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Hilfer-type fractional differential switched inclusions with noninstantaneous impulsive and nonlocal conditions*

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Abstract. In this paper, we study a new class of nonlocal problems for noninstantaneous impulsive Hilfer-type fractional differential switched inclusions in Banach spaces. First, we introduce a mild solution formula for this noninstantaneous impulsive inclusion problem. Second, we show the existence of mild solutions using the Hausdorff measure of noncompactness on the space of piecewise weighted continuous functions. Finally, an example is provided to illustrate the theory.

Keywords: Hilfer fractional switched differential inclusions, noninstantaneous impulsive conditions, nonlocal conditions, existence.

1 Introduction

Fractional differential equations and fractional differential inclusions are important because of their applications in physics, mechanics, and engineering [6, 25]. For existence results for fractional differential equations and inclusions, we refer the reader to [1, 2, 10, 25, 26, 38] and the references therein. Impulsive differential equations and impulsive differential inclusions have applications in physics, biology, engineering, medical

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fields, industry, and technology [8]. They model processes, which, at certain moments, change their state rapidly. For results on mild solutions to impulsive differential equations and inclusions with instantaneous impulses, see [7, 9, 30, 32]. However, the action of instantaneous impulses do not describe certain dynamics of evolution processes in pharmacotherapy. Considering the hemodynamic equilibrium of a person in the case of a decompensation (for example, high or low levels of glucose), one can prescribe some intravenous drugs (insulin), and the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. The impulsive action starts at any arbitrary fixed point and stays active on a finite time interval and, as a result, the authors in [18, 36] introduced noninstantaneous impulsive differential equations.

Abstract nonlocal semilinear initial-value problems was initiated in [11, 12], where existence and uniqueness of mild solutions for nonlocal differential equations without impulsive was discussed (Lipschitz-type conditions were considered). In [27], the authors studied the case where the operator semigroup $T(t)$ is compact and the nonlinearity is single valued. The authors in [3] discuss existence of integral solutions for nonlocal differential inclusions when X is separable, the operator semigroup is compact, and the nonlinearity F is closed valued and lower semicontinuous in its second variable, and, using the measure of noncompactness, the authors in [39] obtain existence results for mild solutions for nonlocal problems when the evolution system is not compact. The authors in [22] considered a nonlocal impulsive differential inclusions governed by a noncompact semigroup, and recently, in [21], the author established the existence of mild solutions to impulsive differential inclusion of first order with nonlocal conditions and governed by a noncompact semigroup when the nonlinearity F satisfies a condition expressed in terms of the Hausdorff measure of noncompactness. For other contributions on nonlocal Cauchy problems, we refer the reader to [4, 17, 35].

Hilfer [19] proposed a generalization of the Riemann–Liouville fractional derivative, the Hilfer fractional derivative, which includes the Riemann–Liouville fractional derivative, and the Caputo fractional derivative. It appears in the theoretical simulation of dielectric relaxation in glass forming materials [16]. The authors in [14] established the existence and uniqueness of global solution in the space of weighted continuous functions for a fractional differential equations involving the Hilfer derivative, the authors in [34] discussed the existence of solutions to nonlocal initial value problems for differential equations with the Hilfer fractional derivative, and the authors in [16] obtained some sufficient conditions to ensure the existence of mild solutions of evolution equation with the Hilfer fractional derivative. In [37], the authors investigated the approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions, the authors in [13] established the approximate controllability of impulsive Hilfer fractional differential inclusions, the author in [24] derived an equivalent definition of the Hilfer derivative, and the authors in [33] studied the controllability of Caputo fractional noninstantaneous impulsive differential inclusions without compactness in reflexive Banach spaces.

The authors in [31] discussed Caputo-type fractional differential switched systems with coupled nonlocal initial and impulsive conditions in a Euclidean space, which extends the classical impulsive switched systems. There are a few papers discussing Hilfer-type fractional differential switched inclusions inserting noninstantaneous impulsive and nonlocal conditions. The aim of the paper is to study the following nonlocal problem for Hilfer-type fractional noninstantaneous impulsive differential inclusions:

$$\begin{aligned}
&D^{\alpha,\beta}_{s_i^+}x(t) \in F\big(t, x(t)\big), \quad \text{a.e. } t \in (s_i, t_{i+1}],\ i = 0, 1, \dots, m,\\
&x\big(t_i^+\big) = g_i\big(t_i, x\big(t_i^-\big)\big), \quad i = 1, \dots, m,\\
&x(t) = g_i\big(t, x\big(t_i^-\big)\big), \quad t \in (t_i, s_i],\ i = 1, \dots, m,\\
&I^{1-\gamma}_{0^+}x\big(0^+\big) = x_0 + g(x),\\
&I^{1-\gamma}_{s_i^+}x\big(s_i^+\big) = g_i\big(s_i, x\big(t_i^-\big)\big), \quad i = 1, \dots, m,
\end{aligned} \tag{1}$$

where $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$, $J = [0, b]$, $b > 0$, $D^{\alpha,\beta}_{s_i^+}$ is the Hilfer derivative with lower limit at s_i of order α and type β, E is a real Banach space, $0 = s_0 < t_1 < s_1 < t_2 < \dots < t_m < s_m < t_{m+1} = b$, $x(t_i^+)$, $x(t_i^-)$ are the right and left limits of x at the point t_i, respectively, $I^{1-\gamma}_{s_i^+}$ is the Riemann–Liouville integral of order $1 - \gamma$ with lower limit at s_i, and $I^{1-\gamma}_{s_i^+}x(s_i^+) = \lim_{t\to s_i^+} I^{1-\gamma}_{s_i^+}x(t)$. In addition, we set $x(t_i^-) = x(t_i)$. Moreover, $F : J \times E \to 2^E - \{\phi\}$ is a multifunction, $g : PC_{1-\gamma}(J, E) \to E$, $g_i : [t_i, s_i] \times E \to E$, $i = 1, 2, \dots, m$, and x_0 is a fixed point of E. The space $PC_{1-\gamma}(J, E)$ will be given in the next section.

The paper is organized as follows. In Section 2, we collect some background material concerning multifunctions and fractional calculus, and we introduce a measure of noncompactness on the space of piecewise weighted continuous functions. In Section 3, we establish the existence of mild solutions of (1), and in Section 4, we give an example to illustrate our theory.

2 Preliminaries

Denote $L^p(J, E) = \{v : J \to E\colon v \text{ is Bochner integrable}\}$ endowed with the norm $\|v\|_{L^p(J,E)} = (\int_J \|v(t)\|^p\,\mathrm{d}t)^{1/p}$, $p \in [1, \infty)$, $P_b(E) = \{B \subseteq E\colon B \text{ is nonempty and bounded}\}$, $P_{\mathrm{cl}}(E) = \{B \subseteq E\colon B \text{ is nonempty, convex and closed}\}$, $P_{\mathrm{ck}}(E) = \{B \subseteq E\colon B \text{ is nonempty, convex and compact}\}$, $\mathrm{conv}(B)$ (respectively, $\overline{\mathrm{conv}}(B)$) be the convex hull (respectively, convex closed hull in E) of a subset B. Let $C(J, E)$ be the Banach space of all E valued continuous functions from J to E with the norm $\|x\|_{C(J,E)} = \sup_{t\in J}\|x(t)\|$. For $a \in [0, b)$ and $0 \leqslant \gamma \leqslant 1$, consider the weighted spaces of continuous functions $C_\gamma([a, b], E) = \{x \in C((a, b], E)\colon (t - a)^\gamma x(t) \in C([a, b], E)\}$ and $C^n_\gamma([a, b], E) = \{x \in C^{n-1}([a, b], E)\colon x^{(n)} \in C_\gamma([a, b], E),\ n \in \mathbb{N}\}$. Now $C_\gamma([a, b], E)$ and $C^n_\gamma([a, b], E)$ are Banach spaces with norms $\|x\|_{C_\gamma([a,b],E)} = \sup_{t\in(a,b]}\|(t-a)^\gamma x(t)\|$ and $\|x\|_{C^n_\gamma([a,b],E)} = \sum_{k=1}^{k=n-1}\|x^{(k)}\|_{C([a,b],E)} + \|x^{(n)}\|_{C_\gamma([a,b],E)}$, respectively.

Let $J_k = (s_k, t_{k+1}]$, $\overline{J_k} = [s_k, t_{k+1}]$ $(k = 0, 1, \dots, m)$, $T_i = (t_i, s_i]$, $\overline{T_i} = [t_i, s_i]$ $(i = 1, 2, \dots, m)$, and consider the Banach space $PC_{1-\gamma}(J, E) = \{x\colon (t - s_k)^{1-\gamma}x \in C(J_k, E),\ \lim_{t\to s_k^+}(t - s_k)^{1-\gamma}x(t) \text{ exists},\ k = 0, 1, \dots, m,\ x \in C(T_i, E), \text{ and}$

$\lim_{t\to t_i^+} x(t)$ exist, $i=1,2,\dots,m\}$ with $\|x\|_{PC_{1-\gamma}(J,E)}=\max\{\sup_{t\in\overline{J_k},\,k=0,1,\dots,m}(t-s_k)^{1-\gamma}\|x(t)\|_E, \sup_{t\in\overline{T_i},\,i=1,\dots,m}\|x(t)\|_E\}$. Denote $C^{\gamma}_{1-\gamma}(J,E) = \{x \in C_{1-\gamma}(J,E),\ D^{\gamma}_{a^+}x \in C_{1-\gamma}(J,E)\}$, $C^{\alpha,\beta}_{1-\gamma}(J,E) = \{x \in C_{1-\gamma}(J,E),\ D^{\alpha,\beta}_{a^+}x \in C_{1-\gamma}(J,E)\}$, $PC^{\gamma}_{1-\gamma}(J,E) = \{x \in PC_{1-\gamma}(J,E),\ D^{\gamma}_{t_k^+}x_{|J_k} \in C_{1-\gamma}(J_k,E), k = 0,1,\dots,m\}$, and $PC^{\alpha,\beta}_{1-\gamma}(J,E) = \{x \in PC_{1-\gamma}(J,E),\ D^{\alpha,\beta}_{t_k^+}x_{|J_k} \in C_{1-\gamma}(J_k,E),\ k = 0,1,\dots,m\}$. Let us recall some facts concerning multifunctions (see [5,20]).

Definition 1. Let X and Y be two topological spaces. A multifunction $\mathscr{G} : X \to P(Y)\setminus\{\emptyset\}$ is said to be upper semicontinuous at $x_0 \in X$, *u.s.c.* for short, if for any open V containing $\mathscr{G}(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that $\mathscr{G}(x) \subseteq V$ for all $x \in N(x_0)$. We say that $\mathscr{G}$ is upper semicontinuous if it is so at every $x_0 \in X$.

Lemma 1. *Let X, Y be two Hausdorff topological spaces and $\mathscr{G} : X \to P(Y)\setminus\{\emptyset\}$.*

(i) *If $\mathscr{G}$ is upper semicontinuous with closed values, then the graph of $\mathscr{G}$ is closed in $X \times Y$, that is to say, if $y_n \in \mathscr{G}(x_n)$, $n \geqslant 1$, and $(x_n, y_n) \to (x, y)$ with respect to the product topology on $X \times Y$, then $y \in \mathscr{G}(x)$.*

(ii) *If $\mathscr{G}$ is a closed and locally compact (i.e. for any $x \in X$, there is a neighborhood $N(x)$ of x such that $\bigcup\{\mathscr{G}(z)\colon z \in N(x)\}$ is relatively compact in Y) with closed values, then $\mathscr{G}$ is u.s.c.*

(iii) *If $\mathscr{G}$ is upper semicontinuous and $\mathcal{K}$ is compact subset of X, then $\mathscr{G}(\mathcal{K}) = \bigcup\{\mathscr{G}(x)\colon x \in \mathcal{K}\}$ is compact in Y.*

We recall some definitions and facts concerning fractional integral and derivatives [6,19,25].

Definition 2. The Riemann–Liouville fractional integral of order $q > 0$ with the lower limit at a for a function $f \in L^p([a,b],E)$, $p \in [1,\infty)$, is defined as follows: $I^q_{a^+}f(t) = (g_q * f)(t) = \int_a^t ((t-s)^{q-1}/\Gamma(q))f(s)\,\mathrm{d}s$, $t \in [a,b]$, where the integration is in the sense of Bochner, Γ is the Euler gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1}\mathrm{e}^{-t}\,\mathrm{d}t$, $g_q(t) = t^{q-1}/\Gamma(q)$ for $t > 0$, $g_q(t) = 0$ for $t \leqslant 0$, and $*$ denotes the convolution of functions. For $q = 0$, we set $I^0_{a^+}f(t) = f(t)$.

Definition 3. Let $q > 0$, m be the smallest integer greater than or equal to q, and $f \in L^1([a,b],E)$ be such that $g_{m-q} * f \in W^{m,1}([a,b],E)$. The Riemann–Liouville fractional derivative of order q with the lower limit zero for f is defined by $D^q_{a^+}f(t) = (\mathrm{d}^m/\mathrm{d}t^m)I^{m-q}_{a^+}f(t) = (\mathrm{d}^m/\mathrm{d}t^m)(g_{m-q} * f)(t)$, where $W^{m,1}([a,b],E) = \{f\colon f(t) = \sum_{k=0}^{m-1} c_k t^k/k! + I^m_{a^+}\varphi(t),\ t \in [a,b],\ \varphi \in L^1([a,b],E)\}$, and $\varphi = f^{(m)}$, and $c_k = f^{(k)}(0)$, $k = 0,1,\dots,m-1$.

Definition 4. The Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leqslant \beta \leqslant 1$ and with lower limit at a for a function $f : [a,b] \to E$ is defined by $D^{\alpha,\beta}_{a^+}f(t) = (I^{\beta(1-\alpha)}_{a^+}D(I^{(1-\beta)(1-\alpha)}_{a^+}f))(t)$, $t \in [a,b]$, provided that the right side is point-wise defined on $[a,b]$; here $D = \mathrm{d}/\mathrm{d}t$.

Linking Definitions 2 and 4, the operator $D^{\alpha,\beta}_{a^+}$ can also be written as $D^{\alpha,\beta}_{a^+}f(t) = I^{\beta(1-\alpha)}_{a^+}D^{\gamma}_{a^+}f(t)$, $\gamma = \alpha+\beta-\alpha\beta$, provided that $D^{\gamma}_{a^+}f$ exist (an example is $f \in C^{\gamma}_{1-\gamma}(J,E)$).

Now we note some properties (the proofs are similar to the scalar case given in [14]).

Lemma 2. *Let $\alpha > 0$, $\beta \geqslant 0$, and $0 \leqslant \gamma < 1$.*

(i) *$I^{\alpha}_{a^+}$ is bounded from $C_\gamma([a,b],E)$ into $C_\gamma([a,b],E)$, and if $\gamma \leqslant \alpha$, then $I^{\alpha}_{a^+}$ is bounded from $C_\gamma([a,b],E)$ into $C([a,b],E)$.*

(ii) *If $f \in C_\gamma([a,b],E)$, then $I^{\alpha}_{a^+}f(a^+) = \lim_{t\to a^+} I^{\alpha}_{a^+}f(t) = 0$, $\gamma < \alpha$.*

(iii) *If $f \in L^1([a,b],E)$, then $I^{\alpha}_{a^+}I^{\beta}_{a^+}f(t) = I^{\alpha+\beta}_{a^+}f(t)$ a.e., in particular, if $f \in C_\gamma([a,b],E)$ or $f \in C([a,b],E)$, then equality holds for every $t \in (a,b]$ or $t \in [a,b]$, respectively.*

(iv) *If $f \in C_\gamma([a,b],E)$ and $I^{1-\alpha}_{a^+}f(t) \in C^1_\gamma([a,b],E)$, then $I^{\alpha}_{a^+}D^{\alpha}_{a^+}f(t) = f(t) - (\lim_{t\to a^+} I^{1-\alpha}_{a^+}f(t)/\Gamma(\alpha))(t-a)^{\alpha-1}$, $t \in [a,b]$.*

Remark 1. If $x \in PC_{1-\gamma}(J,E)$, then for any $k = 0,1,\dots,m$, the following hold:

(i) x is not necessarily defined at s_k, but $\lim_{t\to s_k+}(t-s_k)x(t)$ and $x(s^-_{k+1})$ exist.

(ii) $x(t_{k+1}) = x(t^-_{k+1})$ and $x(t^+_{k+1})$ exists. Moreover, $(t_{k+1}-s_k)^{1-\gamma}\|x(t^-_{k+1})\| \leqslant \|x\|_{PC_{1-\gamma}(J,E)}$.

(iii) If $x_n \to x$ in $PC_{1-\gamma}(J,E)$, then $x_n(t) \to x(t)$, $t \in (t_k, s_k]$, $k = 1,\dots,m$, and $(t-s_k)^{1-\gamma}x_n(t) \to (t-s_k)^{1-\gamma}x(t)$, $t \in (s_k, t_{k+1}]$. Consequently, $x_n(t) \to x(t)$, $t \in (s_i, t_{i+1}]$, and hence $x_n(t_{i+1}) = x_n(t^-_{i+1}) \to x(t_{i+1}) = x(t^-_{i+1})$, $i = 0,1,\dots,m$. It follows that $x_n(t) \to x(t)$ a.e. for $t \in J$.

The function $\chi_{PC_{1-\gamma}(J,E)} : P_b(PC_{1-\gamma}(J,E)) \to [0,\infty)$, defined by

$$\chi_{PC_{1-\gamma}(J,E)}(Z) = \max\Big\{\max_{k=0,1,\dots,m}\chi_{C(\overline{J_k},E)}(Z_{|\overline{J_k}}),\ \max_{i=1,\dots,m}\chi_{C(\overline{T_i},E)}(Z_{|\overline{T_i}})\Big\}, \tag{2}$$

is a measure of noncompactness on $PC_{1-\gamma}(J,E)$, where $Z_{|\overline{J_k}} = \{y^* \in C(\overline{J_k},E)\colon y^*(t) = (t-s_k)^{1-\gamma}y(t),\ t\in J_k,\ y^*(s_k) = \lim_{t\to s_k^+}(t-s_k)^{1-\gamma}y(t),\ y\in Z\}$, and $Z_{|\overline{T_i}} = \{y^* \in C(\overline{T_i},E)\colon y^*(t) = y(t),\ t\in T_i,\ y^*(t_i) = y(t_i^+),\ y\in Z\}$.

Remark 2. Since $D^{\alpha,\beta}_{a^+}x = (I^{\beta(1-\alpha)}_{a^+}D^{\gamma}_{a^+}x))(t)$, it follows from Lemma 2(i) that $C^{\gamma}_{1-\gamma}(J,E) \subseteq C^{\alpha,\beta}_{1-\gamma}(J,E)$. Similarly, $PC^{\gamma}_{1-\gamma}(J,E) \subseteq PC^{\alpha,\beta}_{1-\gamma}(J,E)$.

Lemma 3. *Let $0 < \alpha < 1, 0 \leqslant \beta \leqslant 1, \gamma = \alpha+\beta-\alpha\beta$, and $h \in PC^{\beta(1-\alpha)}_{1-\gamma}(J,E)$. Then the function $u : J \to E$, defined by*

$$u(t) = \begin{cases} \frac{u_0}{\Gamma(\gamma)}t^{\gamma-1} + I^{\alpha}_{0^+}h(t), & t\in(0,t_1], \\ g_i(t, u(t_i^-)), & t\in(t_i,s_i],\ i=1,\dots,m, \\ \frac{(t-s_i)^{\gamma-1}}{\Gamma(\gamma)}g_i(s_i,u(t_i^-)) + I^{\alpha}_{s_i^+}h(t), & t\in(s_i,t_{i+1}],\ i=1,\dots,m, \end{cases} \tag{3}$$

belongs to $PC^{\gamma}_{1-\gamma}(J,E)$*,* $D^{\alpha,\beta}_{s_i+}u(t)$ *exists for any* $t\in(s_i,t_{i+1}]$*,* $i=0,1,\dots,m$*, and satisfies*

$$\begin{aligned}
&D^{\alpha,\beta}_{s_i+}u(t)=h(t),\quad t\in(s_i,t_{i+1}],\ i=0,1,\dots,m,\\
&u\bigl(t_i^+\bigr)=g_i\bigl(t_i,u\bigl(t_i^-\bigr)\bigr),\quad i=1,\dots,m,\\
&u(t)=g_i\bigl(t,u\bigl(t_i^-\bigr)\bigr),\quad t\in(t_i,s_i],\ i=1,\dots,m,\\
&I^{1-\gamma}_{0^+}u\bigl(0^+\bigr)=u_0,\\
&I^{1-\gamma}_{s_i^+}u\bigl(s_i^+\bigr)=g_i\bigl(s_i,u\bigl(t_i^-\bigr)\bigr),\quad i=1,\dots,m,
\end{aligned}\tag{4}$$

where $x_0\in E$*, and* $g_i:[t_i,s_i]\times E\to E$ *is a continuous function for all* $i=1,2,\dots,m$*.*

Proof. For any $t\in(0,t_1]$, from (4),

$$u(t)=\frac{u_0}{\Gamma(\gamma)}t^{\gamma-1}+I^{\alpha}_{0^+}h(t).\tag{5}$$

Note $D^{\gamma}_{a+}(t-a)^{\gamma-1}=0$. By applying the operator $D^{\gamma}_{a^+}$ to both sides of (5), it follows from Definition 3 and Lemma 2(iii) that

$$\begin{aligned}
D^{\gamma}_{a^+}u(t)&=D^{\gamma}_{a+}I^{\alpha}_{a+}h(t)=DI^{1-\gamma}_{a^+}I^{\alpha}_{a+}h(t)=D\,I^{1-\gamma+\alpha}_{a^+}h(t)\\
&=D\,I^{1-\beta(1-\alpha)}_{a^+}=D^{\beta(1-\alpha)}_{a^+}h(t).
\end{aligned}\tag{6}$$

The assumption $h\in PC^{\beta(1-\alpha)}_{1-\gamma}(J,E)$ so $h\in C_{1-\gamma}([0,t_1],E)$ and $DI^{1-\beta(1-\alpha)}_{0^+}h=D^{\beta(1-\alpha)}_{0^+}h\in C_{1-\gamma}([0,t_1],E)$.

Then, it follows from (6) and Lemma 2(i) that $u\in C^{\gamma}_{1-\gamma}([0,t_1],E)$, and hence by Remark 2, $u\in C^{\alpha,\beta}_{1-\gamma}([0,t_1],E)$, so $D^{\alpha,\beta}_{a^+}u(t)$ exists for $t\in(0,t_1]$. Moreover, from Lemma 2(i) $I^{1-\beta(1-\alpha)}_{0^+}h\in C_{1-\gamma}([0,t_1],E)$. Thus h and $I^{1-\beta(1-\alpha)}_{0^+}h$ satisfy the conditions of Lemma 2(iv). By applying $I^{\beta(1-\alpha)}_{0^+}$ to both sides of (6) and using the definition $D^{\alpha,\beta}_{0^+}u(t)$, we get

$$\begin{aligned}
D^{\alpha,\beta}_{0^+}u(t)&=I^{\beta(1-\alpha)}_{0^+}D^{\gamma}_{0^+}u(t)=I^{\beta(1-\alpha)}_{0^+}D^{\beta(1-\alpha)}_{0^+}h(t)\\
&=h(t)-\frac{(I^{1-\beta(1-\alpha)}_{0^+}h)(0^+)}{\Gamma(\beta(1-\alpha))}t^{\beta(1-\alpha)-1},\quad t\in(0,t_1].
\end{aligned}\tag{7}$$

Next, since $1-\gamma<1-\gamma+\alpha=1-\beta(1-\alpha)$, Lemma 2(ii) implies that $(I^{1-\beta(1-\alpha)}_{0^+}h)(0^+)=0$. Hence (7) reduces to $D^{\alpha,\beta}_{a^+}u(t)=h(t)$ on $(0,t_1]$.

Now we show that $I^{1-\gamma}_{0^+}u(0^+)=u_0$. Applying $I^{1-\gamma}_{0^+}$ to both sides of (5), then

$$I^{1-\gamma}_{0^+}u(t)=u_0+I^{1-\gamma+\alpha}_{0^+}h(t)=u_0+\bigl(I^{1-\beta(1-\alpha)}_{0^+}h\bigr)(t),\quad t\in(0,t_1].$$

By taking the limit as $t\to0^+$, we get from Lemma 2(ii) that $I^{1-\gamma}_{0^+}u(0^+)=u_0$.

For any $t\in(t_i,s_i]$, $i=1,\dots,m$, from (4) we have $u(t)=((t-s_i)^{\gamma-1}/\Gamma(\gamma))\times g_i(s_i,u(t_i^-))+I^{\alpha}_{s_i^+}h(t)$. By arguing as above, $u|_{(s_i,t_{i+1}]}\in C^{\beta(1-\alpha)}_{1-\gamma}((s_i,t_{i+1}],E)$,

$D^{\alpha,\beta}_{s_i^+}u(t)$ exists for any $t \in (s_i, t_{i+1}]$, $D^{\alpha,\beta}_{s_i^+}u(t) = h(t)$, $t \in (s_i, t_{i+1}]$, and $I^{1-\gamma}_{s_i^+}u(s_i^+) = g_i(s_i, u(t_i^-))$. This completes the proof. □

Remark 3. If $\beta = 1$, then $D^{\alpha,\beta}_{s_i^+} = {}^c D^{\alpha}_{s_i^+}$, where ${}^cD^{\alpha}_{s_i^+}$ denotes the Caputo fractional derivative of order α with lower limit at s_i^+, and $\gamma = 1$. In this case, (3) becomes

$$u(t) = \begin{cases} u_0 + I^{\alpha}_{0+}h(t), & t \in (0, t_1], \\ g_i\big(t, u\big(t_i^-\big)\big), & t \in (t_i, s_i],\ i = 1, \dots, m, \\ g_i\big(s_i, u\big(t_i^-\big)\big) + I^{\alpha}_{s_i^+}h(t), & t \in (s_i, t_{i+1}],\ i = 1, \dots, m, \end{cases}$$

and (4) becomes

$$\begin{aligned} &D^{\alpha,1}_{s_i+}u(t) = h(t), \quad \text{a.e. } t \in (s_i, t_{i+1}],\ i = 0, 1, \dots, m, \\ &u\big(t_i^+\big) = g_i\big(t_i, u\big(t_i^-\big)\big), \quad i = 1, \dots, m, \\ &u(t) = g_i\big(t, u\big(t_i^-\big)\big), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \\ &u(0) = u_0, \\ &u\big(s_i^+\big) = g_i\big(s_i, u\big(t_i^-\big)\big), \quad i = 1, \dots, m. \end{aligned}$$

Based on Lemma 3, we give a concept of mild solutions of problem (1).

Definition 5. A function $x \in PC_{1-\gamma}(J, E)$ is called a mild solution of problem (1) if there is a $f \in PC^{\beta(1-\alpha)}_{1-\gamma}(J, E)$ such that $f(t) \in F(t, x(t))$ a.e. for $t \in J_k$, $k = 0, 1, \dots, m$, and

$$x(t) = \begin{cases} \frac{t^{\gamma-1}}{\Gamma(\gamma)}(x_0 + g(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, & t \in (0, t_1], \\ g_i(t, x(t_i^-)), & t \in (t_i, s_i],\ i = 1, \dots, m, \\ \frac{(t-s_i)^{\gamma-1}}{\Gamma(\gamma)} g_i(s_i, x(t_i^-)) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1} f(s), \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m. \end{cases}$$

We need the following lemmas in Section 3.

Lemma 4. (*See* [28].) *Let $C \subset L^1(J, E)$ be a countable set such that there is a $h \in L^1(J, E)$ with $f(t) \leqslant h(t)$ for a.e. $t \in J$ and every $f \in C$. Then the function $t \to \chi\{f(t)\colon f \in C\}$ belongs to $L^1(J, E)$ and satisfies $\chi\{\int_0^b f(s)\,\mathrm{d}s\colon f \in C\} \leqslant 2\int_0^b \chi\{f(s)\colon f \in C\}\,\mathrm{d}s$.*

Lemma 5. (*See* [15].) *Let $\chi_{C(J,E)}$ be the Hausdorff measure of noncompactness on $C(J, E)$. If $W \subseteq C(J, E)$ is bounded, then for every $t \in J$, $\chi(W(t)) \leqslant \chi_{C(J,E)}(W)$, where $W(t) = \{x(t)\colon x \in W\}$. Furthermore, if W is equicontinuous on J, then the map $t \to \chi\{x(t)\colon x \in W\}$ is continuous on J and $\chi_{C(J,E)}(W) = \sup_{t\in J} \chi\{x(t)\colon x \in W\}$.*

Lemma 6. (*See* [29, *Thm.* 3.1].) *Let D be a closed convex subset of a Banach space X and $N : D \to P_c(D)$. Assume the graph of N is closed, N maps compact sets into relatively compact sets and that, for some $x_0 \in U$, one has*

$$\begin{aligned} &Z \subseteq D, \quad Z = \operatorname{conv}\bigl(\{x_0\} \cup N(Z)\bigr), \quad \overline{Z} = \overline{C} \text{ with } C \subseteq Z \text{ countable} \\ &\quad \Longrightarrow \quad Z \text{ is relatively compact.} \end{aligned} \tag{8}$$

Then N has a fixed point.

3 Main results

In this section, we present existence results of mild solutions of (1).

Let p be a real number such that $p > 1/\alpha$, $S^p_{F(.,x(.))} = \{z \in L^p(J,E)\colon z(t) \in F(t,x(t))$ a.e. for $t \in J_k,\ k = 0,1,\dots,m\}$. and $I^{\beta(1-\alpha)}(PC_{1-\gamma}(J,E)) = \{f\colon J \to E$, there is a $\delta \in PC_{1-\gamma}[0,b]$ such that $f(t) = I^{\beta(1-\alpha)}_{s_i^+}\delta(t),\ i = 1,2,\dots,m,\ t \in J_k,\ k = 0,1,\dots,m\}$.

We introduce the following assumptions:

(F_1) Let $F : J \times E \to P_{\mathrm{ck}}(E)$ be a multifunction. For every $x \in PC_{1-\gamma}(J,E)$, $S^p_{F(.,x(.))}$ is a nonempty subset of $I^{\beta(1-\alpha)}(PC_{1-\gamma}(J,E))$, and for almost every $t \in J$, $x \to F(t,x)$ is upper semicontinuous.

Note, as an example, from [23, p. 22, Thm. 1.3.1 or p. 26, Lemma 1.3.3], if $F(.,x)$ is measurable for each $x \in E$ (here E is separable) or alternatively $F(.,x)$ is strongly measurable for each $x \in E$ (here E is not necessarily separable), then the multifunction $F(.,x) : J \to P_{\mathrm{ck}}(E)$ has a measurable selection for every $x \in E$. If E is separable, then strongly measurable coincides with measurable. Also from [23, p. 29, Thm. 1.3.5] note that if $F(.,x) : J \to P_{\mathrm{ck}}(E)$ has a strongly measurable selection for every $x \in E$ and if for a.e. $t \in J$, $F(t,.) : E \to P_{\mathrm{ck}}(E)$ is upper semicontinuous, then for every strongly measurable function $x : J \to E$, there exists a strongly measurable selection $z : J \to E$ with $z(t) \in F(t,x(t))$ a.e.

(F_2) There exist a function $\varphi \in L^p(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\Omega : [0,\infty) \to (0,\infty)$ such that $\|F(t,x(t))\| \leqslant \varphi(t)\Omega(\|x\|_{PC_{1-\gamma}(J,E)})$ for $(t,x) \in J \times PC_{1-\gamma}(J,E)$ and

$$\limsup_{n\to\infty} \frac{\Omega(n)}{n} = \upsilon < \infty. \tag{9}$$

(F_2^*) For any natural number n, there is a function $\varphi_n \in L^p(J,\mathbb{R}^+)$ such that $\sup_{\|x\|_{PC_{1-\gamma}(J,E)} \leqslant n} \|F(t,x(t))\| \leqslant \varphi_n(t)$ for a.e. $t \in J$ and

$$\limsup_{n\to\infty} \frac{\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}}{n} = 0. \tag{10}$$

(F_3) There exists a function $\varsigma \in L^p(J,\mathbb{R}^+)$ such that for any bounded subset $D \subseteq E$ and any $k = 0,1,2,\dots,m$, $\chi(F(t,D)) \leqslant (t-s_k)^{1-\gamma}\varsigma(t)\chi(D)$ for a.e. $t \in J_k$ and

$$\|\varsigma\|_{L^p(J,\mathbb{R}^+)}\frac{2\eta b^{1-\gamma}}{\Gamma(\alpha)} < 1, \tag{11}$$

where $\eta = b^{\alpha-1/p}((p-1)/(p\alpha-1))^{(p-1)/p}$, and χ is the Hausdorff measure of non-compactness on E.

(H_g) $g : PC_{1-\gamma}(J,E) \to E$ is continuous, completely continuous, and

$$\limsup_{\|x\|\to\infty}\frac{\|g(x)\|}{\|x\|_{PC_{1-\gamma}(J,E)}} = 0. \tag{12}$$

(H_g^*) $g : PC_{1-\gamma}(J,E) \to E$ is Lipschitz continuous with the Lipschitz constant k and maps convergent sequences in $PC(J,E)$ to strongly convergent sequences in E.

(H) For every $i = 1,2,\dots,m$, $g_i : [t_i,s_i] \times E \to E$ is uniformly continuous on bounded sets and for any $t \in J$, $g_i(t,.)$ maps bounded subsets of E to relatively compact subsets, and there exists a positive constant h_i such that for any $x \in E$,

$$\|g_i(t,x)\| \leqslant h_i(t_i - s_{i-1})^{1-\gamma}\|x\|, \quad t \in [t_i,s_i],\ x \in E.$$

We state the first existence result.

Theorem 1. *Under assumptions* (F_1), (F_2), (F_3), (H_g), and (H), *problem* (1) *has a mild solution provided that*

$$\frac{\eta\upsilon b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)} + h + \frac{h}{\Gamma(\gamma)} < 1, \tag{13}$$

where $h = \sum_{i=0}^{i=m} h_i$.

Proof. In view of (F_1), for every $x \in PC_{1-\gamma}(J,E)$, $S^p_{F(.,x(.))}$ is non empty, and hence we can define a multifunction $R : PC_{1-\gamma}(J,E) \to 2^{PC_{1-\gamma}(J,E)}$ as follows: let $x \in PC_{1-\gamma}(J,E)$, a function $y \in R(x)$ if and only if

$$y(t) = \begin{cases} \frac{t^{\gamma-1}}{\Gamma(\gamma)}(x_0 + g(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,\mathrm{d}s, \quad t \in (0,t_1], \\ g_i(t,x(t_i^-)), \quad t \in (t_i,s_i],\ i = 1,\dots,m, \\ \frac{(t-s_i)^{\gamma-1}}{\Gamma(\gamma)}g_i(s_i,x(t_i^-)) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}f(s)\,\mathrm{d}s, \\ \quad t \in (s_i,t_{i+1}], i = 1,\dots,m, \end{cases} \tag{14}$$

where $f \in S^p_{F(.,x(.))}$. Our goal is to prove, using Lemma 6, that R has a fixed point. The proof will be given in several steps. It is easy to show that the values of R are convex.

Step 1. In this step, we claim that there is a natural number n such that $R(B_n) \subseteq B_n$, where $B_n = \{x \in PC_{1-\gamma}(J,E)\colon \|x\|_{PC_{1-\gamma}(J,E)} \leqslant n\}$. Suppose the contrary.

Then, for any natural number n, there are $x_n, y_n \in PC_{1-\gamma}(J,E)$ with $y_n \in R(x_n)$, $\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant n$, and $\|y_n\|_{PC_{1-\gamma}(J,E)} > n$. Then there exist $(f_n)_{n\geqslant 1} \in S^p_{F(.,x_n(.))}$ such that

$$y_n(t) = \begin{cases} \frac{t^{\gamma-1}}{\Gamma(\gamma)}(x_0 + g(x_n)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_n(s)\,\mathrm{d}s, \quad t \in (0,t_1], \\ g_i(t, x_n(t_i^-)), \quad t \in (t_i,\ s_i], i = 1,\dots,m, \\ \frac{(t-s_i)^{\gamma-1}}{\Gamma(\gamma)} g_i(s_i, x_n(t_i^-)) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1} f_n(s)\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1,\dots,m. \end{cases} \tag{15}$$

From (F_2),

$$\big|f_n(t)\big\| \leqslant \varphi(t)\Omega\big(\|x_n\|_{PC_{1-\gamma}(J,E)}\big), \quad t \in J. \tag{16}$$

Then from Hölder's inequality and (16) we obtain

$$\begin{aligned} \sup_{t\in[0,t_1]} t^{1-\gamma}\big\|y_n(t)\big\| &\leqslant \frac{1}{\Gamma(\gamma)} \sup_{t\in[0,t_1]} \big[\|x_0\| + \big\|g(x_n)\big\|\big] \\ &\quad + \sup_{t\in[0,t_1]} \frac{t^{1-\gamma}\Omega(\|x_n\|_{PC_{1-\gamma}(J,E)})}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s \\ &\leqslant \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + \big\|g(x_n)\big\|\big] + \frac{b^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta. \end{aligned} \tag{17}$$

If $i = 1,2,\dots,m$, then from Remark 1(ii)

$$\begin{aligned} \sup_{t\in[t_i,s_i]} \big\|y_n(t)\big\| = \sup_{t\in[t_i,s_i]} \big\|g_i\big(t, x_n\big(t_i^-\big)\big)\big\| &\leqslant h(t_i - s_{i-1})^{1-\gamma}\big\|x_n\big(t_i^-\big)\big\| \\ &\leqslant h\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant hn. \end{aligned} \tag{18}$$

Similarly, for $i = 1,2,\dots,m$,

$$\begin{aligned} &\sup_{t\in[s_i,t_{i+1}]} (t-s_i)^{1-\gamma}\big\|y_n(t)\big\| \\ &\quad\leqslant \sup_{t\in[s_i,t_{i+1}]} \frac{\|g_i(s_i, x_n(t_i^-))\|}{\Gamma(\gamma)} + \frac{b^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta \\ &\quad\leqslant \frac{hn}{\Gamma(\gamma)} + \frac{b^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta. \end{aligned} \tag{19}$$

From (17), (18), (19) it follows that

$$\begin{aligned} n &< \|y_n\|_{PC_{1-\gamma}(J,E)} \\ &\leqslant \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + \big\|g(x_n)\big\|\big] + \eta\frac{b^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)} + hn + \frac{hn}{\Gamma(\gamma)}. \end{aligned}$$

By dividing both sides by n and passing to the limit as $n \to \infty$, we obtain from (9) and (12) that $1 \leqslant (\eta \upsilon b^{1-\gamma}/\Gamma(\alpha))\|\varphi\|_{L^p(J,\mathbb{R}^+)} + h + h/\Gamma(\gamma)$, which contradicts (13). Thus there is a natural number n_0 such that $R(B_{n_0}) \subseteq B_{n_0}$.

Step 2. Let $K = \{z \in PC_{1-\gamma}(J, E)\colon z \in R(B_{n_0})\}$. We claim that the subsets $K_{|\overline{J_k}}$ ($k = 0, 1, \ldots, m$) and $K_{|\overline{T_i}}$ ($i = 1, 2, \ldots, m$) are equicontinuous, where

$$K_{|\overline{J_k}} = \Big\{z : \overline{J_k} \to E\colon z(t) = (t - s_k)^{1-\gamma} y(t),\ t \in J_k,\\ z(s_k) = \lim_{t \to s_k} (t - s_k)^{1-\gamma} z(t),\ y \in R(x),\ x \in B_{n_0}\Big\},$$

and

$$K_{|\overline{T_i}} = \big\{y^* \in C(\overline{T_i}, E)\colon y^*(t) = y(t),\ t \in [t_i, s_i],\ y^*(t_i) = y(t_i^+),\\ y \in R(x),\ x \in B_{n_0}\big\}.$$

Case 1. Let $z \in K_{|\overline{J_0}}$. Then there is a $x \in B_{n_0}$ ($y \in R(x)$) and $f \in S^p_{F(.,x(.))}$ such that for $t \in (0, t_1]$, $z(t) = t^{1-\gamma}[(t^{\gamma-1}/\Gamma(\gamma))(x_0 + g(x)) + (1/\Gamma(\alpha)) \times \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s]$ and $z(0) = \lim_{t \to 0+} t^{1-\gamma} y(t)$. If $t = 0$, $\delta \in (0, t_1]$, then (see (14)) $\lim_{\delta \to 0+} z(\delta) = \lim_{\delta \to 0+} \delta^{1-\gamma} y(\delta) = z(0)$.

Let t, $t + \delta$ be two points in $(0, t_1]$. Then $\|z(t+\delta) - z(t)\| \leqslant \sum_{i=1}^{i=2} I_i$, where $I_1 = ((t+\delta)^{1-\gamma}/\Gamma(\alpha))\| \int_t^{t+\delta} (t + \delta - s)^{\alpha-1} f(s)\,\mathrm{d}s\|$, and $I_2 = (1/\Gamma(\alpha)) \times \| \int_0^t [(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}] f(s)\,\mathrm{d}s\|$.

It follows that $\lim_{\delta \to 0} I_1 \leqslant \lim_{\delta \to 0}((t+\delta)^{1-\gamma} \Omega(n_0)/\Gamma(\alpha)) \int_t^{t+\delta} (t + \delta - s)^{\alpha-1} \times \varphi(s)\,\mathrm{d}s = 0$ (independently of x).

For I_2, note that for almost $s \in [0, t]$,

$$\big\|\big[(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}\big] f(s)\big\|\\ \leqslant \Omega(n_0)\big|(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} + t^{1-\gamma}(t-s)^{\alpha-1}\big|\varphi(s) \quad \text{a.e.}$$

Since $\varphi \in L^p(J, X)$ and $\int_0^t [(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}]\varphi(s)\,\mathrm{d}s$ exists, then by the Lebesgue dominated convergence theorem we derive that $\lim_{\delta \to 0} I_2 = 0$ (independently of x).

Case 2. Let $y \in K_{|T_i}$, $i = 1, \ldots, m$. Then $y(t) = g_i(t, x(t_i^-))$, $t \in (t_i, s_i]$, $i = 1, \ldots, m$. Let $i \in \{1, 2, \ldots, m\}$ be fixed and t, $t + \delta \in (t_i, s_i]$. Since $\|x\|_{PC_{1-\gamma(J,E)}} \leqslant n_0$, it follows from the uniform continuity of g_i on bounded sets that

$$\lim_{\delta \to 0} \big\|y(t+\delta) - y(t)\big\| = \lim_{\delta \to 0} \big\|g_i\big(t+\delta, x\big(t_i^-\big)\big) - g_i\big(t, x\big(t_i^-\big)\big)\big\| = 0$$

(independently of x). When $t = t_i$, $i = 1, \ldots, m$, let $\delta > 0$ be such that $t_i + \delta \in (t_i, s_i]$ and $\lambda > 0$ such that $t_i < \lambda < t_i + \delta \leqslant s_i$. Then we have $\|y^*(t_i + \delta) - y^*(t_i)\| = \lim_{\lambda \to t_i^+} \|y(t_i + \delta) - y(\lambda)\| = 0$.

Case 3. Let $z \in K_{|J_k}$, $k = 1, \dots, m$. Then there is a $x \in B_{n_0}$ and $f \in S^p_{F(.,x(.))}$ such that for $t \in (s_k, t_{k+1}]$,

$$\begin{aligned} z(t) &= (t - s_k)^{1-\gamma}\left[\frac{(t-s_k)^{\gamma-1}}{\Gamma(\gamma)} g_k\bigl(s_k, x\bigl(t_k^-\bigr)\bigr) + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}(t-s)^{\alpha-1} f(s)\,\mathrm{d}s\right] \\ &= \frac{g_k(s_k, x(t_k^-))}{\Gamma(\gamma)} + \frac{(t-s_k)^{1-\gamma}}{\Gamma(\alpha)}\int_{s_k}^{t}(t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \end{aligned}$$

and $z(s_k) = \lim_{t\to s_k}(t-s_k)^{1-\gamma} z(t)$. Let $k \in \{1, \dots, m\}$ be fixed. If $t = s_k$ and $\delta > 0$, then

$$\begin{aligned} \lim_{\delta\to 0+} z(s_k+\delta) &= \lim_{\delta\to 0+}(s_k+\delta-s_k)^{1-\gamma} y(s_k+\delta) \\ &= \lim_{t\to s_k+}(t-s_k)^{1-\gamma} y(s_k) = z(s_k). \end{aligned}$$

Next, let $t, t+\delta \in (s_k, t_{k+1}]$, $\delta > 0$. Then we have

$$\begin{aligned} &\bigl\|z(t+\delta) - z(t)\bigr\| \\ &= \frac{(t+\delta)^{1-\gamma}}{\Gamma(\alpha)}\left\|\int_{s_k}^{t+\delta}(t+\delta-s)^{\alpha-1} f(s)\,\mathrm{d}s - \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_{s_k}^{t}(t-s)^{\alpha-1} f(s)\,\mathrm{d}s\right\|. \end{aligned}$$

Arguing as in Case 1, we conclude that

$$\lim_{\delta\to 0}\frac{(t+\delta)^{1-\gamma}}{\Gamma(\alpha)}\left\|\int_{s_k}^{t+\delta}(t+\delta-s)^{\alpha-1} f(s)\,\mathrm{d}s - \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_{s_k}^{t}(t-s)^{\alpha-1} f(s)\,\mathrm{d}s\right\| = 0.$$

Step 3. The graph of the multivalued function $R_{|B_{n_0}} : B_{n_0} \to 2^{B_{n_0}}$ is closed. Consider a sequence $\{x_n\}_{n\geqslant 1}$ in B_{n_0} with $x_n \to x$ in B_{n_0} and let $y_n \in R(x_n)$ with $y_n \to y$ in $PC_{1-\gamma}(J, E)$. We need to show that $y \in R(x)$. Recalling the definition of R, for any $n \geqslant 1$, there is a $f_n \in S^p_{F(.,x_n(.))}$ such that (15) holds.

In view of (16), $\|f_n(t)\| \leqslant n_0\varphi(t)$ for every $n \geqslant 1$ and for a.e. $t \in J$. Then $\{f_n\colon n \geqslant 1\}$ is bounded in $L^p(J, E)$. Because $p > 1$, $L^p(J, E)$ is reflexive, and hence we can assume, without loss of generality, that (f_n) converges weakly to a function $f \in L^p(J, E)$. From Mazur's lemma, for every natural number j, there is a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}$, $k = k_0(j), \dots, j$, such that $\sum_{k=j}^{k_0}\lambda_{j,k} = 1$ and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0}\lambda_{j,k} f_k$, $j \geqslant 1$, converges strongly to f in $L^1(J, E)$ as $j \to \infty$.

Take $\overline{y}_n(t) = \sum_{k=n}^{k_0(n)} \lambda_{n,k}\, y_k$. Then

$$\overline{y}_n(t) = \begin{cases} \frac{t^{\gamma-1}}{\Gamma(\gamma)}(x_0 + g(x_n)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_n(s)\,\mathrm{d}s, \quad t \in (0, t_1], \\ g_i(t, x_n(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \\ \frac{(t-s_i)^{\gamma-1}}{\Gamma(\gamma)} g_i(s_i, x_n(t_i^-)) + \frac{1}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} z_n(s)\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m. \end{cases}$$

From Remark 1(ii), the continuity of g, the uniform continuity of g_i on bounded sets, and the Lebesgue dominated convergence theorem we have that $\overline{y}_n(t) \to v(t)$ and

$$v(t) = \begin{cases} \frac{t^{\gamma-1}}{\Gamma(\gamma)}(x_0 + g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \in (0, t_1], \\ g_i(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \\ \frac{(t-s_i)^{\gamma-1}}{\Gamma(\gamma)} g_i(s_i, x(t_i^-)) + \frac{1}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m. \end{cases}$$

Since $y_n \to y$, then $y = v$. Now for a.e. t, $F(t, .)$ is upper semicontinuous with closed convex values, so from [5, Chap. 1, Sect.4] it follows that $f \in S^p_{F(.,x(.))}$, so R is closed.

Step 4. The implication (8) holds with $x_0 = 0$.

Let $Z \subseteq B_{n_0}$, $Z = \operatorname{conv}(\{x_0\} \cup R(Z))$, $\overline{Z} = \overline{C}$ with $C \subseteq Z$ countable. We claim that Z is relatively compact in $PC_{1-\gamma}(J, E)$. Since C is countable and $C \subseteq Z = \operatorname{conv}(\{x_0\} \cup R(Z))$, we can find a countable set $H = \{y_n\colon n \geqslant 1\} \subseteq R(Z)$ with $C \subseteq \operatorname{conv}(\{x_0\} \cup H)$. Now for any $n \geqslant 1$, there exists $x_n \in Z \subseteq B_{n_0}$ with $y_n \in R(x_n)$, so there exists $f_n \in S^p_{F(.,x(.))}$ such that (15) holds. From the definition of $\chi_{PC_{1-\gamma}(J,E)}(Z)$ (2) one obtains

$$\begin{aligned} \chi_{PC_{1-\gamma}(J,E)}(Z) &= \chi_{PC_{1-\gamma}(J,E)}(\overline{Z}) = \chi_{PC_{1-\gamma}(J,E)}(\overline{C}) \\ &= \chi_{PC_{1-\gamma}(J,E)}(C) \leqslant \chi_{PC_{1-\gamma}(J,E)}\bigl(\operatorname{conv}\bigl(\{x_0\} \cup H\bigr)\bigr) \\ &= \chi_{PC_{1-\gamma}(J,E)}(H) \\ &= \max\Bigl\{\max_{k=0,1,\dots,m} \chi_{C(\overline{J_k},E)}(H_{|\overline{J_k}}),\ \max_{i=1,\dots,m} \chi_{C(\overline{T_i},E)}(H_{|\overline{T_i}})\Bigr\}. \end{aligned}$$

Since $Z_{|\overline{J_i}}$ and $Z_{|\overline{T_i}}$ are equicontinuous, then from Lemma 5 the last inequality becomes

$$\begin{aligned} \chi_{PC_{1-\gamma}(J,E)}(Z) \leqslant \max\Bigl\{&\max_{i=0,1,\dots,m} \max_{t \in \overline{J_k}} \chi\bigl\{y_n^*(t)\colon n \geqslant 1\bigr\}, \\ &\max_{i=1,\dots,m} \max_{t \in \overline{T_i}} \chi\bigl\{y_n^*(t)\colon n \geqslant 1\bigr\}\Bigr\}, \end{aligned} \tag{20}$$

where,

$$y_n^*(t) = \begin{cases} t^{1-\gamma}y(t), & t \in (0, t_1], \\ \lim_{t\to 0} t^{1-\gamma}y(t), & t = 0, \\ g_i(t, x_n(t_i^-)), & t \in (t_i, s_i],\ i = 1, \dots, m, \\ y_n(t_i^+), & t = t_i \\ (t - s_i)^{1-\gamma}y(t), & t \in (s_i, t_{i+1}],\ i = 1, \dots, m, \\ \lim_{t\to s_i}(t - s_i)^{\gamma-1}y(t), & t = s_i, i = 1, \dots, m. \end{cases}$$

That is,

$$y_n^*(t) = \begin{cases} \frac{1}{\Gamma(\gamma)}(x_0 + g(x_n)) + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_n(s)\,\mathrm{d}s, & t \in (0, t_1], \\ \lim_{t\to 0} t^{1-\gamma}y(t), & t = 0, \\ g_i(t, x_n(t_i^-)), & t \in (t_i, s_i],\ i = 1, \dots, m, \\ g_i(t_i, x_n(t_i^-)), & t = t_i, \\ \frac{1}{\Gamma(\gamma)} g_i(s_i, x_n(t_i^-)) + \frac{(t-s_i)^{1-\gamma}}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1} f_n(s)\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m, \\ \lim_{t\to s_i^+}(t - s_i)^{\gamma-1}y(t), & t = s_i,\ i = 1, \dots, m. \end{cases}$$

Then using the properties of the measure of noncompactness, one has

$$\chi\{y_n^*(t)\colon n \geqslant 1\} \leqslant \begin{cases} \chi\{\frac{1}{\Gamma(\gamma)}(x_0 + g(x_n))\colon n \geqslant 1\} \\ \quad + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\chi\{\int_0^t (t-s)^{\alpha-1} f_n(s)\,\mathrm{d}s\colon n \geqslant 1\}, & t \in (0, t_1], \\ \chi\{\lim_{t\to 0^+} t^{1-\gamma}y_n(t)\colon n \geqslant 1\}, & t = 0, \\ \chi\{g_i(t, x_n(t_i^-))\colon n \geqslant 1\}, & t \in (t_i,\ s_i],\ i = 1, \dots, m, \\ \chi\{g_i(t_i, x_n(t_i^-))\colon n \geqslant 1\}, & t = t_i,\ i = 1, \dots, m, \\ \chi\{g_i(s_i, x_n(t_i^-))\colon n \geqslant 1\} \\ \quad + \frac{(t-s_i)^{1-\gamma}}{\Gamma(\alpha)}\chi\{\int_{s_m}^t (t-s)^{\alpha-1} f_n(s)\,\mathrm{d}s\colon n \geqslant 1\}, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m, \\ \chi\{\lim_{t\to s_i}(t - s_i)^{1-\gamma}y_n(t)\colon n \geqslant 1\}, \\ \quad t = s_i,\ i = 1, \dots, m. \end{cases} \tag{21}$$

From (H_g) it follows that $\chi\{(x_0 + g(x_n))/\Gamma(\gamma)\colon n \geqslant 1\} = 0$, and hence

$$\begin{aligned} \chi\{y_n^*(0)\colon n \geqslant 1\} &= \chi\Big\{\lim_{t\to 0^+} t^{1-\gamma}y_n(t)\colon n \geqslant 1\Big\} \\ &= \chi\Big\{\frac{1}{\Gamma(\gamma)}\big(x_0 + g(x_n)\big)\colon n \geqslant 1\Big\} = 0. \end{aligned} \tag{22}$$

Moreover, since $x_n(t_i^-) \to x(t_i^-)$, the set $\{x_n(t_i^-)\colon n \geqslant 1\}$ is bounded for every $i = 1, 2, \dots, m$. Then from (H) we get for $i = 1, \dots, m$,

$$\chi\{g_i(t, x_n(t_i^-))\colon n \geqslant 1\} = 0, \quad t \in (t_i, s_i], \tag{23}$$

and

$$\chi\{g_i(t_i, x_n(t_i^-))\colon n \geqslant 1\} = \chi\{g_i(s_i, x_n(t_i^-))\colon n \geqslant 1\} = 0. \tag{24}$$

Similarly, for $i = 1, 2, \dots, m$,

$$\begin{aligned}\chi\{y_n^*(s_i)\colon n \geqslant 1\} &= \chi\Bigl\{\lim_{t \to s_i^+} (t - s_i)^{\gamma-1} y(t)\colon n \geqslant 1\Bigr\} \\ &= \chi\biggl\{\frac{1}{\Gamma(\gamma)} g_i(s_i, x_n(t_i^-))\colon n \geqslant 1\biggr\} = 0. \end{aligned}\tag{25}$$

Then, if $t \in J_0$, using Lemma 4, we obtain that

$$\chi\{y_n^*(t)\colon n \geqslant 1\} \leqslant \frac{2b^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \chi\{f_n(s)\colon n \geqslant 1\}\,\mathrm{d}s. \tag{26}$$

Observe that from (F_3), for a.e. $t \in J_0$, we have

$$\begin{aligned}\chi\{f_n(t)\colon n \geqslant 1\} &\leqslant \chi\{F(t, x_n(t))\colon n \geqslant 1\} \leqslant \varsigma(t) s^{1-\gamma} \chi\{x_k(t)\colon k \geqslant 1\} \\ &= \varsigma(t) \chi\{t^{1-\gamma} x_k(t)\colon k \geqslant 1\} \leqslant \varsigma(t) \chi_{PC_{1-\gamma}(J,E)}(Z),\end{aligned}$$

so it follows from (26) that

$$\max_{t \in J_0} \chi\{y_n^*\colon n \geqslant 1\} \leqslant \chi_{PC_{1-\gamma}(J,E)}(Z) \frac{2\eta b^{1-\gamma}}{\Gamma(\alpha)} \|\varsigma\|_{L^p(J,\mathbb{R}^+)}.$$

Similarly, we can show that for any $k = 1, 2, \dots, m$,

$$\max_{t \in J_k} \chi\{y_n^*(t)\colon n \geqslant 1\} \leqslant \chi_{PC_{1-\gamma}(J,E)}(Z) \frac{2\eta b^{1-\gamma}}{\Gamma(\alpha)} \|\varsigma\|_{L^p(J,\mathbb{R}^+)},$$

which yields with (20)–(25) and (11) that

$$\chi_{PC_{1-\gamma}(J,E)}(Z) \leqslant \chi_{PC_{1-\gamma}(J,E)}(Z) \frac{2\eta b^{1-\gamma}}{\Gamma(\alpha)} \|\varsigma\|_{L^p(J,\mathbb{R}^+)} < \chi_{PC}(Z).$$

Thus $\chi_{PC_{1-\gamma}(J,E)}(Z) = 0$, so Z is relatively compact.

Step 5. R maps compact sets into relatively compact sets.

Let B be a compact subset of B_{n_0}. Let (y_n), $n \geqslant 1$, be a sequence in $R(B)$. Then there is a sequence (x_n), $n \geqslant 1$, in B such that $y_n \in R(x_n)$ (so there exists $f_n \in S^p_{F(.,x_n(.))}$ such that, for $t \in J$, (15) holds). We need to show that the set $Z = \{y_n\colon n \geqslant 1\}$

is relatively compact in $PC_{1-\gamma}(J,E)$. Note that, since B is compact in $PC_{1-\gamma}(J,E)$, then from (F_3) we get for a.e. $t \in J_0$,

$$\begin{aligned}\chi\{f_n(t)\colon n \geqslant 1\} &\leqslant \chi\{F(t,x_n(t))\colon k \geqslant 1\} \leqslant \varsigma(t)\chi\{t^{1-\gamma}x_k(t)\colon k \geqslant 1\} \\ &\leqslant \varsigma(t)\chi_{PC_{1-\gamma}(J,E)}(B) = 0.\end{aligned}$$

Arguing as in the previous step, we see that Z is relatively compact, and hence $R(B)$ is relatively compact.

Now apply Lemma 6. Then there is a $x \in PC_{1-\gamma}(J,E)$ and $f \in S^p_{F(.,x(.))}$ such that

$$x(t) = \begin{cases} \frac{t^{\gamma-1}}{\Gamma(\gamma)}(x_0 + g(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, & t \in (0,t_1], \\ g_i(t, x(t_i^-)), & t \in (t_i, s_i],\ i = 1,\dots,m, \\ \frac{(t-s_i)^{\gamma-1}}{\Gamma(\gamma)} g_i(s_i, x(t_i^-)) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1,\dots,m. \end{cases}$$

Next, in view of (F_1), there is a $\delta \in PC_{1-\gamma}(J,E)$ such that $f(t) = I^{\beta(1-\alpha)}_{s_i^+}\delta(t)$, $t \in J_k$, $k = 0,1,2,\dots,m$. It follows that $D^{\beta(1-\alpha)}_{s_i^+} f(t) = D^{\beta(1-\alpha)}_{s_i^+} I^{\beta(1-\alpha)}_{s_i^+}\delta(t) = \delta(t)$, $t \in J_k$, $k = 0,1,2,\dots,m$. Thus $f \in PC^{\beta(1-\alpha)}_{1-\gamma}(J,E)$, and from Lemma 3 the function x is a solution of (1). $\square$

Remark 4. If, in (12), we assume $\limsup_{\|x\|\to\infty} \|g(x)\|/\|x\|_{PC_{1-\gamma}(J,E)} = 0$ is replaced by $\lim_{\|x\|\to\infty} \|g(x)\|/\|x\|_{PC_{1-\gamma}(J,E)} = 0$, then, in (9), we could replace $\limsup_{n\to\infty} \Omega(n)/n = \upsilon < \infty$ with $\liminf_{n\to\infty} \Omega(n)/n = \upsilon < \infty$.

Next, we present the following affine results.

Theorem 2. *Under assumptions* (F_1), (F_2^*), (F_3), (H_g), *and* (H), *problem* (1) *has a mild solution provided that*

$$h + \frac{h}{\Gamma(\gamma)} < 1. \tag{27}$$

Proof. The proof is similar to Theorem 1. The only difference is to show that there is a natural number n such that $R(B_n) \subseteq B_n$ under (F_2^*). Suppose the contrary holds. Then, for any natural number n, there are $x_n, y_n \in PC_{1-\gamma}(J,E)$ with $y_n \in R(x_n)$, $\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant n$ and $\|y_n\|_{PC_{1-\gamma}(J,E)} > n$, and y_n is defined by (15).

From Hölder's inequality we obtain that

$$\begin{aligned}\sup_{t\in[0,t_1]} t^{1-\gamma}\|y_n(t)\| &\leqslant \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + \|g(x_n)\|\big] \\ &\quad + \frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} b^{\alpha-1/p}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \\ &= \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + \|g(x_n)\|\big] + \eta\frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}.\end{aligned} \tag{28}$$

Similarly, we get for $i = 1, 2, \dots, m$ that

$$\sup_{t\in[s_i,t_{i+1}]}(t-s_i)^{1-\gamma}\big\|y_n(t)\big\| \leqslant \sup_{t\in[s_i,t_{i+1}]}\frac{\|g_i(s_i, x_n(t_i^-))\|}{\Gamma(\gamma)} + \frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta$$
$$\leqslant \frac{hn}{\Gamma(\gamma)} + \frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta. \tag{29}$$

It follows from (18), (28), and (29) that

$$n < \|y_n\|_{PC_{1-\gamma}(J,E)}$$
$$\leqslant \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + \big\|g(x_n)\big\|\big] + \eta\frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} + hn + \frac{hn}{\Gamma(\gamma)}.$$

By dividing both side by n and passing to the limit as $n \to \infty$, we get $1 \leqslant h + h/\Gamma(\gamma)$, which contradicts (27). This completes the proof. □

Theorem 3. *Under assumptions* (F_1), (F_2^*), (F_3), (H_g^*), *and* (H), *problem* (1) *has a mild solution provided that*

$$\frac{k}{\Gamma(\gamma)} + h + \frac{h}{\Gamma(\gamma)} < 1. \tag{30}$$

Proof. Like Theorem 2, we only need to note that

$$\sup_{t\in[0,t_1]} t^{1-\gamma}\|y_n(t)\| \leqslant \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + k\big\|x_n\big\|_{PC_{1-\gamma}(J,E)} + \big\|g(0)\big\|\big]$$
$$+ \frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}b^{\alpha-1/p}\left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p}$$
$$= \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + kn + \big\|g(0)\big\|\big] + \eta\frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}.$$

From this inequality, (18), and (29) one obtains

$$n < \|y_n\|_{PC_{1-\gamma}(J,E)}$$
$$\leqslant \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + kn + \big\|g(0)\big\|\big] + \eta\frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} + hn + \frac{hn}{\Gamma(\gamma)}.$$

Dividing both side by n and passing to the limit as $n \to \infty$, we get $1 \leqslant k/\Gamma(\gamma) + h + h/\Gamma(\gamma)$, which contradicts (30). The proof is complete. □

4 An example

Let $J = [0, 3]$, $s_0 = 0$, $t_1 = 1$, $s_1 = 2$, $t_2 = 3$, $\alpha = 1/4$, $\beta = 1/8$, and $\gamma = \alpha+\beta-\alpha\beta = 11/32$.

Consider $F : [0,3] \times E \to P_{\mathrm{ck}}(E)$ defined by

$$F(t,x) = \begin{cases} \{0\}, & t = 0, \\ t^{-1/16} Z, & t \in (0,1], \\ t^2 Z, & t \in (1,2], \\ (t-2)^{-1/8} Z, & t \in (2,3], \end{cases} \tag{31}$$

where Z is a convex compact subset of E. Note that $1-\gamma = 21/32$. For any $z \in Z$, consider $\delta_z : J \to E$ defined by

$$\delta_z(t) = \begin{cases} \frac{\Gamma(\frac{1}{16})}{\Gamma(\frac{27}{32})} t^{-5/32} \|z\|, & t \in (0,1], \\ t^2 \|z\|, & t \in (1,2], \\ \frac{\Gamma(\frac{28}{32})}{\Gamma(\frac{25}{32})} (t-2)^{-7/32} \|z\|, & t \in (2,3]. \end{cases}$$

It is clear that, for any $z \in Z$, $\delta_z(t) \in PC_{1-\gamma}(J,E)$. If $f \in S^p_{F(.,x(.))}$ and $x \in PC_{1-\gamma}(J,E)$, then

$$f(t) = \begin{cases} t^{-1/16} \|z_0\|, & t \in (0,1], \\ (t-2)^{-1/8} \|z_0\|, & t \in (2,3], \end{cases}$$

where $z_0 \in Z$. Note that

$$\begin{aligned} I_{0^+}^{\beta(1-\alpha)} \delta_{z_0}(t) &= \begin{cases} \frac{\Gamma(\frac{1}{16})}{\Gamma(\frac{27}{32})} \|z_0\| I_{0^+}^{\beta(1-\alpha)} t^{-5/32} = \frac{\Gamma(\frac{1}{16})}{\Gamma(\frac{27}{32})} \|z_0\| \frac{\Gamma(\frac{27}{32})}{\Gamma(\frac{1}{16})} t^{-1/16}, \\ \quad t \in (0,1] \\ \frac{\Gamma(\frac{28}{32})}{\Gamma(\frac{25}{32})} \|z_0\| I_{0^+}^{\beta(1-\alpha)} (t-2)^{-7/32} = \frac{\Gamma(\frac{28}{32})}{\Gamma(\frac{25}{32})} \|z_0\| \frac{\Gamma(\frac{25}{32})}{\Gamma(\frac{28}{32})} (t-2)^{-1/8}, \\ \quad t \in (2,3]. \end{cases} \\ &= f(t). \end{aligned}$$

Then (F_1) is satisfied. In order to show that $(F_2)^*$ holds for any natural number n, we let

$$\varphi_n(t) = \begin{cases} t^{-1/16} \sigma, & t \in (0,1], \\ t^2 \sigma, & t \in (1,2], \\ (t-2)^{-1/8} \sigma, & t \in (2,3], \end{cases}$$

where $\sigma = \max\{\|z\|: z \in Z\}$.

Clearly $\varphi_n \in L^p(J, \mathbb{R}^+)$ and $\sup_{\|x\|_{PC_{1-\gamma}(J,E)} \leqslant n} \|F(t,x(t))\| \leqslant \varphi_n(t)$ for a.e. $t \in J$, and (10) is satisfied.

Next, let $g_1 : [t_1, s_1] \times E \to E$ be given by

$$g_1(t,x) = (t_1 - s_0)^{1-\gamma} K(x), \tag{32}$$

where $K : D(K) = E \to E$ is a linear bounded completely continuous operator. Then (H) is satisfied, and note $h = \|K\|$. Let $g : PC_{1-\gamma}(J, E) \to E$ be given by

$$g(x) = c_1 K\bigl(x(t_1)\bigr), \tag{33}$$

where c_1 is a real number. For any $x, y \in PC_{1-\gamma}(J, E)$, $\|g(x) - g(y)\| \leqslant \|K\| |c_1| \|x - y\|$.

Moreover, if $x_n \to x$ in $PC(J, E)$, then $x_n(t_1) \to x(t_1)$, and the complete continuity of K implies that $K(x_n(t_1)) \to K(x(t_1))$ in E, and hence $g(x_n) \to g(x)$. Then (H_g^*) holds, and note $k = \|K\| \, |c_1|$.

Now consider (1), where F, g_i, and g are given in (31)–(33). Then from Theorem 3, problem (1) has a mild solution, provided that $\|K\|(|c_1|/\Gamma(\gamma) + 1 + 1/\Gamma(\gamma)) < 1$.

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