
\begin{aligned}
S(t) &\leqslant S(t_0)\frac{\mathrm{e}^{\mu_1 t_0}}{\mathrm{e}^{\mu_1 t}} + \frac{A^+}{\mu_1^-}\frac{\mathrm{e}^{\mu_1 t} - \mathrm{e}^{\mu_1 t_0}}{\mathrm{e}^{\mu_1 t}} \leqslant S(t_0) + \frac{A^+}{\mu_1^-} =: M_1, \\
I(t) &\leqslant I(t_0)\frac{\mathrm{e}^{(\gamma^- + \mu_1^- + \delta^-)t_0}}{\mathrm{e}^{(\gamma^- + \mu_1^- + \delta^-)t}} + \beta^+ M_1 \frac{\mathrm{e}^{(\gamma^- + \mu_1^- + \delta^-)t} - \mathrm{e}^{(\gamma^- + \mu_1^- + \delta^-)t_0}}{\mathrm{e}^{(\gamma^- + \mu_1^- + \delta^-)t}} \\
&\leqslant I(t_0) + \frac{\beta^+ M_1}{\gamma^- + \mu_1^- + \delta^-} =: M_2, \\
R(t) &\leqslant R(t_0)\frac{\mathrm{e}^{\mu_1 t_0}}{\mathrm{e}^{\mu_1 t}} + \frac{M_2(\gamma^+ + \vartheta^+)}{\mu_1^-}\frac{\mathrm{e}^{\mu_1 t} - \mathrm{e}^{\mu_1 t_0}}{\mathrm{e}^{\mu_1 t}} \\
&\leqslant R(t_0) + \frac{M_2(\gamma^+ + \vartheta^+)}{\mu_1^-} =: M_3, \\
B(t) &\leqslant B(t_0)\frac{\mathrm{e}^{\mu_2 t_0}}{\mathrm{e}^{\mu_2 t}} + \frac{M_3 \eta^+}{\mu_2^-}\frac{\mathrm{e}^{\mu_2 t} - \mathrm{e}^{\mu_2 t_0}}{\mathrm{e}^{\mu_2 t}} \leqslant B(t_0) + \frac{M_3 \eta^+}{\mu_2^-} =: M_4
\end{aligned}
$$

for all $t \in (t_0, T^*)$. It follows that $S(t)$, $I(t)$, $R(t)$, and $B(t)$ are bounded on (t_0, T^*). From Theorem 1.2.1 in [6] we easily obtain $T^* = \infty$. □

Lemma 2. *Let*

$$
L^S = \sup_{t\in\mathbb{R}} \frac{A(t)}{\mu_1(t)} \geqslant l^S = \inf_{t\in\mathbb{R}} \frac{A(t)}{\mu_1(t) + \beta(t)} > 0,
$$

$$
l^I = \inf_{t\in\mathbb{R}} \frac{K(t)\mu_2(t)}{\eta(t)}\left[\frac{\frac{\beta(t)\eta(t)}{K(t)\mu_2(t)} \inf_{t\in\mathbb{R}} \frac{A(t)}{\mu_1(t)+\beta(t)} - \vartheta(t)}{\gamma(t) + \mu_1(t) + \delta(t)} - 1\right] > 0,
$$

$$
L^I = \inf_{t\in\mathbb{R}} \left\{\frac{\eta(t)}{\mu_2(t)} \inf_{t\in\mathbb{R}} \frac{K(t)\mu_2(t)}{\eta(t)}\left[\frac{\frac{\eta(t)}{K(t)\mu_2(t)}\beta(t) \inf_{t\in\mathbb{R}} \frac{A(t)}{\mu_1(t)+\beta(t)} - \vartheta(t)}{\gamma(t) + \mu_1(t) + \delta(t)} - 1\right]\right\} > 0,
$$

and let $(S(t), I(t), R(t), B(t))$ *be a solution of system* (1) *with initial conditions* (2). *Then*

$$
l^S \leqslant \liminf_{t\to\infty} S(t) \leqslant \limsup_{t\to\infty} S(t) \leqslant L^S,
$$

$$
\liminf_{t\to\infty} I(t) \geqslant l^I, \qquad \liminf_{t\to\infty} B(t) \geqslant L^I, \qquad \liminf_{t\to\infty} R(t) > 0.
$$

Proof. From Lemma 1 we can find that the solution $(S(t), I(t), R(t), B(t))$ is positive and bounded on (t_0, ∞). By the fluctuation lemma [17, Lemma A.1], there exist sequences

$$
\{t_p^1\}_{p\geqslant 1}, \quad \{t_p^2\}_{p\geqslant 1}, \quad \{t_p^3\}_{p\geqslant 1}, \quad \{t_p^4\}_{p\geqslant 1}, \quad \{t_p^5\}_{p\geqslant 1}
$$

such that, as $p \to \infty$,

$$
\begin{aligned}
&t_p^1 \to \infty, \qquad S(t_p^1) \to \limsup_{t\to\infty} S(t), \qquad S'(t_p^1) \to 0, \\
&t_p^2 \to \infty, \qquad S(t_p^2) \to \liminf_{t\to\infty} S(t), \qquad S'(t_p^2) \to 0,
\end{aligned} \tag{12a}
$$

$$
\begin{aligned}
&t_p^3 \to \infty, \qquad I\bigl(t_p^3\bigr) \to \liminf_{t\to\infty} I(t), \qquad I'\bigl(t_p^3\bigr) \to 0,\\
&t_p^4 \to \infty, \qquad B\bigl(t_p^4\bigr) \to \liminf_{t\to\infty} B(t), \qquad B'\bigl(t_p^4\bigr) \to 0,\\
&t_p^5 \to \infty, \qquad R\bigl(t_p^5\bigr) \to \liminf_{t\to\infty} R(t), \qquad R'\bigl(t_p^5\bigr) \to 0,
\end{aligned} \tag{12b}
$$

The first two lines in (12) yield

$$
\begin{aligned}
S'\bigl(t_p^1\bigr) &= A\bigl(t_p^1\bigr) - \mu_1\bigl(t_p^1\bigr)S\bigl(t_p^1\bigr) - \frac{\beta(t_p^1)S(t_p^1)B(t_p^1)}{K(t_p^1)+B(t_p^1)}\\
&\leqslant \mu_1\bigl(t_p^1\bigr)\left[\frac{A(t_p^1)}{\mu_1(t_p^1)} - S(t_p^1)\right]\\
&\leqslant \mu_1\bigl(t_p^1\bigr)\left[\sup_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)} - S\bigl(t_p^1\bigr)\right],\\
S'\bigl(t_p^2\bigr) &= A\bigl(t_p^2\bigr) - \mu_1\bigl(t_p^2\bigr)S\bigl(t_p^2\bigr) - \frac{\beta(t_p^2)S(t_p^2)B(t_p^2)}{K(t_p^2)+B(t_p^2)}\\
&\geqslant A\bigl(t_p^2\bigr) - \mu_1\bigl(t_p^2\bigr)S\bigl(t_p^2\bigr) - \beta\bigl(t_p^2\bigr)S\bigl(t_p^2\bigr)\\
&= A\bigl(t_p^2\bigr) - S\bigl(t_p^2\bigr)\bigl[\mu_1\bigl(t_p^2\bigr) + \beta\bigl(t_p^2\bigr)\bigr]\\
&\geqslant \bigl[\mu_1\bigl(t_p^2\bigr) + \beta\bigl(t_p^2\bigr)\bigr]\left[\inf_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)+\beta(t)} - S\bigl(t_p^2\bigr)\right],
\end{aligned}
$$

and

$$
\frac{S'(t_p^1)}{\mu_1(t_p^1)} \leqslant \sup_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)} - S\bigl(t_p^1\bigr), \tag{13}
$$

$$
\frac{S'(t_p^2)}{\mu_1(t_p^2)+\beta(t_p^2)} \geqslant \inf_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)+\beta(t)} - S\bigl(t_p^2\bigr). \tag{14}
$$

Letting $p\to\infty$ in (13), (14) implies that

$$
l^S \leqslant \liminf_{t\to\infty} S(t) \leqslant \limsup_{t\to\infty} S(t) \leqslant L^S.
$$

Furthermore, we prove that there exists positive constants l and L such that

$$
\liminf_{t\to\infty} I(t) \geqslant l^I, \qquad \liminf_{t\to\infty} B(t) \geqslant L^I.
$$

Otherwise, $\liminf_{t\to\infty} I(t) = 0$, $\liminf_{t\to\infty} B(t) = 0$. For each $t \geqslant t_0$, we define

$$
\begin{aligned}
m(t) = \max&\Bigl\{\xi\colon\ \xi\leqslant t,\ I(\xi) = \min_{t_0\leqslant s\leqslant t} I(s)\Bigr\}\\
&\cap \max\Bigl\{\xi\colon\ \xi\leqslant t,\ B(\xi) = \min_{t_0\leqslant s\leqslant t} B(s)\Bigr\}.
\end{aligned}
$$

Notice that $m(t) \to \infty$ as $t \to \infty$ and that

$$\liminf_{t\to\infty} I\bigl(m(t)\bigr) = 0, \qquad \liminf_{t\to\infty} B\bigl(m(t)\bigr) = 0.$$

However, $I(m(t)) = \min_{t_0 \leqslant s \leqslant t} I(\xi)$, $B(m(t)) = \min_{t_0 \leqslant s \leqslant t} B(\xi)$, and so $I'(m(t)) \leqslant 0$, $B'(m(t)) \leqslant 0$ for all $m(t) > t_0$. Let $\epsilon > 0$ and $t_0^* > t_0$ be such that

$$\inf_{t\in\mathbb{R}} \frac{K(t)\mu_2(t)}{\eta(t)} \left[\frac{\frac{\beta(t)\eta(t)}{K(t)\mu_2(t)}(\inf_{t\in\mathbb{R}} \frac{A(t)}{\mu_1(t)+\beta(t)} - \epsilon) - \vartheta(t)}{\gamma(t) + \mu_1(t) + \delta(t)} - 1 \right] > 0,$$

and

$$S(t) > \inf_{t\in\mathbb{R}} \frac{A(t)}{\mu_1(t) + \beta(t)} - \epsilon \quad \text{for all } t \geqslant t_0^*.$$

Since $0 \geqslant B'(m(t)) = \eta(m(t))I(m(t)) - \mu_2(m(t))B(m(t))$ and $\mu_2(m(t))$ is positive, then

$$B\bigl(m(t)\bigr) \geqslant \frac{\eta(m(t))}{\mu_2(m(t))} I\bigl(m(t)\bigr). \tag{15}$$

According to (1), (3), and (15), we have

$$\begin{aligned}
0 &\geqslant I'\bigl(m(t)\bigr) \\
&= \frac{\beta(m(t))S(m(t))B(m(t))}{K(m(t)) + B(m(t))} - \bigl(\gamma(m(t)) + \mu_1\bigl(m(t)\bigr) + \delta\bigl(m(t)\bigr)\bigr) I\bigl(m(t)\bigr) \\
&\quad - \frac{\vartheta(m(t))I(m(t))}{1 + \alpha(m(t))I(m(t))} \\
&= \frac{\beta(m(t))S(m(t))}{\frac{K(m(t))}{B(m(t))} + 1} - \bigl(\gamma(m(t)) + \mu_1\bigl(m(t)\bigr) + \delta\bigl(m(t)\bigr)\bigr) I\bigl(m(t)\bigr) \\
&\quad - \frac{\vartheta(m(t))I(m(t))}{1 + \alpha(m(t))I(m(t))} \\
&\geqslant \frac{\beta(m(t))S(m(t))}{\frac{K(m(t))\mu_2(m(t))}{\eta(m(t))I(m(t))} + 1} - \bigl(\gamma\bigl(m(t)\bigr) + \mu_1\bigl(m(t)\bigr) + \delta\bigl(m(t)\bigr)\bigr) I\bigl(m(t)\bigr) \\
&\quad - \frac{\vartheta(m(t))I(m(t))}{1 + \alpha(m(t))I(m(t))} \\
&\geqslant I\bigl(m(t)\bigr) \Biggl\{ \frac{\frac{\eta(m(t))}{K(m(t))\mu_2(m(t))}\beta(m(t))S(m(t)) - \vartheta(m(t))}{1 + \frac{\eta(m(t))}{K(m(t))\mu_2(m(t))} I(m(t))} \\
&\quad - \bigl(\gamma\bigl(m(t)\bigr) + \mu_1\bigl(m(t)\bigr) + \delta\bigl(m(t)\bigr)\bigr) \Biggr\} \quad \text{for all } m(t) \geqslant t_0^*.
\end{aligned}$$

Thus, for all $m(t) \geqslant t_0^*$,

$$\gamma\bigl(m(t)\bigr) + \mu_1\bigl(m(t)\bigr) + \delta\bigl(m(t)\bigr) \geqslant \frac{\frac{\eta(m(t))}{K(m(t))\mu_2(m(t))}\beta(m(t))S(m(t)) - \vartheta(m(t))}{1 + \frac{\eta(m(t))}{K(m(t))\mu_2(m(t))} I(m(t))}$$

and

$$\begin{aligned}
I\big(m(t)\big) &\geqslant \frac{K(m(t))\mu_2(m(t)))}{\eta(m(t))}\left[\frac{\frac{\eta(m(t))}{K(m(t))\mu_2(m(t))}\beta(m(t))S(m(t))-\vartheta(m(t))}{\gamma(m(t))+\mu_1(m(t))+\delta(m(t))}-1\right] \\
&\geqslant \frac{K(m(t))\mu_2(m(t))}{\eta(m(t))} \\
&\quad\times\left[\frac{\frac{\eta(m(t))}{K(m(t))\mu_2(m(t))}\beta(m(t))(\inf_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)+\beta(t)}-\epsilon)-\vartheta(m(t))}{\gamma(m(t))+\mu_1(m(t))+\delta(m(t))}-1\right] \\
&\geqslant \inf_{t\in\mathbb{R}}\left\{\frac{K(m(t))\mu_2(m(t))}{\eta(m(t))}\right. \\
&\quad\left.\times\left[\frac{\frac{\eta(m(t))}{K(m(t))\mu_2(m(t))}\beta(m(t))(\inf_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)+\beta(t)}-\epsilon)-\vartheta(m(t))}{\gamma(m(t))+\mu_1(m(t))+\delta(m(t))}-1\right]\right\} \\
&\geqslant \inf_{t\in\mathbb{R}}\left\{\frac{K(t)\mu_2(t)}{\eta(t)}\left[\frac{\frac{\eta(t)}{K(t)\mu_2(t)}\beta(t)(\inf_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)+\beta(t)}-\epsilon)-\vartheta(t)}{\gamma(t)+\mu_1(t)+\delta(t)}-1\right]\right\} \\
&> 0.
\end{aligned}$$

By the continuity and boundedness of the coefficient functions in (1), we can select a subsequence, still denoted by $\{t_p^i\}_{p=1}^{\infty}$, such that

$$\begin{gathered}
\lim_{p\to\infty}S\big(t_p^i\big),\quad \lim_{p\to\infty}K\big(t_p^i\big),\quad \lim_{p\to\infty}\gamma\big(t_p^i\big)+\mu_1\big(t_p^i\big)+\delta\big(t_p^i\big),\quad \lim_{p\to\infty}\mu_2\big(t_p^i\big), \\
\lim_{p\to\infty}\alpha\big(t_p^i\big),\quad \lim_{p\to\infty}\vartheta\big(t_p^i\big),\quad \text{and}\quad \lim_{p\to\infty}\beta\big(t_p^i\big)\quad \text{exist for all } i=3,4,5.
\end{gathered}\tag{16}$$

From (1), (3), (12), and (16) we get

$$\frac{I'(t_p^3)}{I(t_p^3)}\geqslant\frac{\frac{\eta(t_p^3)}{K(t_p^3)\mu_2(t_p^3)}\beta(t_p^3)S(t_p^3)-\vartheta(t_p^3)}{1+\frac{\eta(t_p^3)}{K(t_p^3)\mu_2(t_p^3)}I(t_p^3)}-\big(\gamma\big(t_p^3\big)+\mu_1\big(t_p^3\big)+\delta\big(t_p^3\big)\big).\tag{17}$$

Letting $p\to\infty$ in (16) and (17) implies that

$$\begin{aligned}
\liminf_{t\to\infty}I(t)=\lim_{p\to\infty}I\big(t_p^3\big)&\geqslant\lim_{p\to\infty}\frac{K(t_p^3)\mu_2(t_p^3)}{\eta(t_p^3)}\left[\frac{\frac{\eta(t_p^3)}{K(t_p^3)\mu_2(t_p^3)}\beta(t_p^3)S(t_p^3)-\vartheta(t_p^3)}{\gamma(t_p^3)+\mu_1(t_p^3)+\delta(t_p^3)}-1\right] \\
&\geqslant\inf_{t\in\mathbb{R}}\left\{\frac{K(t)\mu_2(t)}{\eta(t)}\left[\frac{\frac{\eta(t)}{K(t)\mu_2(t)}\beta(t)\inf_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)+\beta(t)}-\vartheta(t)}{\gamma(t)+\mu_1(t)+\delta(t)}-1\right]\right\} \\
&=l^I>0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
0\geqslant R'\big(t_p^4\big)&=\gamma\big(t_p^4\big)I\big(t_p^4\big)+\frac{\vartheta(t_p^4)I(t_p^4)}{1+\alpha(t_p^4)I(t_p^4)}-\mu_1\big(t_p^4\big)R\big(t_p^4\big), \\
&\geqslant\gamma\big(t_p^4\big)I\big(t_p^4\big)-\mu_1\big(t_p^4\big)R\big(t_p^4\big),
\end{aligned}$$

which yields

$$\liminf_{t\to\infty} R(t) = \liminf_{p\to\infty} R\bigl(t_p^4\bigr) \geqslant \inf_{t\in\mathbb{R}} \frac{\gamma(t)}{\mu_1(t)} \liminf_{t\to\infty} I(t) > 0.$$

Similarly, we also have

$$\begin{aligned}
&\liminf_{t\to\infty} B(t)\\
&\quad= \liminf_{p\to\infty} B\bigl(t_p^5\bigr) \geqslant \inf_{t\in\mathbb{R}} \frac{\eta(t)}{\mu_2(t)} \liminf_{t\to\infty} I(t)\\
&\quad\geqslant \inf_{t\in\mathbb{R}}\left\{\frac{\eta(t)}{\mu_2(t)} \inf_{t\in\mathbb{R}} \frac{K(t)\mu_2(t)}{\eta(t)}\left[\frac{\frac{\eta(t)}{K(t)\mu_2(t)}\beta(t)\inf_{t\in\mathbb{R}}\frac{A(t)}{\mu_1(t)+\beta(t)} - \vartheta(t)}{\gamma(t)+\mu_1(t)+\delta(t)} - 1\right]\right\}\\
&\quad= L^I > 0.
\end{aligned}$$

The proof of Lemma 2 is now completed. □

Lemma 3. *Assume that*

$$\sup_{t\in\mathbb{R}}\left\{-\mu_1(t) + \frac{K(t)\beta(t)L^S}{(K(t)+L^I)^2}\right\} < 0, \tag{18}$$

$$\sup_{t\in\mathbb{R}}\left\{-\bigl[\bigl(\gamma(t)+\mu_1(t)+\delta(t)\bigr)\bigr] + \beta(t) + \frac{K(t)\beta(t)L^S}{(K(t)+L^I)^2}\right\} < 0, \tag{19}$$

$$\sup_{t\in\mathbb{R}}\bigl\{\eta(t)-\mu_2(t)\bigr\} < 0 \tag{20}$$

and the assumptions of Lemma 2 *hold. Let*

$$\bigl(S(t), I(t), R(t), B(t)\bigr), \qquad \bigl(\check{S}(t), \check{I}(t), \check{R}(t), \check{B}(t)\bigr)$$

be the solutions of system (1) *with initial conditions* (2)*. Then there exist* $\check{t}_0 \geqslant t_0$ *and positive constants* ζ *and* k *such that, for all* $t \geqslant \check{t}_0$,

$$\bigl|S(t)-\check{S}(t)\bigr| \leqslant k\mathrm{e}^{-\zeta t}, \quad \bigl|I(t)-\check{I}(t)\bigr| \leqslant k\mathrm{e}^{-\zeta t}, \quad \bigl|B(t)-\check{B}(t)\bigr| \leqslant k\mathrm{e}^{-\zeta t}. \tag{21}$$

Moreover, there exist constants $t_R \geqslant \check{t}_0$ *and* k_R *such that*

$$\bigl|R(t)-\check{R}(t)\bigr| \leqslant k_R\mathrm{e}^{-\zeta t} \quad \text{for all } t \geqslant t_R. \tag{22}$$

Proof. Let, for all $t \in [t_0, \infty)$.

$$x(t) = \bigl(x_1(t), x_2(t), x_4(t)\bigr) = \bigl(S(t)-\check{S}(t),\, I(t)-\check{I}(t),\, B(t)-\check{B}(t)\bigr).$$

Then (1) gives

$$x_1'(t) = -\left[\mu_1(t) - \frac{\beta(t)B(t)}{K(t)+B(t)}\right]x_1(t) - \frac{K(t)\beta(t)\check{S}(t)}{(K(t)+B(t))(K(t)+\check{B}(t))}x_4(t),$$

$$\begin{aligned}
x_2'(t) &= -\left[\gamma(t) + \mu_1(t) + \delta(t) + \frac{\vartheta(t)}{(1+\alpha(t)I(t))(1+\alpha(t)\check{I}(t))}\right]x_2(t) \\
&\quad + \frac{\beta(t)B(t)}{K(t)+B(t)}x_1(t) + \frac{K(t)\beta(t)\check{S}(t)}{(K(t)+B(t))(K(t)+\check{B}(t))}x_4(t), \\
x_4'(t) &= \eta(t)x_2(t) - \mu_2(t)x_4(t),
\end{aligned}$$

which implies

$$\begin{aligned}
x_1(t) &= \mathrm{e}^{-\int_{\check{t}_0}^t [\mu_1(\theta) - \frac{\beta(\theta)B(\theta)}{K(\theta)+B(\theta)}]\,\mathrm{d}\theta} x_1(\check{t}_0) \\
&\quad + \int_{\check{t}_0}^{t} \mathrm{e}^{-\int_v^t [\mu_1(\theta) - \frac{\beta(\theta)B(\theta)}{K(\theta)+B(\theta)}]\,\mathrm{d}\theta} \left[-\frac{K(v)\beta(v)\check{S}(v)x_4(v)}{(K(v)+B(v))(K(v)+\check{B}(v))}\right]\mathrm{d}v, \qquad (23) \\
x_2(t) &= \mathrm{e}^{-\int_{\check{t}_0}^t [\gamma(\theta)+\mu_1(\theta)+\delta(\theta)+\frac{\vartheta(\theta)}{(1+\alpha(\theta)I(\theta))(1+\alpha(\theta)\check{I}(\theta))}]\,\mathrm{d}\theta} x_2(\check{t}_0) \\
&\quad + \int_{\check{t}_0}^{t} \mathrm{e}^{-\int_v^t [\gamma(\theta)+\mu_1(\theta)+\delta(\theta)+\frac{\vartheta(\theta)}{(1+\alpha(\theta)I(\theta))(1+\alpha(\theta)\check{I}(\theta))}]\,\mathrm{d}\theta} \\
&\qquad \times \left[\frac{\beta(v)B(v)x_1(v)}{K(v)+B(v)} + \frac{K(v)\beta(v)\check{S}(v)x_4(v)}{(K(v)+B(v))(K(v)+\check{B}(v))}\right]\mathrm{d}v, \qquad (24)
\end{aligned}$$

and

$$x_2(t) = \mathrm{e}^{-\int_{\check{t}_0}^t \mu_2(\theta)\,\mathrm{d}\theta} x_4(\check{t}_0) + \int_{\check{t}_0}^{t} \mathrm{e}^{-\int_v^t \mu_2(\theta)\,\mathrm{d}\theta} \eta(v)x_2(v)\,\mathrm{d}v \qquad (25)$$

for all $t \geqslant \check{t}_0$. Let $\epsilon < \min\{L^I, L^S\}$ be a positive constant such that

$$\begin{gathered}
\sup_{t\in\mathbb{R}}\left\{-\mu_1(t) + \frac{K(t)\beta(t)(L^S+\epsilon)}{(K(t)+L^I-\epsilon)^2}\right\} < 0, \\
\sup_{t\in\mathbb{R}}\left\{-\left[\left(\gamma(t)+\mu_1(t)+\delta(t)\right)\right] + \beta(t) + \frac{K(t)\beta(t)L^S}{(K(t)+L^I-\epsilon)^2}\right\} < 0, \\
\sup_{t\in\mathbb{R}}\left\{\eta(t) - \mu_2(t)\right\} < 0.
\end{gathered}$$

This can be achieved because of (18), (19), and (20). Consequently, we can choose positive constants ζ and τ such that

$$\sup_{t\in\mathbb{R}}\left\{\zeta - \mu_1(t) + \frac{K(t)\beta(t)(L^S+\epsilon)}{(K(t)+L^I-\epsilon)^2}\right\} < -\tau, \qquad (26)$$

$$\sup_{t\in\mathbb{R}}\left\{\zeta-\left[\left(\gamma(t)+\mu_1(t)+\delta(t)\right)\right]+\beta(t)+\frac{K(t)\beta(t)L^S}{(K(t)+L^I-\epsilon)^2}\right\}<-\tau, \tag{27}$$

$$\sup_{t\in\mathbb{R}}\left\{\zeta+\eta(t)-\mu_2(t)\right\}<-\tau. \tag{28}$$

From Lemma 2 we can choose $\check{t}_0 \geqslant t_0$ such that, for all $t \geqslant \check{t}_0$,

$$S(t)\leqslant L^S+\epsilon, \qquad \check{S}(t)\leqslant L^S+\epsilon, \qquad \check{I}(t)\geqslant l^I-\epsilon, \qquad \check{B}(t)\geqslant L^I-\epsilon.$$

Let $\|x\|_0 = \max\{\sup_{t\in[t_0,\check{t}_0]}|x_1(t)|, \sup_{t\in[t_0,\check{t}_0]}|x_2(t)|, \sup_{t\in[t_0,\check{t}_0]}|x_4(t)|\}$, and $k_0>1$ be a constant. It is obvious that

$$\left\|x(\check{t}_0)\right\|<\|x\|_0+\epsilon<k_0\left(\|x\|_0+\epsilon\right)=k_0\left(\|x\|_0+\epsilon\right)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\check{t}_0}.$$

In the following, we will show

$$\left\|x(t)\right\|<k_0\left(\|x\|_0+\epsilon\right)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta t} \quad \text{for all } t\geqslant\check{t}_0. \tag{29}$$

Otherwise, one of the following three cases must occur:

Case 1. There exists $\theta_1>0$ such that

$$\left\|x_1(\theta_1)\right\|=k_0\left(\|x\|_0+\epsilon\right)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_1}, \tag{30}$$

$$\left\|x(t)\right\|<k_0\left(\|x\|_0+\epsilon\right)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta t} \quad \text{for all } t\in[\check{t}_0,\theta_1). \tag{31}$$

If Case 1 holds, then in view of (23), (26), (30), and (31), we have

$$\begin{aligned}
\left|x_1(\theta_1)\right| &= \Bigg|\mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}[\mu_1(\theta)-\frac{\beta(\theta)B(\theta)}{K(\theta)+B(\theta)}]\,\mathrm{d}\theta}x_1(\check{t}_0) \\
&\quad+\int_{\check{t}_0}^{\theta_1}\mathrm{e}^{-\int_v^{\theta_1}[\mu_1(\theta)-\frac{\beta(\theta)B(\theta)}{K(\theta)+B(\theta)}]\,\mathrm{d}\theta}\left[-\frac{K(v)\beta(v)\check{S}(v)x_4(v)}{(K(v)+B(v))(K(v)+\check{B}(v))}\right]\mathrm{d}v\Bigg| \\
&\leqslant \mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}\mu_1(\theta)\,\mathrm{d}\theta}\left|x_1(\check{t}_0)\right|+\int_{\check{t}_0}^{\theta_1}\mathrm{e}^{-\int_v^{\theta_1}\mu_1(\theta)\,\mathrm{d}\theta}\frac{K(v)\beta(v)(L^S+\epsilon)|x_4(v)\|}{(K(v)+L^I-\epsilon)^2}\,\mathrm{d}v \\
&\leqslant \mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}\mu_1(\theta)\,\mathrm{d}\theta}\left(\|x\|_0+\epsilon\right) \\
&\quad+\int_{\check{t}_0}^{\theta_1}\mathrm{e}^{-\int_v^{\theta_1}\mu_1(\theta)\,\mathrm{d}\theta}\frac{K(v)\beta(v)(L^S+\epsilon)}{(K(v)+L^I-\epsilon)^2}k_0\left(\|x\|_0+\epsilon\right)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta v}\,\mathrm{d}v \\
&= k_0\left(\|x\|_0+\epsilon\right)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_1} \\
&\quad\times\left[\frac{1}{k_0}\mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}(\mu_1(\theta)-\zeta)\,\mathrm{d}\theta}+\int_{\check{t}_0}^{\theta_1}\mathrm{e}^{-\int_v^{\theta_1}(\mu_1(\theta)-\zeta)\,\mathrm{d}\theta}\frac{K(v)\beta(v)(L^S+\epsilon)}{(K(v)+L^I-\epsilon)^2}\,\mathrm{d}v\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant k_0\bigl(\|x\|_0+\epsilon\bigr)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_1} \\
&\quad\times\left[\frac{1}{k_0}\mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}(\mu_1(\theta)-\zeta)\,\mathrm{d}\theta}+\int_{\check{t}_0}^{\theta_1}\mathrm{e}^{-\int_{v}^{\theta_1}(\mu_1(\theta)-\zeta)\,\mathrm{d}\theta}\bigl(\mu_1(v)-\zeta\bigr)\,\mathrm{d}v\right] \\
&= k_0\bigl(\|x\|_0+\epsilon\bigr)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_1}\left[1-\left(1-\frac{1}{k_0}\right)\mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}(\mu_1(\theta)-\zeta)\,\mathrm{d}\theta}\right] \\
&< k_0\bigl(\|x\|_0+\epsilon\bigr)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_1},
\end{aligned}
$$

which contradicts (30) and proves (29).

Case 2. There exists $\theta_2 > 0$ such that

$$
\bigl\|x_2(\theta_2)\bigr\| = k_0\bigl(\|x\|_0+\epsilon\bigr)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_2}, \tag{32}
$$

$$
\bigl\|x(t)\bigr\| < k_0\bigl(\|x\|_0+\epsilon\bigr)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta t} \quad \text{for all } t\in[\check{t}_0,\theta_2). \tag{33}
$$

If Case 2 holds, then in view of (24), (27), (32), and (33), we have

$$
\begin{aligned}
\bigl|x_2(\theta_2)\bigr| &= \Biggl|\mathrm{e}^{-\int_{\check{t}_0}^{t}[\gamma(\theta)+\mu_1(\theta)+\delta(\theta)+\frac{\vartheta(\theta)}{(1+\alpha(\theta)I(\theta))(1+\alpha(\theta)\check{I}(\theta))}]\,\mathrm{d}\theta}x_2(\check{t}_0) \\
&\quad+\int_{\check{t}_0}^{t}\mathrm{e}^{-\int_{v}^{t}[\gamma(\theta)+\mu_1(\theta)+\delta(\theta)+\frac{\vartheta(\theta)}{(1+\alpha(\theta)I(\theta))(1+\alpha(\theta)\check{I}(\theta))}]\,\mathrm{d}\theta} \\
&\qquad\times\left[\frac{\beta(v)B(v)x_1(v)}{K(v)+B(v)}+\frac{K(v)\beta(v)\check{S}(v)x_4(v)}{(K(v)+B(v))(K(v)+\check{B}(v))}\right]\mathrm{d}v\Biggr| \\
&\leqslant \mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}[\gamma(\theta)+\mu_1(\theta)+\delta(\theta)]\,\mathrm{d}\theta}\bigl|x_2(\check{t}_0)\bigr| \\
&\quad+\int_{\check{t}_0}^{\theta_2}\mathrm{e}^{-\int_{v}^{\theta_2}[\gamma(\theta)+\mu_1(\theta)+\delta(\theta)]\,\mathrm{d}\theta} \\
&\qquad\times\left[\beta(v)\bigl|x_1(v)\bigr|+\frac{K(v)\beta(v)(L^S+\epsilon)|x_4(v)|}{(K(v)+L^I-\epsilon)^2}\right]\mathrm{d}v \\
&\leqslant \mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}[\gamma(\theta)+\mu_1(\theta)+\delta(\theta)]\,\mathrm{d}\theta}\bigl(\|x\|_0+\epsilon\bigr) \\
&\quad+\int_{\check{t}_0}^{\theta_2}\mathrm{e}^{-\int_{v}^{\theta_2}[\gamma(\theta)+\mu_1(\theta)+\delta(\theta)]\,\mathrm{d}\theta} \\
&\qquad\times\left[\beta(v)+\frac{K(v)\beta(v)(L^S+\epsilon)}{(K(v)+L^I-\epsilon)^2}\right]k_0\bigl(\|x\|_0+\epsilon\bigr)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta v}\,\mathrm{d}v \\
&\leqslant \mathrm{e}^{-\int_{\check{t}_0}^{\theta_1}[\gamma(\theta)+\mu_1(\theta)+\delta(\theta)]\,\mathrm{d}\theta}\bigl(\|x\|_0+\epsilon\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&+\int\limits_{\check{t}_0}^{\theta_2} \mathrm{e}^{-\int_v^{\theta_2}[\gamma(\theta)+\mu_1(\theta)+\delta(\theta)]\,\mathrm{d}\theta} \\
&\qquad \times \big[\gamma(v)+\mu_1(v)+\delta(v)-\zeta\big]k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta v}\,\mathrm{d}v \\
&= k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_2}\Bigg[\frac{1}{k_0}\mathrm{e}^{-\int_{\check{t}_0}^{\theta_2}(\gamma(\theta)+\mu_1(\theta)+\delta(\theta)-\zeta)\,\mathrm{d}\theta} \\
&\quad + \int\limits_{\check{t}_0}^{\theta_2} \mathrm{e}^{-\int_v^{\theta_2}(\gamma(\theta)+\mu_1(\theta)+\delta(\theta)-\zeta)\,\mathrm{d}\theta}\big(\gamma(v)+\mu_1(v)+\delta(v)-\zeta\big)\,\mathrm{d}v\Bigg] \\
&= k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_2}\bigg[1-\bigg(1-\frac{1}{k_0}\bigg)\mathrm{e}^{-\int_{\check{t}_0}^{\theta_2}(\gamma(\theta)+\mu_1(\theta)+\delta(\theta)-\zeta)\,\mathrm{d}\theta}\bigg] \\
&< k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_2},
\end{aligned}
$$

which contradicts (32) and proves (29).

Case 3. There exists $\theta_4 > 0$ such that

$$\big\|x_4(\theta_4)\big\| = k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_4}, \tag{34}$$

$$\big\|x(t)\big\| < k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta t} \quad \text{for all } t\in[\check{t}_0,\theta_4). \tag{35}$$

If Case 3 holds, then in view of (25), (28), (34), and (35), we have

$$
\begin{aligned}
\big|x_4(t)\big| &= \Bigg|\mathrm{e}^{-\int_{\check{t}_0}^{\theta_4}\mu_2(\theta)\,\mathrm{d}\theta}x_4(\check{t}_0) + \int\limits_{\check{t}_0}^{\theta_4}\mathrm{e}^{-\int_v^{\theta_4}\mu_2(\theta)\,\mathrm{d}\theta}\eta(v)x_2(v)\,\mathrm{d}v\Bigg| \\
&\leqslant \mathrm{e}^{-\int_{\check{t}_0}^{\theta_4}\mu_2(\theta)\,\mathrm{d}\theta}\big|x_4(\check{t}_0)\big| + \int\limits_{\check{t}_0}^{\theta_4}\mathrm{e}^{-\int_v^{\theta_4}\mu_2(\theta)\,\mathrm{d}\theta}\eta(v)\big|x_2(v)\big|\,\mathrm{d}v \\
&\leqslant \mathrm{e}^{-\int_{\check{t}_0}^{\theta_4}\mu_2(\theta)\,\mathrm{d}\theta}\big(\|x\|_0+\epsilon\big) + \int\limits_{\check{t}_0}^{\theta_4}\mathrm{e}^{-\int_v^{\theta_4}\mu_2(\theta)\,\mathrm{d}\theta}\eta(v)k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta v}\,\mathrm{d}v \\
&= k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_4} \\
&\quad \times \Bigg[\frac{1}{k_0}\mathrm{e}^{-\int_{\check{t}_0}^{\theta_4}(\mu_2(\theta)-\zeta)\,\mathrm{d}\theta} + \int\limits_{\check{t}_0}^{\theta_4}\mathrm{e}^{-\int_v^{\theta_4}(\mu_2(\theta)-\zeta)\,\mathrm{d}\theta}\eta(v)\,\mathrm{d}v\Bigg] \\
&\leqslant k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_4} \\
&\quad \times \Bigg[\frac{1}{k_0}\mathrm{e}^{-\int_{\check{t}_0}^{\theta_4}(\mu_2(\theta)-\zeta)\,\mathrm{d}\theta} + \int\limits_{\check{t}_0}^{\theta_4}\mathrm{e}^{-\int_v^{\theta_4}(\mu_2(\theta)-\zeta)\,\mathrm{d}\theta}\big(\mu_2(\theta)-\zeta\big)\,\mathrm{d}v\Bigg]
\end{aligned}
$$

$$= k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_4}\left[1-\left(1-\frac{1}{k_0}\right)\mathrm{e}^{-\int_{\check{t}_0}^{\theta_4}\left(\mu_2(\theta)-\zeta\right)\mathrm{d}\theta}\right]$$
$$< k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta\theta_4},$$

which contradicts (34) and proves (29).

Letting $\epsilon\to 0^+$, it follows from (29) that

$$\big\|x(t)\big\| < k_0\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta t}\quad\text{for all } t\geqslant\check{t}_0,$$

which proves (21).

Now, we prove that (22) holds. Without loss of generality, we assume that

$$\|R\|_\infty=\sup_{t\geqslant t_0}\big|R(t)-\check{R}(t)\big|>0.$$

Let

$$x_3(t)=R(t)-\check{R}(t)\quad\text{for all } t\in(t_0,\infty).$$

Then

$$x_3'(t)=\gamma(t)x_2(t)+\frac{\vartheta(t)x_2(t)}{(1+\alpha(t)I(t))(1+\alpha(t)\check{I}(t))}-\mu_1(t)x_3(t)$$

and

$$\begin{aligned}x_3(t)&=\mathrm{e}^{-\int_{\check{t}_0}^{t}\mu_1(\theta)\,\mathrm{d}\theta}x_3(\check{t}_0)\\&\quad+\int_{\check{t}_0}^{t}\mathrm{e}^{-\int_v^t\mu_1(\theta)\,\mathrm{d}\theta}\left[\gamma(v)x_2(v)+\frac{\vartheta(v)x_2(v)}{(1+\alpha(v)I(v))(1+\alpha(v)\check{I}(v))}\right]\mathrm{d}v\end{aligned}\tag{36}$$

for all $t\geqslant\tilde{t}_0\geqslant\check{t}_0$. For any $\epsilon>0$, since $\mu_1^- -\zeta>0$, we can choose $t_R\geqslant\check{t}_0$ and $k_0^*>k_0$ such that

$$-(\mu_1^- -\zeta)+\frac{(\gamma^+ +\vartheta^+)k_0\|x\|_0\mathrm{e}^{\zeta\check{t}_0}}{k_0^*(\|R\|_\infty+\epsilon)\mathrm{e}^{\zeta t_R}}\leqslant-(\mu_1^- -\zeta)+\frac{(\gamma^+ +\vartheta^+)k_0\|x\|_0\mathrm{e}^{\zeta\check{t}_0}}{k_0^*\|R\|_\infty\mathrm{e}^{\zeta t_R}}<0.\tag{37}$$

Consequently,

$$\big|x_3(t)\big|<\|R\|_\infty+\epsilon<k_0^*\big(\|R\|_\infty+\epsilon\big)=k_0^*\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta t_R}\mathrm{e}^{-\zeta t_R}.$$

Now, we will show

$$\big|x_3(t)\big|<k_0^*\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta t_R}\mathrm{e}^{-\zeta t}\quad\text{for all } t\geqslant t_R.\tag{38}$$

Otherwise, there must exists $\theta^*>t_R$ such that

$$\big\|x_3(\theta^*)\big\|=k_0^*\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta t_R}\mathrm{e}^{-\zeta\theta^*},\tag{39}$$
$$\big\|x_3(t)\big\|<k_0^*\big(\|x\|_0+\epsilon\big)\mathrm{e}^{\zeta t_R}\mathrm{e}^{-\zeta t}\quad\text{for all } t\in[t_R,\theta^*).\tag{40}$$

From (36), (37), (39), and (40), we have

$$\begin{aligned}
\left|x_3(\theta^*)\right| &= \Bigg| \mathrm{e}^{-\int_{t_R}^{\theta^*} \mu_1(\theta)\,\mathrm{d}\theta} x_3(t_R) \\
&\quad + \int_{t_R}^{\theta^*} \mathrm{e}^{-\int_v^{\theta_1} \mu_1(\theta)\,\mathrm{d}\theta} \left[\gamma(v)x_2(v) + \frac{\vartheta(v)x_2(v)}{(1+\alpha(v)I(v))(1+\alpha(v)\check{I}(v))}\right] \mathrm{d}v \Bigg| \\
&\leqslant \mathrm{e}^{-\mu_1^-(\theta^*-t_R)}\left|x_3(t_R)\right| + \int_{t_R}^{\theta^*} \mathrm{e}^{-\mu_1^-(\theta^*-v)}\left(\gamma^+ + \vartheta^+\right)\left|x_2(v)\right| \mathrm{d}v \\
&\leqslant \mathrm{e}^{-\mu_1^-(\theta^*-t_R)}\left(\|R\|_\infty + \epsilon\right) \\
&\quad + \int_{t_R}^{\theta^*} \mathrm{e}^{-\mu_1^-(\theta^*-v)}(\gamma^+ + \vartheta^+)k_0\left(\|x\|_0 + \epsilon\right)\mathrm{e}^{\zeta\check{t}_0}\mathrm{e}^{-\zeta v}\,\mathrm{d}v \\
&\leqslant k_0^*\left(\|R\|_\infty + \epsilon\right)\mathrm{e}^{\zeta t_R}\mathrm{e}^{-\zeta\theta^*} \\
&\quad \times \left[\frac{1}{k_0^*}\mathrm{e}^{-(\theta^*-t_R)(\mu_1^- - \zeta)} + \int_{t_R}^{\theta^*} \mathrm{e}^{-(\theta^*-t_R)(\mu_1^- - \zeta)} \frac{(\gamma^+ + \vartheta^+)k_0\|x\|_0\mathrm{e}^{\zeta\check{t}_0}}{k_0^*\left(\|R\|_\infty + \epsilon\right)\mathrm{e}^{\zeta t_R}}\,\mathrm{d}v\right] \\
&\leqslant k_0^*\left(\|R\|_\infty + \epsilon\right)\mathrm{e}^{\zeta t_R}\mathrm{e}^{-\zeta\theta^*} \\
&\quad \times \left[\frac{1}{k_0^*}\mathrm{e}^{-(\theta^*-t_R)(\mu_1^- - \zeta)} + \int_{t_R}^{\theta^*} \mathrm{e}^{-(\theta^*-t_R)(\mu_1^- - \zeta)}\left(\mu_1^- - \zeta\right)\mathrm{d}v\right] \\
&= k_0^*\left(\|R\|_\infty + \epsilon\right)\mathrm{e}^{\zeta t_R}\mathrm{e}^{-\zeta\theta^*}\left[1 - \left(1 - \frac{1}{k_0^*}\right)\mathrm{e}^{-(\theta^*-t_R)(\mu_1^- - \zeta)}\right] \\
&< k_0^*\left(\|R\|_\infty + \epsilon\right)\mathrm{e}^{\zeta t_R}\mathrm{e}^{-\zeta\theta^*},
\end{aligned}$$

which contradicts (39). Hence, (38) holds. Letting $\epsilon \to 0^+$, we deduce from (38) that (22) holds, which ends the proof. □

Remark 1. Lemma 3 shows that a T-periodic solution $(\check{s}(t), \check{I}(t), \check{R}(t), \check{B}(t))$ of (1) is globally exponentially stable.

3 Main results

Theorem 2. *Under the assumptions of Lemma* 3, *system* (1) *has exactly one positive T-periodic solution, which is globally exponentially stable.*

Proof. Let $(\widetilde{S}(t), \widetilde{I}(t), \widetilde{R}(t), \widetilde{B}(t))$ be a solution of (1) with initial conditions

$$\widetilde{S}(t_0) > 0, \qquad \widetilde{I}(t_0) > 0, \qquad \widetilde{R}(t_0) \geqslant 0, \qquad \widetilde{B}(t_0) \geqslant 0,$$

By Lemmas 2 and 3, the solution $(\widetilde{S}(t), \widetilde{I}(t), \widetilde{R}(t), \widetilde{B}(t))$ is bounded, and

$$\liminf_{t\to\infty} \widetilde{S}(t) > 0, \qquad \liminf_{t\to\infty} \widetilde{I}(t) > 0, \qquad \liminf_{t\to\infty} \widetilde{R}(t) \geqslant 0, \qquad \liminf_{t\to\infty} \widetilde{B}(t) \geqslant 0.$$

By the periodicity of the coefficients of system (1), one can easily see that, for any nonnegative integer h, $(\widetilde{S}(t+hT), \widetilde{I}(t+hT), \widetilde{R}(t+hT), \widetilde{B}(t+hT))$ is a solution of system (1) with initial values

$$\big(\widetilde{S}(t_0+hT), \widetilde{I}(t_0+hT), \widetilde{R}(t_0+hT), \widetilde{B}(t_0+hT)\big).$$

In particular, $(\widehat{S}(t), \widehat{I}(t), \widehat{R}(t), \widehat{B}(t)) = (\widetilde{S}(t+T), \widetilde{I}(t+T), \widetilde{R}(t+T), \widetilde{B}(t+T))$ is a solution of (1) with initial values

$$\big(\widehat{S}(t_0), \widehat{I}(t_0), \widehat{R}(t_0), \widehat{B}(t_0)\big) = \big(\widetilde{S}(t_0+T), \widetilde{I}(t_0+T), \widetilde{R}(t_0+T), \widetilde{B}(t_0+T)\big).$$

It follows from Lemma 3 that there exist $\widehat{t_0} > t_0$ and $\widehat{k}$ such that, for an nonnegative integer h and $t + hT \geqslant \widehat{t_0}$,

$$\begin{aligned}
\big|\widetilde{S}\big(t+(h+1)T\big) - \widetilde{S}(t+hT)\big| &= \big|\widehat{S}(t+hT) - \widetilde{S}(t+hT)\big| \leqslant \hat{K}\mathrm{e}^{-\zeta(t+hT)},\\
\big|\widetilde{I}\big(t+(h+1)T\big) - \widetilde{I}(t+hT)\big| &= \big|\widehat{I}(t+hT) - \widetilde{I}(t+hT)\big| \leqslant \hat{K}\mathrm{e}^{-\zeta(t+hT)},\\
\big|\widetilde{R}\big(t+(h+1)T\big) - \widetilde{R}(t+hT)\big| &= \big|\widehat{R}(t+hT) - \widetilde{R}(t+hT)\big| \leqslant \hat{K}\mathrm{e}^{-\zeta(t+hT)},\\
\big|\widetilde{B}\big(t+(h+1)T\big) - \widetilde{B}(t+hT)\big| &= \big|\widehat{B}(t+hT) - \widetilde{B}(t+hT)\big| \leqslant \hat{K}\mathrm{e}^{-\zeta(t+hT)}.
\end{aligned} \tag{41}$$

Now, we show that $(\widetilde{S}(t+hT), \widetilde{I}(t+hT), \widetilde{R}(t+hT), \widetilde{B}(t+hT))_q$ is convergent on any compact interval as $q \to \infty$. Let $[a,b] \subset \mathbb{R}$ be an arbitrary interval. Choose a nonnegative integer q_0 such that $t + q_0T \geqslant \widehat{t_0}$ for $t \in [a,b]$. Then for $t \in [a,b]$ and $q > q_0$ we have

$$\begin{aligned}
\widetilde{S}(t+qT) &= \widetilde{S}(t+q_0T) + \sum_{h=q_0}^{q-1} \big[\widetilde{S}\big(t+(h+1)T\big) - \widetilde{S}(t+hT)\big],\\
\widetilde{I}(t+qT) &= \widetilde{I}(t+q_0T) + \sum_{h=q_0}^{q-1} \big[\widetilde{I}\big(t+(h+1)T\big) - \widetilde{I}(t+hT)\big],\\
\widetilde{R}(t+qT) &= \widetilde{R}(t+q_0T) + \sum_{h=q_0}^{q-1} \big[\widetilde{R}\big(t+(h+1)T\big) - \widetilde{R}(t+hT)\big],\\
\widetilde{B}(t+qT) &= \widetilde{B}(t+q_0T) + \sum_{h=q_0}^{q-1} \big[\widetilde{B}\big(t+(h+1)T\big) - \widetilde{B}(t+hT)\big],
\end{aligned}$$

which, together with (41), implies that $\{(\widetilde{S}(t+hT), \widetilde{I}(t+hT), \widetilde{R}(t+hT), \widetilde{B}(t+hT))\}_q$ converges uniformly to a continuous function, say $(S^*(t), I^*(t), R^*(t), B^*(t))$, on $[a,b] \subset \mathbb{R}$.

Because of arbitrariness of $[a,b]$, $(\widetilde{S}(t+qT), \widetilde{I}(t+qT), \widetilde{R}(t+qT), \widetilde{B}(t+qT)) \to (S^*(t), I^*(t), R^*(t), B^*(t))$ as $q\to\infty$ for $t\in\mathbb{R}$. Moreover, $(S^*(t), I^*(t), R^*(t), B^*(t))$ is bounded and

$$\begin{aligned} S^* &\geqslant \liminf_{t\to\infty} \widetilde{S}(t) > 0, \qquad & I^* &\geqslant \liminf_{t\to\infty} \widetilde{I}(t) > 0, \\ R^* &\geqslant \liminf_{t\to\infty} \widetilde{R}(t) \geqslant 0, \qquad & B^* &\geqslant \liminf_{t\to\infty} \widetilde{B}(t) \geqslant 0 \end{aligned}$$

for all $t\in\mathbb{R}$.

It remains to show that $(S^*(t), I^*(t), R^*(t), B^*(t))$ is a T-periodic solution of system (1). The periodicity is obvious since

$$\begin{aligned} S^*(t+T) &= \lim_{q\to\infty} \widetilde{S}\big((t+T)+qT\big) = \lim_{q+1\to\infty} \widetilde{S}\big(t+(q+1)T\big) = S^*(t), \\ I^*(t+T) &= \lim_{q\to\infty} \widetilde{I}\big((t+T)+qT\big) = \lim_{q+1\to\infty} \widetilde{I}\big(t+(q+1)T\big) = I^*(t), \\ R^*(t+T) &= \lim_{q\to\infty} \widetilde{R}\big((t+T)+qT\big) = \lim_{q+1\to\infty} \widetilde{R}\big(t+(q+1)T\big) = R^*(t), \\ B^*(t+T) &= \lim_{q\to\infty} \widetilde{B}\big((t+T)+qT\big) = \lim_{q+1\to\infty} \widetilde{B}\big(t+(q+1)T\big) = B^*(t) \end{aligned}$$

for all $t\in\mathbb{R}$. Now, note that $(\widetilde{S}(t+qT), \widetilde{I}(t+qT), \widetilde{R}(t+qT), \widetilde{B}(t+qT))$ is a solution to (1), i.e.

$$\begin{aligned} \widetilde{S}(t-qT) - \widetilde{S}(t_0+qT) &= \int_{t_0}^{t} \biggl[A(s+qT) - \mu_1(s+qT)\widetilde{S}(s+qT) \\ &\quad - \frac{\beta(s+qT)\widetilde{S}(s+qT)\widetilde{B}(s+qT)}{K(s+qT)+\widetilde{B}(s+qT)}\biggr]\,\mathrm{d}s, \\ \widetilde{I}(t-qT) - \widetilde{S}(t_0+qT) &= \int_{t_0}^{t} \biggl[\frac{\beta(s+qT)\widetilde{S}(s+qT)\widetilde{B}(s+qT)}{K(s+qT)+\widetilde{B}(s+qT)} \\ &\quad - \big(\gamma(s+qT)+\mu_1(s+qT)+\delta(s+qT)\big)\widetilde{I}(s+qT) \\ &\quad - \frac{\vartheta(s+qT)\widetilde{I}(s+qT)}{1+\alpha(s+qT)\widetilde{I}(s+qT)}\biggr]\,\mathrm{d}s, \\ \widetilde{R}(t-qT) - \widetilde{S}(t_0+qT) &= \int_{t_0}^{t} \biggl[\gamma(s+qT)\widetilde{I}(s+qT) + \frac{\vartheta(s+qT)\widetilde{I}(s+qT)}{1+\alpha(s+qT)\widetilde{I}(s+qT)} \\ &\quad - \mu_1(s+qT)\widetilde{R}(s+qT)\biggr]\,\mathrm{d}s, \\ \widetilde{B}(t-qT) - \widetilde{S}(t_0+qT) &= \int_{t_0}^{t} \big[\eta(s+qT)\widetilde{I}(s+qT) - \mu_2(s+qT)\widetilde{B}(s+qT)\big]\,\mathrm{d}s \end{aligned}$$

for $t \geqslant t_0$. Letting $q \to \infty$ gives

$$S^*(t) - S^*(t_0) = \int\limits_{t_0}^{t} \left[A(s) - \mu_1(s)S^*(s) - \frac{\beta(s)S^*(s)B^*(s)}{K(s)+B^*(s)}\right] \mathrm{d}s,$$

$$I^*(t) - I^*(t_0) = \int\limits_{t_0}^{t} \left[\frac{\beta(s)S^*(s)B^*(s)}{K(s)+B^*(s)} - \big(\gamma(s)+\mu_1(s)+\delta(s)\big)I^*(s) - \frac{\vartheta(s)I^*(s)}{1+\alpha(s)I^*(s)}\right] \mathrm{d}s,$$

$$R^*(t) - R^*(t_0) = \int\limits_{t_0}^{t} \left[\gamma(s)I^*(s) + \frac{\vartheta(s)I^*(s)}{1+\alpha(s)I^*(s)} - \mu_1(s)R^*S\right] \mathrm{d}s,$$

$$B^*(t) - B^*(t_0) = \int\limits_{t_0}^{t} \left[\eta(s)I^*(s) - \mu_2(s)B^*S\right] \mathrm{d}s$$

for $t \geqslant t_0$, so $(S^*(t), I^*(t), R^*(t), B^*(t))$ is a solution to (1) on $[t_0, \infty)$.

Lastly, by Lemma 3, $(S^*(t), I^*(t), R^*(t), B^*(t))$ is globally exponentially stable. □

Remark 2. Assume that all parameters are constants. Then the autonomous cholera model (1) has exactly one endemic equilibrium, which is globally exponentially stable.

4 Simulations

In this section, we will illustrate the existence and global exponential stability of positive periodic solutions for system (1) by simulations. Let $A(t) = 20(1+0.75|\sin(2\pi t/365)|)$, $\beta(t) = 0.005(1 + 0.75|\cos(2\pi t/365)|)$, and other parameters are listed in Table 1. Then system (1) satisfies all the conditions in Theorem 2. Hence, system (1) has exactly one positive 511.4-periodic solution $(S^*(t), I^*(t), R^*(t), B^*(t))$. Moreover, it is globally exponentially stable with exponential convergence rate $\zeta \approx 0.036$. This fact is confirmed by the numerical simulations in Fig. 1. The experimental environment of numerical simulation is Matlab 2.9a.

Table 1. Estimation of parameters.

Parameters	Meaning	Values	Reference
$\eta(t)$	Contribution of infected individuals to the population of V. cholera	0.032 cells/L-per day	[3]
$\mu_1(t)$	Natural death rate of human	5.48×10^{-5}/day	[14]
$\mu_2(t)$	Rate of loss of V. cholera	0.33/day	[3]
$\gamma(t)$	Recovery rate	0.04/day	[7, 8]
$\delta(t)$	Disease-induced death rate	0.015/day	[7]
$K(t)$	Concentration of V. cholera in water	4000 cells/L	Assumed
$\vartheta(t)$	Maximum recovery rate	0.05	Assumed
$\alpha(t)$	50% saturation	120	Assumed

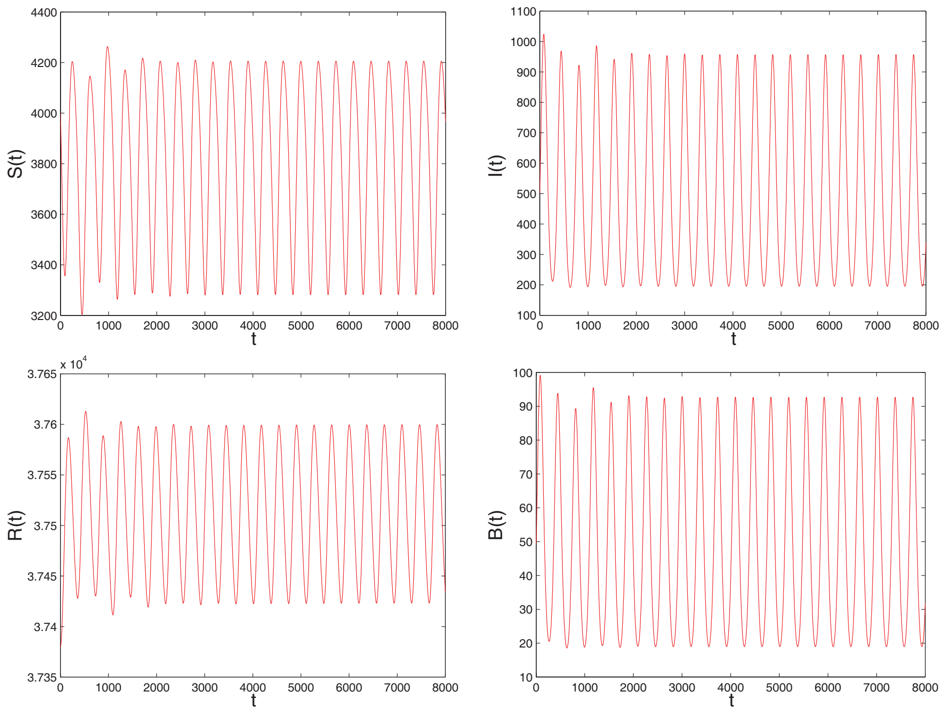

Figure 1. Numerical solutions of (1) for $(S(0), I(0), R(0), B(0)) = (4000, 500, 37400, 50)$.

Remark 3. To the best of our knowledge, there is no result on the global exponential stability of positive periodic solutions for the cholera model with periodic incidence rate and saturated treatment function. We also mention that the results in (see [1, 5, 9, 13]) can not be applied to the global exponential stability of positive periodic solutions for system (1). Here we employ a novel proof to establish some criteria, which guarantee the existence and global exponential stability of positive periodic solutions for the cholera model.

5 Discussion

In this paper, we considered a non-autonomous cholera epidemic model, which involves almost periodic incidence rate and saturated treatment function. By using the differential inequality technique and Lyapunov functional method, we obtained the existence and global exponential stability of almost periodic solutions for the addressed SIR model, which improve and supplement existing ones. Also, an example and its numerical simulations are given to demonstrate our theoretical results.

As we all known, spatial diffusion plays an important role in epidemic spread [11, 18, 19, 24]. We will study the cholera models with spatial diffusion in the future.

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