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Extension of the discrete universality theorem for zeta-functions of certain cusp forms*

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Abstract. In the paper, an universality theorem on the approximation of analytic functions by generalized discrete shifts of zeta functions of Hecke-eigen cusp forms is obtained. These shifts are defined by using the function having continuous derivative satisfying certain natural growth conditions and, on positive integers, uniformly distributed modulo 1.

Keywords: Hecke-eigen cusp form, uniform distribution modulo 1, universality, zeta-function of cusp form.

1 Introduction

In [18], S.M. Voronin discovered the universality property of the Riemann zeta-function $\zeta(s)$, $s = \sigma + \mathrm{i}t$, on the approximation of a wide class of analytic functions by shifts $\zeta(s + \mathrm{i}\tau)$, $\tau \in \mathbb{R}$. Later, it turned out that some other zeta and L-functions also are universal in the Voronin sense, among them, zeta-functions of certain cusp forms. We recall their definition.

Let

$$SL(2,\mathbb{Z}) \stackrel{\mathrm{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \colon a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

be the full modular group. The function $F(z)$ is called a holomorphic cusp form of weight κ for $SL(2,\mathbb{Z})$ if $F(z)$ is holomorphic for $\operatorname{Im} z > 0$, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z),$$

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and, at infinity, has the Fourier series expansion

$$F(z)=\sum_{m=1}^{\infty} c(m)\mathrm{e}^{2\pi \mathrm{i} m z}.$$

We assume additionally that the cusp form $F(z)$ is a normalized Hecke-eigen cusp form, i.e., is an eigen form of all Hecke operators

$$T_m F(z)=m^{\kappa-1} \sum_{\substack{a,d>0\\ ad=m}} \frac{1}{d^\kappa} \sum_{b\,(\mathrm{mod}\,d)} F\left(\frac{az+b}{d}\right), \quad m\in\mathbb{N}.$$

Then it is known that the Fourier coefficients $c(m)\neq 0$. Therefore, after normalization, we can assume that $c(1)=1$.

The zeta-function $\zeta(s,F)$ associated to a normalized Hecke-eigen cusp form $F(z)$ of weight κ is defined, for $\sigma>(\kappa+1/2$, by the Dirichlet series

$$\zeta(s,F)=\sum_{m=1}^{\infty} \frac{c(m)}{m^s}$$

and can be analytically continued to an entire function. Moreover, as the Riemann zeta-function, the function $\zeta(s,F)$, for $\sigma>(\kappa+1)/2$, has the Euler product expansion over primes

$$\zeta(s,F)=\prod_p \left(1-\frac{\alpha(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying $\alpha(p)+\beta(p)=c(p)$.

The universality of $\zeta(s,F)$ was obtained in [7]. Let $D_F=\{s\in\mathbb{C}\colon \kappa/2<\sigma<(\kappa+1)/2\}$. Denote by $\mathcal{K}_F$ the class of compact subsets of the strip D_F with connected complements and by $H_0(K)$, $K\in\mathcal{K}_F$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Let $\mathrm{meas}\,A$ stand for the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then the main theorem of [7] is of the following form.

Theorem 1. *Suppose that $K\in\mathcal{K}_F$ and $f(s)\in H_0(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]\colon \sup_{s\in K}\big|\zeta(s+\mathrm{i}\tau,\,F)-f(s)\big|<\varepsilon\Big\}>0.$$

Generalizations of Theorem 1 were given in [8] and [6].

The discrete version of universality for zeta-functions was proposed by A. Reich. In [16], he obtained a discrete universality theorem for Dedekind zeta-functions. In his theorem, τ takes values from the arithmetic progression $\{kh\colon k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$, where $h>0$ is a fixed number. The first discrete universality theorem for $\zeta(s,F)$ attached to a new form $F(z)$, under a certain arithmetical hypothesis for the number h, was proved in [9]. In [10], this hypothesis was removed, and the following statement was obtained.

Theorem 2. *Let $\#A$ denote the cardinality of a set A. Suppose that $K \in \mathcal{K}_F$, $f(s) \in H_0(K)$, and $h > 0$ is an arbitrary fixed number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N\colon \sup_{s\in K} \big|\zeta(s+\mathrm{i}kh,\, F) - f(s)\big| < \varepsilon\Big\} > 0.$$

There exists a problem to prove analogues of Theorem 2 for the sets different from the progression $\{kh\colon k \in \mathbb{N}_0\}$. The first attempt in this direction, in the case of the Riemann zeta-function, was made in [2], where the arithmetical progression was replaced by the set $\{k^\alpha h\colon k \in \mathbb{N}_0\}$ with a fixed α, $0 < \alpha < 1$. An analogue of the theorem from [2] for the function $\zeta(s, F)$ was given in [5]. Ł. Pańkowski investigating the joint universality of Dirichlet L-functions extended [15] the theorem of [2] for all non-integers $\alpha > 0$ and more general sets of the type $\{hk^\alpha \log^\beta k\}$, where

$$\beta = \begin{cases} \mathbb{R} & \text{if } \alpha \notin \mathbb{Z},\\ (-\infty, 0] \cup (1, \infty) & \text{if } \alpha \in \mathbb{N}. \end{cases}$$

The aim of this paper is to prove a discrete universality theorem for the function $\zeta(s, F)$ when τ in $\zeta(s + \mathrm{i}\tau,\, F)$ runs over some general sequence of real numbers.

For the definition of a class of sequences for τ, we will use the notion of uniform distribution modulo 1. Let $\{u\}$ denote the fractional part of $u \in \mathbb{R}$, and let χ_I be the indicator function of the set I. We remind that a sequence $\{x_k\colon k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every interval $I = [a, b) \subset [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I\big(\{x_k\}\big) = b - a.$$

Let $k_0 \in \mathbb{N}$. We say that a function $\varphi \in U(k_0)$ if the following hypotheses are satisfied:

(i) $\varphi(t)$ is a real-valued positive increasing function on $[k_0 - 1/2, \infty)$.

(ii) $\varphi(t)$ has a continuous derivative $\varphi'(t)$ on $[k_0 - 1/2, \infty)$ satisfying the estimate

$$\varphi(2t) \max_{t\leqslant u\leqslant 2t} \frac{1}{\varphi'(u)} \ll t.$$

(iii) A sequence $\{a\varphi(k)\colon k \geqslant k_0\} \subset \mathbb{R}$ with every $a \in \mathbb{R}\backslash\{0\}$ is uniformly distributed modulo 1.

For example, the function $\varphi(t) = t \log^\alpha t$ with $0 < \alpha < 1$ is an element of the class $U(2)$ because the sequence $\{ak \log^\alpha k\}$ is uniformly distributed modulo 1 [3, Exercise 3.14]. On the other hand, this sequence does not belong to the set of sequences of [15].

Theorem 3. *Suppose that $\varphi \in U(k_0)$. Let $K \in \mathcal{K}_F$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N\colon \sup_{s\in K} \big|\zeta\big(s+\mathrm{i}\varphi(k),\, F\big) - f(s)\big| < \varepsilon\Big\} > 0.$$

It is known [11,12] that universality theorems have a modified form. Thus, Theorem 3 can be stated in the following form.

Theorem 4. *Suppose that $\varphi \in U(k_0)$. Let $K \in \mathcal{K}_F$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N\colon \sup_{s\in K} \big|\zeta\big(s+\mathrm{i}\varphi(k),\, F\big) - f(s)\big| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

2 Auxiliary results

For the proof of universality for the function $\zeta(s,F)$, we will use the probabilistic approach. Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X. Let P_n, $n \in \mathbb{N}$, and P be the probability measures on $(X, \mathcal{B}(X))$. We remind that P_n, as $n \to \infty$, converges weakly to P if, for every real continuous bounded function g on X,

$$\lim_{n\to\infty} \int_X g \,\mathrm{d}P_n = \int_X g \,\mathrm{d}P.$$

Denote by $H(D_F)$ the space of analytic functions on D_F endowed with the topology of uniform convergence on compacta. The proof of universality theorems is based on the weak convergence for

$$P_{N,F}(A) \stackrel{\mathrm{def}}{=} \frac{1}{N-k_0+1} \#\big\{k_0 \leqslant k \leqslant N\colon \zeta\big(s+\mathrm{i}\varphi(k), F\big) \in A\big\}, \quad A \in \mathcal{B}\big(H(D_F)\big),$$

as $N \to \infty$.

For the statement of a limit theorem for $P_{N,F}$, we need some notation. Let $\mathbb{P}$ be the set of all prime numbers, and let γ denote the unit circle on the complex plane. Define the set

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group, therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathbb{P}$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D_F)$-valued random element $\zeta(s,\omega,F)$ by the formula

$$\zeta(s,\omega,F) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

Let $P_{\zeta,F}$ stand for the distribution of $\zeta(s,\omega,F)$, i.e.,

$$P_{\zeta,F}(A) = m_H\big\{\omega \in \Omega\colon \zeta(s,\omega,F) \in A\big\}, \quad A \in \mathcal{B}\big(H(D_F)\big).$$

Now we state the main result of this section.

Theorem 5. *Suppose that $\varphi \in U(k_0)$. Then $P_{N,F}$ converges weakly to $P_{\zeta,F}$ as $N \to \infty$. Moreover, the support of $P_{\zeta,F}$ is the set $S_F = \{g \in H(D_F)\colon\ g(s) \neq 0$ or $g(s) \equiv 0\}$.*

We divide the proof of Theorem 5 into several lemmas. We start with the Weyl criterion.

Lemma 1. *A sequence $\{x_k\colon\ k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for all $m \in \mathbb{Z} \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{e}^{2\pi \mathrm{i} m x_k} = 0.$$

Proof of the lemma can be found, for example, in [3].

For $A \in \mathcal{B}(\Omega)$, define

$$Q_N(A) = \frac{1}{N - k_0 + 1} \#\bigl\{k_0 \leqslant k \leqslant N\colon\ \bigl(p^{-\mathrm{i}\varphi(k)}\colon\ p \in \mathbb{P}\bigr) \in A\bigr\}.$$

Lemma 2. *Suppose that $\varphi \in U(k_0)$. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We apply the Fourier transform method. It is well known that the dual group of Ω is isomorphic to the group

$$\mathcal{D} = \bigoplus_p \mathbb{Z}_p,$$

where $\mathbb{Z}_p = \mathbb{Z}$ for all $p \in \mathbb{P}$. An element $\underline{k} = \{k_p\colon\ k_p \in \mathbb{Z},\ p \in \mathbb{P}\}$ of $\mathcal{D}$, where only a finite number of integers k_p are distinct from zero, acts on Ω by

$$\omega \to \omega^{\underline{k}} = {\prod_{p\in\mathbb{P}}}' \omega^{k_p}(p),$$

where the sign "$'$" means that only a finite number of integers k_p are distinct from zero. Hence, the characters are of the form

$${\prod_{p\in\mathbb{P}}}' \omega^{k_p}(p),$$

therefore, the Fourier transform $g_N(\underline{k})$ of Q_N is given by the formula

$$g_N(\underline{k}) = \int_{\Omega} {\prod_{p\in\mathbb{P}}}' \omega^{k_p}(p)\, \mathrm{d}Q_N.$$

Thus, by the definition of Q_N,

$$\begin{aligned} g_N(\underline{k}) &= \frac{1}{N - k_0 + 1} \sum_{k=k_0}^{N} {\prod_{p\in\mathbb{P}}}' p^{-ik_p\varphi(k)} \\ &= \frac{1}{N - k_0 + 1} \sum_{k=k_0}^{N} \exp\biggl\{-\mathrm{i}\varphi(k) {\sum_{p\in\mathbb{P}}}' k_p \log p\biggr\}. \end{aligned} \qquad (1)$$

Obviously,

$$g_N(\underline{0}) = 1. \tag{2}$$

Since the set $\{\log p\colon\ p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$, we have that $\sum'_{p\in\mathbb{P}} k_p \log p \neq 0$ for $\underline{k} \neq \underline{0}$. Therefore, since $\varphi \in U(k_0)$, in the case $\underline{k} \neq \underline{0}$, the sequence

$$\left\{ \frac{\varphi(k)}{2\pi} \sum_{p\in\mathbb{P}}{}' k_p \log p\colon\ k \geqslant k_0 \right\}$$

is uniformly distributed modulo 1. Thus, by Lemma 1 with $m = -1$ and (1), we find that, for $\underline{k} \neq \underline{0}$,

$$\lim_{N\to\infty} g_N(\underline{k}) = 0.$$

This and (2) show that $g_N(\underline{k})$, as $N \to \infty$, converges to the Fourier transform of the Haar measure m_H, and the lemma is a consequence of a continuity theorem for probability measures on compact groups. □

Lemma 2 implies a limit theorem in the space of analytic functions for a certain absolutely convergent Dirichlet series. This theorem is very important for proving Theorem 5, therefore, we give its precise statement.

We extend the functions $\omega(p)$, $p \in \mathbb{P}$, to the set $\mathbb{N}$ by

$$\omega(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Let $\theta > 1/2$ be a fixed number. For $m, n \in \mathbb{N}$, define the series

$$\zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m) v_n(m)}{m^s} \quad \text{and} \quad \zeta_n(s, \omega, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m) v_n(m)}{m^s},$$

where

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\}.$$

Then, the latter series are absolutely convergent for $\sigma > \kappa/2$. Let the function $u_{n,F}\colon\ \Omega \to H(D_F)$ be given by the formula $u_{n,F}(\omega) = \zeta_n(s, \omega, F)$. Since the series for $\zeta_n(s, \omega, F)$ is absolutely convergent for $\sigma > \kappa/2$, the function $u_{n,F}$ is continuous, thus, it is $(\mathcal{B}(\Omega), \mathcal{B}(H(D_F)))$-measurable. Hence, $\widehat{P}_{n,F} = m_H u_{n,F}^{-1}$, where

$$\widehat{P}_{n,F}(A) = m_H u_{n,F}^{-1}(A) = m_H\bigl(u_{n,F}^{-1} A\bigr), \quad A \in \mathcal{B}\bigl(H(D_F)\bigr),$$

is a probability measure on $(H(D_F), \mathcal{B}(H(D_F)))$. For $A \in \mathcal{B}(H(D_F))$, define

$$P_{N,n,F}(A) = \frac{1}{N - k_0 + 1} \#\bigl\{k_0 \leqslant k \leqslant N\colon\ \zeta_n\bigl(s + \mathrm{i}\varphi(k),\ F\bigr) \in A\bigr\}.$$

The above remarks, Lemma 2, and Theorem 5.1 of [1] lead to

Lemma 3. *Suppose that* $\varphi \in U(k_0)$. *Then* $P_{N,n,F}$ *converges weakly to* $\widehat{P}_{n,F}$ *as* $N \to \infty$.

Our next aim is to prove that $P_{N,F}$, as $N \to \infty$, converges weakly to the limit measure P_F of $\widehat{P}_{n,F}$ as $n \to \infty$. For this, we need some mean square results for the function $\zeta(s, F)$.

Lemma 4. *Suppose that* $\varphi \in U(k_0)$, *and* σ, $\kappa/2 < \sigma < (\kappa + 1)/2$, *is fixed. Then, for all* $\tau \in \mathbb{R}$,

$$\int\limits_{k_0-1/2}^{T} \bigl|\zeta\bigl(\sigma + \mathrm{i}\tau + \mathrm{i}\varphi(t),\, F\bigr)\bigr|^2 \,\mathrm{d}t \ll T\bigl(1 + |\tau|\bigr).$$

Proof. It is well known that, for fixed σ, $\kappa/2 < \sigma < (\kappa + 1)/2$,

$$\int\limits_{0}^{T} \bigl|\zeta(\sigma + \mathrm{i}t,\, F)\bigr|^2 \,\mathrm{d}t \ll T. \tag{3}$$

Let $X > 1$. Since the function $\varphi(t)$ is increasing and continuously differentiable, we have that

$$\begin{aligned}
&\int\limits_{X}^{2X} \bigl|\zeta\bigl(\sigma + \mathrm{i}\tau + \mathrm{i}\varphi(t),\, F\bigr)\bigr|^2 \,\mathrm{d}t \\
&\quad = \int\limits_{X}^{2X} \frac{1}{\varphi'(t)} \bigl|\zeta\bigl(\sigma + \mathrm{i}\tau + \mathrm{i}\varphi(t),\, F\bigr)\bigr|^2 \,\mathrm{d}\bigl(\varphi(t)\bigr) \\
&\quad \ll \max_{X \leqslant t \leqslant 2X} \frac{1}{\varphi'(t)} \int\limits_{X}^{2X} \mathrm{d}\Biggl(\int\limits_{0}^{|\tau|+\varphi(t)} \bigl|\zeta(\sigma + \mathrm{i}u,\, F)\bigr|^2 \,\mathrm{d}u \Biggr).
\end{aligned} \tag{4}$$

By estimate (3),

$$\int\limits_{0}^{|\tau|+\varphi(t)} \bigl|\zeta(\sigma + \mathrm{i}u,\, F)\bigr|^2 \,\mathrm{d}u \ll |\tau| + \varphi(t).$$

Since $\varphi \in U(k_0)$, the latter estimate together with (4) shows that

$$\begin{aligned}
\int\limits_{X}^{2X} \bigl|\zeta\bigl(\sigma + \mathrm{i}\tau + \mathrm{i}\varphi(t),\, F\bigr)\bigr|^2 \,\mathrm{d}t &\ll \bigl(|\tau| + \varphi(2X)\bigr) \max_{X \leqslant t \leqslant 2X} \frac{1}{\varphi'(t)} \\
&\ll X\bigl(1 + |\tau|\bigr).
\end{aligned}$$

Now, taking $X = 2^{-k-1}T$ and summing over k, gives the lemma. □

Lemma 4 together with Gallagher's lemma, which connects the continuous and discrete mean squares of some functions, allows to estimate the discrete mean square

$$I_N(\sigma, t, F) = \sum_{k=k_0}^{N} \bigl|\zeta\bigl(\sigma + \mathrm{i}t + \mathrm{i}\varphi(k),\, F\bigr)\bigr|^2.$$

For convenience, we state Gallagher's lemma, see [14, Lemma 1.4].

Lemma 5. *Suppose that T_0, $T \geqslant \delta > 0$ are real numbers, and $\mathcal{T} \neq \emptyset$ is a finite set in the interval $[T_0 + \delta/2,\, T_0 + T - \delta/2]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1.$$

Let $S(x)$ be a complex-valued continuous function on $[T_0,\, T + T_0]$ having a continuous derivative on $(T_0, T + T_0)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t)\bigl|S(t)\bigr|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} \bigl|S(x)\bigr|^2 \,\mathrm{d}x + \left(\int_{T_0}^{T_0+T} \bigl|S(x)\bigr|^2 \,\mathrm{d}x \int_{T_0}^{T_0+T} \bigl|S'(x)\bigr|^2 \,\mathrm{d}x \right)^{1/2}.$$

Lemma 6. *Suppose that $\varphi \in U(k_0)$, and σ, $\kappa/2 < \sigma < (\kappa+1)/2$, is fixed. Then, for $t \in \mathbb{R}$,*

$$I_N(\sigma, t, F) \ll N\bigl(1 + |t|\bigr).$$

Proof. An application of the Cauchy integral formula and Lemma 4 gives, for $\kappa/2 < \sigma < (\kappa+1)/2$, the bound

$$\int_{k_0-1/2}^{N+1/2} \bigl|\zeta'\bigl(\sigma + \mathrm{i}t + \mathrm{i}\varphi(t),\, F\bigr)\bigr|^2 \,\mathrm{d}t \ll N\bigl(1 + |t|\bigr). \tag{5}$$

Actually, in view of the Cauchy integral formula,

$$\zeta'\bigl(\sigma + \mathrm{i}t + \mathrm{i}\varphi(\tau),\, F\bigr) = \frac{1}{2\pi\mathrm{i}} \int_L \frac{\zeta(z + \mathrm{i}t + \mathrm{i}\varphi(\tau),\, F)}{(z-\sigma)^2} \,\mathrm{d}z,$$

where L is the circle with a center σ lying in D. Then

$$\begin{aligned} \bigl|\zeta'\bigl(\sigma + \mathrm{i}t + \mathrm{i}\varphi(\tau),\, F\bigr)\bigr|^2 &= \frac{1}{4\pi^2} \left| \int_L \frac{\zeta(z + \mathrm{i}t + \mathrm{i}\varphi(\tau),\, F)}{(z-\sigma)^2} \,\mathrm{d}z \right|^2 \\ &\ll \int_L \frac{|\mathrm{d}z|}{|z-\sigma|^4} \int_L \bigl|\zeta\bigl(z + \mathrm{i}t + \mathrm{i}\varphi(\tau),\, F\bigr)\bigr|^2 \,|\mathrm{d}z| \\ &\ll \int_L \bigl|\zeta\bigl(z + \mathrm{i}t + \mathrm{i}\varphi(\tau),\, F\bigr)\bigr|^2 \,|\mathrm{d}z|. \end{aligned}$$

Hence, in view of Lemma 4,

$$\begin{aligned}
&\int\limits_{k_0-1/2}^{N+1/2} \bigl|\zeta'\bigl(\sigma+\mathrm{i}t+\mathrm{i}\varphi(\tau),\, F\bigr)\bigr|^2\,\mathrm{d}\tau \\
&\quad\ll \int\limits_L |\mathrm{d}z| \int\limits_{k_0-1/2}^{N+1/2} \bigl|\zeta\bigl(\operatorname{Re} z+\mathrm{i}\operatorname{Im} z+\mathrm{i}t+\mathrm{i}\varphi(\tau),\, F\bigr)\bigr|^2\,\mathrm{d}\tau \\
&\quad\ll N\bigl(1+|t|\bigr).
\end{aligned}$$

We apply Lemma 5 with $\mathcal{T} = \{k\colon\ k\in\mathbb{N},\, k_0\leqslant k\leqslant N\}$, $T_0 = k_0-1/2$, $T = N-k_0+1$, and $\delta = 1$. Then, clearly, $N_\delta(x) = 1$, and, in view of Lemma 5 with $S(\tau) = \zeta(\sigma+\mathrm{i}t+\mathrm{i}\varphi(\tau),\, F)$, we have

$$\begin{aligned}
&I_N(\sigma,t,F) \\
&\quad\ll \int\limits_{k_0-1/2}^{N+1/2} \bigl|\zeta\bigl(\sigma+\mathrm{i}t+\mathrm{i}\varphi(\tau),\, F\bigr)\bigr|^2\,\mathrm{d}\tau \\
&\qquad+\Biggl(\,\int\limits_{k_0-1/2}^{N+1/2} \bigl|\zeta\bigl(\sigma+\mathrm{i}t+\mathrm{i}\varphi(\tau),\, F\bigr)\bigr|^2\,\mathrm{d}\tau \int\limits_{k_0-1/2}^{N+1/2} \bigl|\zeta'\bigl(\sigma+\mathrm{i}t+\mathrm{i}\varphi(\tau),\, F\bigr)\bigr|^2\,\mathrm{d}\tau\Biggr)^{1/2}.
\end{aligned}$$

This, Lemma 4, and estimate (5) prove the lemma. □

Now we are ready to approximate $\zeta(s,F)$ by $\zeta_n(s,F)$ in the mean. For $g_1, g_2 \in H(D_F)$, let

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|},$$

where $\{K_l\colon\ l\in\mathbb{N}\}\subset D_F$ is a sequence of compact subsets such that

$$D_F = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for $l\in\mathbb{N}$, and if $K\subset D_F$ is a compact subset, then $K\subset K_l$ for some $l\in\mathbb{N}$. Then ρ is the metric in $H(D_F)$ inducing its topology of uniform convergence on compacta.

Lemma 7. *Suppose that* $\varphi\in U(k_0)$. *Then*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N-k_0+1}\sum_{k=k_0}^{N}\rho\bigl(\zeta\bigl(s+\mathrm{i}\varphi(k),\, F\bigr),\, \zeta_n\bigl(s+\mathrm{i}\varphi(k),\, F\bigr)\bigr) = 0.$$

Proof. Let $\theta > 1/2$ be from the definition of $v_n(m)$, and

$$l_n(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)n^s,$$

where $\Gamma(s)$ is the Euler gamma-function. Then the function $\zeta_n(s, F)$ has the representation [7]

$$\zeta_n(s, F) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\theta-\mathrm{i}\infty}^{\theta+\mathrm{i}\infty} \zeta(s+z,\, F) l_n(z) \frac{\mathrm{d}z}{z}, \quad \sigma > \frac{\kappa}{2}.$$

Let K be an arbitrary compact subset of D. Then, using the above integral representation and the residue theorem, we find that

$$\begin{aligned} &\frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s\in K} \bigl|\zeta\bigl(s+\mathrm{i}\varphi(k),\, F\bigr) - \zeta_n\bigl(s+\mathrm{i}\varphi(k),\, F\bigr)\bigr| \\ &\quad \ll \int\limits_{-\infty}^{\infty} \bigl|l_n\bigl(\widehat{\sigma}+\mathrm{i}\tau\bigr)\bigr| \left(\frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \bigl|\zeta\bigl(\sigma+\mathrm{i}t+\mathrm{i}\tau+\mathrm{i}\varphi(k),\, F\bigr)\bigr|\right) \mathrm{d}\tau, \end{aligned} \tag{6}$$

where $\widehat{\sigma} < 0$, $\kappa/2 < \sigma < (\kappa+1)/2$, and t is bounded by a constant depending on K. Now an application of Lemma 6 and (6) implies the equality

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s\in K} \bigl|\zeta\bigl(s+\mathrm{i}\varphi(k)\bigr) - \zeta_n\bigl(s+\mathrm{i}\varphi(k)\bigr)\bigr| = 0.$$

This and the definition of the metric ρ prove the lemma. □

Proof of Theorem 5. Let θ_N be a random variable defined on a certain probability space with the measure μ and having the distribution

$$\mu\bigl\{\theta_N = \varphi(k)\bigr\} = \frac{1}{N-k_0+1}, \quad k = k_0, \dots, N.$$

Consider the $H(D_F)$-valued random element

$$X_{N,n,F} = X_{N,n,F}(s) = \zeta_n(s+\mathrm{i}\theta_N, F).$$

We recall that $\widehat{P}_{n,F}$ is the limit measure in Lemma 3. Then, in view of Lemma 3,

$$X_{N,n,F} \xrightarrow[N\to\infty]{\mathcal{D}} \widehat{X}_{n,F}, \tag{7}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution, and $\widehat{X}_{n,F}$ is the $H(D_F)$-valued random element with distribution $\widehat{P}_{n,F}$. Using the absolute convergence of the series for $\zeta_n(s, F)$ and (7), we prove by using the method of [4] that the family of probability measures

$\{\widehat{P}_{n,F}\colon n \in \mathbb{N}\}$ is tight. Hence, by Theorem 6.1 of [1], it is relatively compact. Therefore, each subsequence of $\{\widehat{P}_{n,F}\}$ contains a subsequence $\{\widehat{P}_{n_r,F}\}$, which converges weakly to a certain probability measure P_F on $(H(D_F), \mathcal{B}(H(D_F)))$ as $r \to \infty$. Thus

$$\widehat{X}_{n_r,F} \xrightarrow[r\to\infty]{\mathcal{D}} P_F. \tag{8}$$

On the probability space of the random variable θ_N, define the $H(D_F)$-valued random element

$$X_{N,F} = X_{N,F}(s) = \zeta(s + \mathrm{i}\theta_N,\, F).$$

Then the application of Lemma 7 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mu\bigl(\rho(X_{N,F}, X_{N,n,F}) \geqslant \varepsilon\bigr) \\
&\quad = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N - k_0 + 1} \\
&\qquad\qquad \times \#\bigl\{k_0 \leqslant k \leqslant N\colon \rho\bigl(\zeta\bigl(s + \mathrm{i}\varphi(k),\, F\bigr), \zeta_n\bigl(s + \mathrm{i}\varphi(k),\, F\bigr)\bigr) \geqslant \varepsilon\bigr\} \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N - k_0 + 1)\varepsilon} \sum_{k=k_0}^{N} \rho\bigl(\zeta\bigl(s + \mathrm{i}\varphi(k),\, F\bigr), \zeta_n\bigl(s + \mathrm{i}\varphi(k),\, F\bigr)\bigr) = 0.
\end{aligned}$$

From this, (7), (8), and Theorem 4.2 of [1] it follows that

$$X_{N,F} \xrightarrow[N\to\infty]{\mathcal{D}} P_F. \tag{9}$$

This means that $P_{N,F}$ converges weakly to P_F as $N \to \infty$. On the other hand, (9) shows that the measure P_F is independent of the sequence $\{\widehat{P}_{n_r,F}\}$. Since the family $\{\widehat{P}_{n,F}\}$ is relatively compact, hence we have, by Theorem 2.3 of [1], that

$$\widehat{X}_{n,F} \xrightarrow[n\to\infty]{\mathcal{D}} P_F,$$

or equivalently, $\widehat{P}_{n,F}$ converges weakly to P_F as $n \to \infty$.

It remains to identity the measure P_F. For this, usually, elements of the ergodic theory are applied. However, we use a very simple observation. It is known [7, 17] that

$$\frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0, T]\colon \zeta(s + \mathrm{i}\tau,\, F) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H(D_F)\bigr),$$

as $T \to \infty$, converges weakly to the limit measure P_F of $\widehat{P}_{n,F}$ and that $P_F = P_{\zeta,F}$. Moreover, the support of $P_{\zeta,F}$ is the set S_F. Therefore, $P_{N,F}$ also converges weakly to $P_{\zeta,F}$ as $N \to \infty$. □

3 Proofs of universality theorems

Proof of Theorem 3. Define

$$G_\varepsilon = \biggl\{g \in H(D)\colon \sup_{s\in K} \bigl|g(s) - \mathrm{e}^{p(s)}\bigr| < \frac{\varepsilon}{2}\biggr\},$$

where $p(s)$ is a polynomial. By Theorem 5, the function $\mathrm{e}^{p(s)}$ is an element of the support of the measure $P_{\zeta,F}$. Therefore,

$$P_{\zeta,F}(G_\varepsilon) > 0. \tag{10}$$

By Theorem 5 and the equivalent of weak convergence of probability measures in terms of open sets [1, Thm. 2.1],

$$\liminf_{N\to\infty} P_{N,F}(G_\varepsilon) \geqslant P_{\zeta,F}(G_\varepsilon).$$

This, the definitions of $P_{N,F}$ and G_ε, and (10) show that

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1} \# \left\{ k_0 \leqslant k \leqslant N\colon \sup_{s\in K} \left|\zeta\bigl(s+\mathrm{i}\varphi(k),\, F\bigr) - f(s)\right| < \frac{\varepsilon}{2} \right\} > 0. \tag{11}$$

By the Mergelyan theorem on the approximation of analytic functions by polynomials [13], we can choose the polynomial $p(s)$ to satisfy the inequality

$$\sup_{s\in K} \left|f(s) - \mathrm{e}^{p(s)}\right| < \frac{\varepsilon}{2}. \tag{12}$$

This inequality together with (11) proves Theorem 3. □

Proof of Theorem 4. Define the set

$$\widehat{G}_\varepsilon = \left\{ g \in H(D)\colon \sup_{s\in K} \left|g(s) - f(s)\right| < \varepsilon \right\}.$$

Then we have that the boundary $\partial\widehat{G}_\varepsilon$ of $\widehat{G}_\varepsilon$ is the set

$$\left\{ g \in H(D)\colon \sup_{s\in K} \left|g(s) - f(s)\right| = \varepsilon \right\}.$$

Hence, $\partial\widehat{G}_{\varepsilon_1} \cap \partial\widehat{G}_{\varepsilon_2} = \emptyset$ for $\varepsilon_1 \neq \varepsilon_2$. Therefore, the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,F}$ for all but at most countably many $\varepsilon > 0$. Using Theorem 5 and the equivalent of weak convergence of probability measures in terms of continuity sets [1, Thm. 2.1], we obtain that

$$\lim_{N\to\infty} P_{N,F}(\widehat{G}_\varepsilon) = P_{\zeta,F}(\widehat{G}_\varepsilon) \tag{13}$$

for all but at most countably many $\varepsilon > 0$. In view of (12), if $g \in G_\varepsilon$, then $g \in \widehat{G}_\varepsilon$. Thus, $G_\varepsilon \subset \widehat{G}_\varepsilon$. Therefore, in virtue of (10), $P_{\zeta,F}(\widehat{G}_\varepsilon) > 0$. Combining this with (13) and the definitions of $P_{N,F}$ and $\widehat{G}_\varepsilon$ proves Theorem 4. □

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