

Triple positive solutions for semipositone fractional differential equations m -point boundary value problems with singularities and p - q -order derivatives*

Xingqiu Zhang^{a,1}, Zhuyan Shao^a, Qiuyan Zhong^b, Zengqin Zhao^c

^aSchool of Medical Information Engineering, Jining Medical College, Rizhao 276826, Shandong, China
zhxq197508@163.com; 317096018@qq.com

^bCenter for Information Technology, Jining Medical College, Jining 272067, Shandong, China
zhqy197308@163.com

^cSchool of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China
zqzhaoy@163.com

Received: February 12, 2018 / **Revised:** June 8, 2018 / **Published online:** October 31, 2018

Abstract. In this paper, by means of Leggett–Williams and Guo–Krasnosel’skii fixed point theorems, together with height functions of the nonlinearity on different bounded sets, triple positive solutions are obtained for some fractional differential equations with p - q -order derivatives involved in multi-point boundary value conditions. The nonlinearity may not only take negative infinity but also may permit singularities on both the time and the space variables.

Keywords: fractional differential equations, semipositone, triple positive solution, singularity on space variable, multi-point BVP.

1 Introduction

The purpose of this paper is to obtain the existence result of triple positive solutions for the following fractional differential equation (FDE for short):

$$\begin{aligned} D_{0+}^{\alpha} u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^p u(1) = \sum_{i=1}^m a_i D_{0+}^q u(\xi_i), \end{aligned} \quad (1)$$

*This research was supported by a Project of Shandong Province Higher Educational Science and Technology Program (J18KA217), the National Natural Science Foundation of China (11571197, 11571296, 11871302), the Foundation for NSFC cultivation project of Jining Medical University (2016-05), the Natural Science Foundation of Jining Medical University (JY2015BS07, 2017JYQD22) and the Natural Science Foundation of Shandong Province of China (ZR2015AL002).

¹Corresponding author.

where D_{0+}^{α} represents Riemann–Liouville derivative of order α , $n - 1 < \alpha \leq n$ ($n \geq 3$), $a_i \geq 0$ ($i = 1, 2, \dots, m, m \in \mathbb{N}^+$), $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1$, $p, q \in \mathbb{R}$, $1 \leq p \leq n - 2$, $0 \leq q \leq p$ with $\Delta = \Gamma(\alpha)/\Gamma(\alpha - p) - (\Gamma(\alpha)/\Gamma(\alpha - q)) \sum_{i=1}^m a_i \xi_i^{\alpha - q - 1} > 0$. The nonlinearity allows to take negative value and be unbounded below, that is, the problems considered here are so called semipositone problems. In addition, $f(t, u)$ may be singular both at $t = 0, 1$ and/or $u = 0$.

Recently, great efforts have been made to investigate FDEs nonlocal problems for their better effect in describing important phenomena in science, engineering, biology, economics and so on. Many excellent works can be found in the literature (see [1–4, 6–13, 15–40] and the references therein). In a recent paper, when f permits singularity on t and is semipositone, Henderson and Luca [9] gave an existence result of at least one positive solution for fractional differential equation eigenvalue problems subject to the boundary conditions given in (1). The main tool is the famous Guo–Krasnosel’skii fixed point theorem. In another paper [10], under the assumption that f is either nonsingular or singular on t , by means of fixed point index theory, they established an existence result of at least one positive solution for some system of nonlinear ordinary fractional differential equations with some coupled multi-point boundary conditions. By virtue of height functions on some special bounded sets, Pu et al. [21] considered existence and multiplicity of positive solutions for BVP (1) on the premise that the nonlinearity is semipositone and permits singularity with respect to space variable. Very recently, Zhang and Zhong [36] obtained an existence result of triple positive solutions for some fractional differential equations integral boundary value problems.

This paper is a continuation of our paper [36]. We concentrate on investigating triple positive solutions for semipositone BVP (1) by Leggett–Williams and Guo–Krasnosel’skii fixed point theorems. Compared with the existing works, this paper admits some new features. Firstly, the notable difference with the traditional results on triple solutions (see [1, 4, 19, 37] for instance) lies in that the nonlinearity f possesses singularities with respect to the space variable. At present, there are relatively few results on triple solutions for integer-order differential equations when Leggett–Williams fixed point theorem is used under this circumstance, not to mention fractional differential equations. Secondly, not only the method exploited here is different in essence from that in reference [21], but also the bounded sets and height functions constructed in this paper are quite different from those in [21]. Finally, the problem discussed in this paper is different from that in [36]. The nonlinearity permits taking negative infinity. Thus, the problem studied in this paper is so called semipositone problem.

2 Preliminaries and several lemmas

Two typical Banach spaces $E = C[0, 1]$ and $L^1(0, 1)$ are involved in this article, where $E = C[0, 1]$ and $L^1(0, 1)$ represent the spaces of the continuous functions and Lebesgue integrable functions equipped with the norms $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and $\|u\|_1 = \int_0^1 |u(t)| dt$, respectively.

Definition 1. (See [14].) A functional $\zeta : P \rightarrow [0, +\infty)$ is called a concave positive functional on a cone P if

$$\zeta(tx + (1-t)y) \geq t\zeta(x) + (1-t)\zeta(y) \quad \forall x, y \in P, 0 \leq t \leq 1.$$

Lemma 1. (See [9].) Let $\Delta \neq 0$. Given $y \in C(0, 1) \cap L^1(0, 1)$, the solution of the following differential equation

$$\begin{aligned} D_{0+}^{\alpha} u(t) + y(t) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^p u(1) = \sum_{i=1}^m a_i D_{0+}^q u(\xi_i), \end{aligned} \quad (2)$$

can be written as

$$u(t) = \int_0^1 G(t, s)y(s) ds, \quad t \in [0, 1],$$

where

$$G(t, s) = G_1(t, s) + \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m a_i G_2(\xi_i, s), \quad (3)$$

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-p-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_2(t, s) = \frac{1}{\Gamma(\alpha-q)} \begin{cases} t^{\alpha-q-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-q-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-q-1}(1-s)^{\alpha-p-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 2. (See [9].) Assume that $a_i > 0$ ($i = 1, 2, \dots, m$) and $\Delta > 0$. Then the Green function G of (2) given by (3) is a continuous function on $[0, 1] \times [0, 1]$ and satisfies the inequalities:

- (i) $G(t, s) \leq J(s)$, for all $t, s \in [0, 1]$, where $J(s) = h_1(s) + \sum_{i=1}^m a_i G_2(\xi_i, s)/\Delta$, $h_1(s) = (1-s)^{\alpha-p-1}(1-(1-s)^p)/\Gamma(\alpha)$, $s \in [0, 1]$;
- (ii) $G(t, s) \geq t^{\alpha-1}J(s)$ for all $t, s \in [0, 1]$;
- (iii) $G(t, s) \leq \sigma t^{\alpha-1}$ for all $t, s \in [0, 1]$, where $\sigma = 1/\Gamma(\alpha) + \sum_{i=1}^m a_i \xi_i^{\alpha-q-1}/(\Delta\Gamma(\alpha-q))$.

Lemma 3. (See [21].) Suppose that $w(t) \in C[0, 1]$ be the solution of

$$\begin{aligned} D_{0+}^{\alpha} u(t) + k(t) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^p u(1) = \sum_{i=1}^m a_i D_{0+}^q u(\xi_i), \end{aligned}$$

where $k \in L^1(0, 1)$, $k(t) > 0$. Then $w(t) \leq \sigma \|k\|_1 t^{\alpha-1}$, $0 \leq t \leq 1$.

In this paper, let

$$K = \{u \in E: u(t) \geq t^{\alpha-1}\|u\|, t \in [0, 1]\}.$$

Obviously, K is a cone in E . For simplicity, denote

$$K_r = \{u \in K: \|u\| < r\}$$

and

$$\begin{aligned} K(\zeta, a, b) &= \{u \in K: a \leq \zeta(u), \|u\| \leq b\}, \\ \mathring{K}(\zeta, a, b) &= \{u \in K: a < \zeta(u), \|u\| \leq b\}. \end{aligned} \quad (4)$$

We make the following assumptions throughout this paper:

- (H1) $f \in C((0, 1) \times (0, +\infty), (-\infty, +\infty))$, there exists a function $k \in L^1(0, 1)$, $k(t) > 0$, such that $f(t, u) \geq -k(t)$ for all $t \in (0, 1)$, $u > 0$.
- (H2) For any positive numbers $r_1 < r_2$, there exists a nonnegative continuous function $\gamma_{r_1, r_2} \in L^1(0, 1)$ such that

$$|f(t, u)| \leq \gamma_{r_1, r_2}(t), \quad 0 < t < 1, \quad r_1 t^{\alpha-1} \leq u \leq r_2,$$

with

$$\int_0^1 (1-s)^{\alpha-p-1} \gamma_{r_1, r_2}(s) ds < +\infty.$$

Lemma 4. (See [14].) Suppose that $T : \overline{K}_c \rightarrow K$ is completely continuous and there exist a concave positive functional ζ with $\zeta(u) \leq \|u\|$ ($u \in K$) and numbers $b > a > 0$ ($b \leq c$) satisfying the following conditions:

- (i) $\{u \in K(\zeta, a, b): \zeta(u) > a\} \neq \emptyset$, and $\zeta(Tu) > a$ if $u \in K(\zeta, a, b)$;
- (ii) $Tu \in \overline{K}_c$ if $u \in K(\zeta, a, c)$;
- (iii) $\zeta(Tu) > a$ for all $u \in K(\zeta, a, c)$ with $\|Tu\| > b$.

Then $i(T, \mathring{K}(\zeta, a, c), \overline{K}_c) = 1$.

Lemma 5. (See [5].) Let K be a cone in Banach space X , and $T : K \rightarrow K$ be a completely continuous operator. Let a, b, c be three positive numbers with $a < b < c$.

- (i) If $\|Tu\| > \|u\|$ for $u \in \partial(K_a)$ and $\|Tu\| \leq \|u\|$ for $u \in \partial(K_b)$, then

$$i(T, \overline{K}_b \setminus \overline{K}_a, \overline{K}_b) = 1,$$

- (ii) If $\|Tu\| > \|u\|$ for $u \in \partial(K_a)$ and $\|Tu\| < \|u\|$ for $u \in \partial(K_b)$, then

$$i(T, K_b \setminus \overline{K}_a, \overline{K}_c) = 1.$$

3 Main result

Suppose that $0 < a^* < b^* < 1$. In applications, a^* and b^* can be chosen according to the properties of $f(t, u)$. Denote $\sigma^* = \min_{t \in [a^*, b^*]} t^{\alpha-1}$. For any $r, r_1, r_2 > 0$ with $r_1 < r_2$, define the height functions as follows:

$$\begin{aligned}\widehat{\varphi}(t, r) &= \max\{f(t, u): (r - \sigma\|k\|_1)t^{\alpha-1} \leq u \leq r\} + k(t), \\ \widehat{\psi}(t, r) &= \min\{f(t, u): (r - \sigma\|k\|_1)t^{\alpha-1} \leq u \leq r\} + k(t), \\ \widehat{\varphi}(t, r_1, r_2) &= \max\{f(t, u): (r_1 - \sigma\|k\|_1)t^{\alpha-1} \leq u \leq r_2\} + k(t), \\ \widehat{\psi}(t, r_1, r_2) &= \min\{f(t, u): (r_1 - \sigma\|k\|_1)t^{\alpha-1} \leq u \leq r_2\} + k(t).\end{aligned}$$

Theorem 1. *Suppose that (H1) and (H2) hold. In addition, there exist five positive numbers $\sigma\|k\|_1 < e_1 < e_2 < e_3 < e_4 \leq e_5$ with $e_4 \geq e_3\sigma^{*-1}$ satisfying*

$$\begin{aligned}\text{(A1)} \quad & \int_0^1 J(s)\widehat{\varphi}(s, e_2) \, ds < e_2; \\ \text{(A2)} \quad & \int_0^1 J(s)\widehat{\psi}(s, e_1) \, ds \geq e_1; \\ \text{(A3)} \quad & \int_0^1 J(s)\widehat{\varphi}(s, e_3, e_5) \, ds \leq e_5; \\ \text{(A4)} \quad & \int_{a^*}^{b^*} J(s)\widehat{\psi}(s, e_3, e_4) \, ds > \sigma^{*-1}e_3.\end{aligned}$$

Then, BVP (1) has at least three positive solutions $\hat{u}_1, \hat{u}_2, \hat{u}_3$ with $e_1 - \sigma\|k\|_1 \leq \|\hat{u}_1\| \leq e_2, e_3 - \sigma\|k\|_1 \leq \|\hat{u}_2\| \leq e_5, e_2 - \sigma\|k\|_1 \leq \|\hat{u}_3\| \leq e_5$ and

$$\min_{t \in [a^*, b^*]} \hat{u}_2(t) \geq e_3 - \sigma\|k\|_1, \quad \min_{t \in [a^*, b^*]} \hat{u}_3(t) \leq e_3.$$

Proof. First, consider the following modified approximating BVP (MABVP for short):

$$\begin{aligned}D_{0+}^\alpha u(t) + f(t, \chi_n(u-w)(t)) + k(t) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^\beta(1) = \sum_{i=1}^m a_i D_{0+}^\alpha u(\xi_i),\end{aligned}$$

where

$$\chi_n(u) = \begin{cases} u, & u \geq \frac{1}{n}, \\ \frac{1}{n}, & u < \frac{1}{n}. \end{cases}$$

Define operators T_n ($n \in \mathbb{N}^+$) as follows:

$$(T_n u)(t) = \int_0^1 G(t, s)[f(s, \chi_n(u-w)(s)) + k(s)] \, ds, \quad 0 \leq t \leq 1, \quad n \in \mathbb{N}^+.$$

In the sequel, we will give the proof by the following three steps.

(I) We show that for any $\sigma\|k\|_1 < r_1 < r_2$ and sufficiently large n , operators $T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \rightarrow K$ is completely continuous.

The proof is as the same as the first part of Theorem 6 in [3], we omit it here. Thus, for sufficiently large n , the same conclusion is valid for $T_n : \overline{K}_{e_5} \setminus K_{e_1} \rightarrow K$. Define $\zeta(u) = \min_{t \in [a^*, b^*]} u(t)$ for any $u \in K$. In the following, $K(\zeta, e_3, e_4)$, $\overset{\circ}{K}(\zeta, e_3, e_4)$, $K(\zeta, e_3, e_5)$ have the same meaning as those in (4).

(II) We demonstrate that for sufficiently large n , T_n has three fixed points. (See Fig. 1 for better comprehension.)

First, we are in position to show that for sufficiently large n ,

$$i(T_n, \overset{\circ}{K}(\zeta, e_3, e_5), \overline{K}_{e_5}) = 1. \tag{5}$$

Set $u_0(t) \equiv (e_3 + e_4)/2$. Then $u_0 \in \overset{\circ}{K}(\zeta, e_3, e_4)$, which means that $\overset{\circ}{K}(\zeta, e_3, e_4) \neq \emptyset$. If $u \in K(\zeta, e_3, e_4)$, we have

$$e_3 \leq \min_{t \in [a^*, b^*]} u(t) \leq \max_{t \in [0, 1]} u(t) = \|u\| \leq e_4.$$

By the construction of cone K and Lemma 3 we know that $0 < (e_3 - \sigma\|k\|_1)t^{\alpha-1} \leq u(t) - w(t) \leq e_4, t \in [0, 1]$, and

$$(e_3 - \sigma\|k\|_1)t^{\alpha-1} \leq \max\left\{u(t) - w(t), \frac{1}{n}\right\} \leq e_4, \quad n > N_1 = \left[\frac{1}{e_3}\right] + 1,$$

which means

$$(e_3 - \sigma\|k\|_1)t^{\alpha-1} \leq \chi_n(u - w)(t) \leq e_4, \quad 0 < t < 1, \quad n > N_1. \tag{6}$$

It follows from Lemma 2 and (A4) that

$$\begin{aligned} \zeta(T_n u) &= \min_{t \in [a^*, b^*]} (T_n u)(t) \geq \min_{t \in [a^*, b^*]} t^{\alpha-1} \|T_n u\| = \sigma^* \|T_n u\| \\ &= \sigma^* \max_{t \in [0, 1]} \int_0^1 G(t, s) [f(s, \chi_n(u - w)(s)) + k(s)] ds \\ &\geq \sigma^* \max_{t \in [0, 1]} t^{\alpha-1} \int_0^1 J(s) [f(s, \chi_n(u - w)(s)) + k(s)] ds \\ &\geq \sigma^* \int_{a^*}^{b^*} J(s) \hat{\psi}(s, e_3, e_4) ds > e_3. \end{aligned}$$

If $u \in K(\zeta, e_3, e_5)$, then

$$e_3 \leq \min_{t \in [a^*, b^*]} u(t) \leq \max_{t \in [0, 1]} u(t) = \|u\| \leq e_5.$$

Hence, $e_3 t^{\alpha-1} \leq u(t) \leq e_5$ for $t \in [0, 1]$. Similar to (6), one has

$$(e_3 - \sigma\|k\|_1)t^{\alpha-1} \leq \chi_n(u - w)(t) \leq e_5, \quad 0 < t < 1, \quad n > N_1. \tag{7}$$

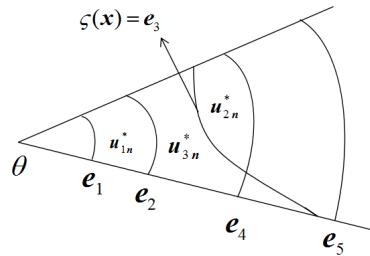


Figure 1. Illustration of proof.

This, together with Lemma 2 and (A3), means that

$$\begin{aligned} \|T_n u\| &= \max_{t \in [0,1]} \int_0^1 G(t,s) [f(s, \chi_n(u-w)(s)) + k(s)] ds \\ &\leq \int_0^1 J(s) [f(s, \chi_n(u-w)(s)) + k(s)] ds \\ &\leq \int_0^1 J(s) \widehat{\varphi}(s, e_3, e_5) ds \leq e_5, \quad n > N_1. \end{aligned} \tag{8}$$

Consequently, $T_n u \in \overline{K}_{e_5}$.

For $u \in K(\zeta, e_3, e_5)$ with $\|T_n u\| > e_4$, noticing that $e_4 \geq e_3 \sigma^{*-1}$, we have $\|T_n u\| > e_3 \sigma^{*-1}$. Therefore,

$$\zeta(T_n u) = \min_{t \in [a^*, b^*]} (T_n u)(t) \geq \sigma^* \|T_n u\| > \sigma^* e_3 \sigma^{*-1} = e_3.$$

Thus, for $n > N_1$, we know from Lemma 4 that (5) holds.

If $u \in \partial(K_{e_5})$, then $\|u\| = e_5$ and $e_3 t^{\alpha-1} \leq e_5 t^{\alpha-1} \leq u(t) \leq e_5, t \in [0, 1]$. Thus, (7) holds. By (7), (A3) and Lemma 2, similar to the proof of (8), for any $n > N_1$, one gets

$$\begin{aligned} \|T_n u\| &= \max_{t \in [0,1]} \int_0^1 G(t,s) [f(s, \chi_n(u-w)(s)) + k(s)] ds \\ &\leq \int_0^1 J(s) [f(s, \chi_n(u-w)(s)) + k(s)] ds \\ &\leq \int_0^1 J(s) \widehat{\varphi}(s, e_3, e_5) ds \leq e_5 \quad \forall u \in \partial(K_{e_5}). \end{aligned} \tag{9}$$

If $u \in \partial(K_{e_2})$, then $\|u\| = e_2$ and $e_2 t^{\alpha-1} \leq u(t) \leq e_2, t \in [0, 1]$. Thus, we have $0 < (e_2 - \sigma\|k\|_1)t^{\alpha-1} \leq u(t) - w(t) \leq e_2$ for $t \in [0, 1]$ and

$$(e_2 - \sigma\|k\|_1)t^{\alpha-1} \leq \max \left\{ u(t) - w(t), \frac{1}{n} \right\} \leq e_2, \quad n > N_2 = \left[\frac{1}{e_2} \right] + 1,$$

i.e.,

$$(e_2 - \sigma\|k\|_1)t^{\alpha-1} \leq \chi_n(u - w)(t) \leq e_2, \quad 0 < t < 1, n > N_2.$$

By (A1) and Lemma 2, for any $n > N_2$, one has

$$\begin{aligned} \|T_n u\| &= \max_{t \in [0,1]} \int_0^1 G(t,s) [f(s, \chi_n(u-w)(s)) + k(s)] ds \\ &\leq \int_0^1 J(s) \widehat{\varphi}(s, e_2) ds < e_2 \quad \forall u \in \partial(K_{e_2}). \end{aligned} \tag{10}$$

If $u \in \partial(K_{e_1})$, then $e_1 t^{\alpha-1} \leq u(t) \leq e_1, t \in [0, 1]$. Thus, we have $0 < (e_1 - \sigma\|k\|_1) \times t^{\alpha-1} \leq u(t) - w(t) \leq e_1$ for $t \in [0, 1]$ and

$$(e_1 - \sigma\|k\|_1)t^{\alpha-1} \leq \max \left\{ u(t) - w(t), \frac{1}{n} \right\} \leq e_1, \quad n > N_3 = \left[\frac{1}{e_1} \right] + 1,$$

i.e.,

$$(e_1 - \sigma\|k\|_1)t^{\alpha-1} \leq \chi_n(u - w)(t) \leq e_1, \quad 0 < t < 1, n > N_3.$$

By (A2) and Lemma 2, for any $n > N_3$, one gets

$$\begin{aligned} \|T_n u\| &= \max_{t \in [0,1]} \int_0^1 G(t,s) [f(s, \chi_n(u-w)(s)) + k(s)] ds \\ &\geq \max_{t \in [0,1]} t^{\alpha-1} \int_0^1 J(s) [f(s, \chi_n(u-w)(s)) + k(s)] ds \\ &\geq \int_0^1 J(s) \widehat{\psi}(s, e_1) ds \geq e_1 \quad \forall u \in \partial(K_{e_1}). \end{aligned} \tag{11}$$

By (9), (10), (11) and Lemma 5, for any $n > N = \max\{N_1, N_2, N_3\}$, we both have (5) and the following two equalities:

$$i(T_n, \overline{K}_{e_5} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) = 1, \quad i(T_n, K_{e_2} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) = 1. \tag{12}$$

It is clear, (A1) implies that T_n has no fixed point on $\partial(K_{e_2})$. In addition, for $u \in K(\zeta, e_3, e_4)$, we have that $\zeta(T_n u) > e_3$, and for $u \in K(\zeta, e_3, e_5)$ with $\|T_n u\| > e_4$,

we also have that $\zeta(T_n u) > e_3$. This is to say, T_n has no fixed point on $K(\zeta, e_3, e_5) \setminus \mathring{K}(\zeta, e_3, e_5)$. Thus, for $n > N$, it follows from (5), (12) and the addition property of the topological degree that

$$\begin{aligned} & i(T_n, \overline{K}_{e_5} \setminus (K(\zeta, e_3, e_5) \cup \overline{K}_{e_2}), \overline{K}_{e_5}) \\ &= i(T_n, \overline{K}_{e_5} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) - i(T_n, K_{e_2} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) \\ &\quad - i(T_n, \mathring{K}(\zeta, e_3, e_5), \overline{K}_{e_5}) \\ &= -1. \end{aligned}$$

As a consequence, for $n > N$, T_n has at least three fixed points $u_{1n}^* \in K_{e_2} \setminus \overline{K}_{e_1}$, $u_{2n}^* \in \mathring{K}(\zeta, e_3, e_5)$, $u_{3n}^* \in \overline{K}_{e_5} \setminus (K(\zeta, e_3, e_5) \cup \overline{K}_{e_2})$ satisfying $e_1 \leq \|u_{1n}^*\| < e_2$, $e_3 \leq \|u_{2n}^*\| \leq e_5$, $e_2 \leq \|u_{3n}^*\| < e_5$ with

$$\min_{t \in [a^*, b^*]} u_{2n}^*(t) > e_3, \quad \min_{t \in [a^*, b^*]} u_{3n}^*(t) < e_3.$$

(III) We prove that BVP (1) has triple positive solutions.

Taking into account the construction of the cone K , one has that, for $n > N$ and $i = 1, 2, 3$,

$$u_{in}^*(t) \geq \|u_{in}^*\| t^{\alpha-1} \geq e_1 t^{\alpha-1} \geq \sigma \|k\|_1 t^{\alpha-1} \geq w(t), \quad t \in [0, 1], \quad (13)$$

and

$$u_{in}^*(t) = \int_0^1 G(t, s) [f(s, \chi_n(u_{in}^* - w)(s)) + k(s)] ds, \quad t \in [0, 1]. \quad (14)$$

It is easy to know from (H2) that $\{u_{in}^* : n > N\}$ ($i = 1, 2, 3$) are bounded and equicontinuous on $[0, 1]$. Thus, Arzelà–Ascoli theorem implies that there exist a subsequence N_0 of N and corresponding continuous functions u_i^* ($i = 1, 2, 3$) such that u_{in}^* converges to u_i^* ($i = 1, 2, 3$) uniformly on $[0, 1]$ as $n \rightarrow \infty$ through N_0 . Let $n \rightarrow \infty$ on both sides of (14), one has

$$u_i^*(t) = \int_0^1 G(t, s) [f(s, \chi(u_i^* - w)(s)) + k(s)] ds, \quad t \in [0, 1], \quad i = 1, 2, 3, \quad (15)$$

and

$$e_1 \leq \|u_1^*\| \leq e_2, e_3 \leq \|u_2^*\| \leq e_5, e_2 \leq \|u_3^*\| \leq e_5 \quad (16)$$

with $\min_{t \in [a^*, b^*]} u_2^*(t) \geq e_3$, $\min_{t \in [a^*, b^*]} u_3^*(t) \leq e_3$. It can be easily seen from (13) that $u_i^*(t) \geq t^{\alpha-1} \|u_i^*\| \geq e_1 t^{\alpha-1} \geq \sigma \|k\|_1 t^{\alpha-1} \geq w(t)$ ($i = 1, 2, 3$). Let $\hat{u}_i(t) = u_i^*(t) - w(t)$, then we know from (15) that $\hat{u}_i(t)$ ($i = 1, 2, 3$) are positive solutions for BVP (1). This, together with (16) and Lemma 3, implies that $e_1 - \sigma \|k\|_1 \leq \|\hat{u}_1\| \leq e_2$, $e_3 - \sigma \|k\|_1 \leq \|\hat{u}_2\| \leq e_5$, $e_2 - \sigma \|k\|_1 \leq \|\hat{u}_3\| \leq e_5$ and $\min_{t \in [a^*, b^*]} \hat{u}_2(t) \geq e_3 - \sigma \|k\|_1$, $\min_{t \in [a^*, b^*]} \hat{u}_3(t) \leq e_3$. \square

4 An example

Consider the following singular fractional differential equations:

$$\begin{aligned}
 D_{0+}^{11/3} u(t) + f(t, u(t)) - \frac{1}{10^4 \sqrt[3]{t}} &= 0, \quad 0 < t < 1, \\
 u(0) = u'(0) = u''(0) &= 0, \\
 D_{0+}^{3/2} u(1) = \frac{1}{2} D_{0+}^{4/3} u\left(\frac{1}{4}\right) + \frac{2}{3} D_{0+}^{4/3} u\left(\frac{1}{2}\right) + \frac{1}{4} D_{0+}^{4/3} u\left(\frac{4}{5}\right),
 \end{aligned}
 \tag{17}$$

where $f(t, u(t)) = \theta(u(t))/(30 \sqrt[3]{t(1-t)^2})$ and

$$\theta(u) = \begin{cases} u^{1/3} + u^{-1/5}, & 0 < u \leq 1, \\ u^8 + 1, & 1 < u \leq 5, \\ u^{1/3} + 390626 - 5^{1/3}, & u \geq 5. \end{cases}$$

Clearly, $\alpha = 11/3, n = 4, p = 3/2, q = 4/3, m = 3, \xi_1 = 1/4, \xi_2 = 1/2, \xi_3 = 4/5, a_1 = 1/2, a_2 = 2/3, a_3 = 1/4$. It is easy to see that (H1) holds for $k(t) = 1/(10^4 \sqrt[3]{t})$ and (H2) is valid for $\gamma_{r_1, r_2}(t) = (1/(30 \sqrt[3]{t(1-t)^2}))[r_2^{1/3} + (r_1 t^{8/3})^{-1/5} + r_2^8 + r_2^{1/3} + 390627 - 5^{1/3}]$. After direct calculation, we have $\alpha - p - 1 = 7/6, \alpha - q - 1 = 4/3, \Gamma(\alpha) = 4.0122, \Gamma(\alpha - p) = 1.0823, \Gamma(\alpha - q) = 1.1906, \sum_{i=1}^3 a_i \xi_i^{\alpha - q - 1} = 0.5290, \Delta = 1.9244, \sigma = 0.4801, \|k\|_1 = 1.5 \cdot 10^{-4}, \sigma \|k\|_1 = 7.202 \cdot 10^{-5}$. Take $a^* = 2/3, b^* = 1, e_1 = 10^{-3}, e_2 = 1/2, e_3 = 5, e_4 = 20, e_5 = 10^5$. Then $\sigma^* = (2/3)^{8/3} = 0.3392, \sigma^{*-1} = 2.9481, e_4 > \sigma^{*-1} e_3$. We have

$$\begin{aligned}
 &\int_0^1 J(s) \widehat{\varphi}(s, e_2) \, ds \\
 &\leq \int_0^1 \left[\frac{(1-s)^{\alpha-p-1} (1-(1-s)^p)}{\Gamma(\alpha)} + \frac{\sum_{i=1}^m a_i \xi_i^{\alpha-q-1} (1-s)^{\alpha-p-1}}{\Delta \Gamma(\alpha-q)} \right] \\
 &\quad \times \frac{1}{30 \sqrt[3]{s(1-s)^2}} \max \{ (u^{1/3} + u^{-1/5}): (e_2 - \sigma \|k\|_1) s^{\alpha-1} \leq u \leq e_2 \} \, ds \\
 &= \int_0^1 \left[\frac{(1-s)^{7/6} (1-(1-s)^{3/2})}{\Gamma(\alpha)} + \frac{\sum_{i=1}^3 a_i \xi_i^{4/3}}{\Delta \Gamma(\alpha-q)} (1-s)^{7/6} \right] \\
 &\quad \times \frac{1}{30 \sqrt[3]{s(1-s)^2}} \max \{ (u^{1/3} + u^{-1/5}): 0.4999 s^{8/3} \leq u \leq 2^{-1} \} \, ds \\
 &< \frac{1}{30} \int_0^1 \frac{[0.2492(1-s)^{7/6} - 0.2492(1-s)^{8/3} + 0.2309(1-s)^{7/6}]}{s^{1/3}(1-s)^{2/3}} \\
 &\quad \times [2^{-1/3} + (0.4999 s^{8/3})^{-1/5}] \, ds
 \end{aligned}$$

$$\begin{aligned}
&< \frac{1}{30} \int_0^1 \frac{[0.4801(1-s)^{7/6} - 0.2492(1-s)^{8/3}](0.7937 + 1.1487s^{-8/15})}{s^{1/3}(1-s)^{2/3}} ds \\
&= \frac{1}{30} \left[0.4801 \cdot 0.7937 \int_0^1 \frac{(1-s)^{1/2}}{s^{1/3}} ds - 0.2492 \cdot 0.7937 \int_0^1 \frac{(1-s)^2}{s^{1/3}} ds \right. \\
&\quad \left. + 0.4801 \cdot 1.1487 \int_0^1 \frac{(1-s)^{1/2}}{s^{13/15}} ds - 0.2492 \cdot 1.1487 \int_0^1 \frac{(1-s)^2}{s^{13/15}} ds \right] \\
&= \frac{1}{30} (0.4801 \cdot 0.7937 \cdot 1.1088 - 0.2492 \cdot 0.7937 \cdot 0.6750 \\
&\quad + 0.4801 \cdot 1.1487 \cdot 6.9506 - 0.2492 \cdot 1.1487 \cdot 6.2040) \\
&= 0.0782 < \frac{1}{2} = e_2.
\end{aligned}$$

Hence, (A1) is satisfied. By Lemma 2 we have

$$\begin{aligned}
&\int_0^1 J(s) \widehat{\psi}(s, e_1) ds \\
&= \int_0^1 \left[\frac{(1-s)^{\alpha-p-1}(1-(1-s)^p)}{\Gamma(\alpha)} + \frac{\sum_{i=1}^m a_i \xi_i^{\alpha-q-1} (1-s)^{\alpha-p-1}}{\Delta \Gamma(\alpha-q)} \right] \widehat{\psi}(s, e_1) ds \\
&= \int_0^1 \left[\frac{(1-s)^{7/6}(1-(1-s)^{3/2})}{\Gamma(\alpha)} + \frac{\sum_{i=1}^3 a_i \xi_i^{4/3}}{\Delta \Gamma(\alpha-q)} (1-s)^{7/6} \right] \frac{1}{30 \sqrt[3]{s(1-s)^2}} \\
&\quad \times \min\{u^{1/3} + u^{-1/5}\} : 9.2798 \cdot s^{8/3} \cdot 10^{-4} \leq u \leq 10^{-3} \} ds \\
&= \frac{1}{30} \int_0^1 \frac{[0.2492(1-s)^{7/6} - 0.2492(1-s)^{8/3} + 0.2309 \cdot (1-s)^{7/6}]}{s^{1/3}(1-s)^{2/3}} \\
&\quad \times [(9.2798 \cdot s^{8/3} \cdot 10^{-4})^{1/3} + (10^{-3})^{-1/5}] ds \\
&> \frac{1}{30} \left(0.4801 \int_0^1 \frac{(1-s)^{1/2}}{s^{1/3}} ds - 0.2492 \int_0^1 \frac{(1-s)^2}{s^{1/3}} ds \right) \cdot 10^{3/5} \\
&= \frac{1}{30} (0.4801 \cdot 1.1088 - 0.2492 \cdot 0.6750) \cdot 3.9811 \\
&= 0.0483 > \frac{1}{10^3} = e_1.
\end{aligned}$$

Thus, (A2) is verified. On the other hand,

$$\begin{aligned}
 & \int_0^1 J(s) \widehat{\varphi}(s, e_3, e_5) \, ds \\
 &= \int_0^1 \left[\frac{(1-s)^{\alpha-p-1} (1-(1-s)^p)}{\Gamma(\alpha)} + \frac{\sum_{i=1}^m a_i \xi_i^{\alpha-q-1} (1-s)^{\alpha-p-1}}{\Delta\Gamma(\alpha-q)} \right] \\
 & \quad \times \frac{1}{30 \sqrt[3]{s(1-s)^2}} (10^{5/3} + 390626 - 5^{1/3}) \, ds \\
 &< \frac{1}{30} \int_0^1 \frac{[0.4801(1-s)^{7/6} - 0.2492(1-s)^{8/3}]}{s^{1/3}(1-s)^{2/3}} \, ds \cdot 3.9067 \cdot 10^5 \\
 &= 4.7418 \cdot 10^3 < 10^5 = e_5.
 \end{aligned}$$

As a result, (A3) is valid. We also have

$$\begin{aligned}
 & \int_{2/3}^1 J(s) \widehat{\psi}(s, e_3, e_4) \, ds \\
 &= \int_{2/3}^1 \left[\frac{(1-s)^{7/6} (1-(1-s)^{3/2})}{\Gamma(\alpha)} + \frac{\sum_{i=1}^3 a_i \xi_i^{4/3}}{\Delta\Gamma(\alpha-q)} (1-s)^{7/6} \right] \widehat{\psi}(s, e_3, e_4) \, ds \\
 &> \int_{2/3}^1 \left[\frac{(1-s)^{7/6} (1-(1-s)^{3/2})}{\Gamma(\alpha)} + \frac{\sum_{i=1}^3 a_i \xi_i^{4/3}}{\Delta\Gamma(\alpha-q)} (1-s)^{7/6} \right] \\
 & \quad \times \frac{1}{30 \sqrt[3]{s(1-s)^2}} (5^8 + 1) \, ds \\
 &> \frac{1}{30} \int_{2/3}^1 \frac{\sum_{i=1}^3 a_i \xi_i^{4/3}}{\Delta\Gamma(\alpha-q)} (1-s)^{7/6} s^{-1/3} (1-s)^{-2/3} \, ds \\
 &> \frac{1}{30} \cdot 390626 \int_{2/3}^1 \frac{\sum_{i=1}^3 a_i \xi_i^{4/3}}{\Delta\Gamma(\alpha-q)} (1-s)^{7/6} (1-s)^{-2/3} \, ds \\
 &= \frac{1}{30} \cdot 390626 \cdot 0.2309 \int_{2/3}^1 (1-s)^{1/2} \, ds \\
 &= \frac{1}{30} \cdot 390626 \cdot 0.2309 \cdot 0.1283 \\
 &= 385.7363 > 14.7405 = \sigma^{-1} e_3.
 \end{aligned}$$

This is to say that (A4) is checked. Thus, Theorem 1 guarantees that BVP (17) has at least three positive solutions $\hat{u}_1, \hat{u}_2, \hat{u}_3$ satisfying $9.2798 \cdot 10^{-4} \leq \|\hat{u}_1\| \leq 1/2$, $4.9999 \leq \|\hat{u}_2\| \leq 10^5$, $0.4999 \leq \|\hat{u}_3\| \leq 10^5$ with $\min_{t \in [a^*, b^*]} \hat{u}_2(t) \geq 4.9999$, $\min_{t \in [a^*, b^*]} \hat{u}_3(t) \leq 5$.

Acknowledgment. The authors would like to thank the referee for his/her valuable comments and suggestions.

References

1. Z. Bai, H. Lv, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311**(2):495–505, 2005.
2. Y. Cui, W. Ma, Q. Sun, X. Su, New uniqueness results for boundary value problem of fractional differential equation, *Nonlinear Anal. Model. Control*, **23**(1):31–39, 2018.
3. Y. Cui, Y. Zou, An existence and uniqueness theorem for a second order nonlinear system with coupled integral boundary value conditions, *Appl. Math. Comput.*, **256**:438–444, 2015.
4. M. Günendi, İ. Yaslan, Positive solutions of higher-order nonlinear multi-point fractional equations with integral boundary conditions, *Fract. Calc. Appl. Anal.*, **19**(4):989–1009, 2016.
5. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, CA, 1988.
6. L. Guo, L. Liu, Y. Wu, Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions, *Nonlinear Anal. Model. Control*, **21**(5):635–650, 2016.
7. L. Guo, L. Liu, Y. Wu, Iterative unique positive solutions for singular p -Laplacian fractional differential equation system with several parameters, *Nonlinear Anal. Model. Control*, **23**(2): 182–203, 2018.
8. J. Henderson, R. Luca, Positive solutions for a system of nonlocal fractional boundary value problems, *Fract. Calc. Appl. Anal.*, **16**(4):985–1008, 2013.
9. J. Henderson, R. Luca, Existence of positive solutions for a singular fractional boundary value problem, *Nonlinear Anal. Model. Control*, **22**(1):99–114, 2017.
10. J. Henderson, R. Luca, Systems of Riemann–Liouville fractional equations with multi-point boundary conditions, *Appl. Math. Comput.*, **309**:303–323, 2017.
11. Y. Jia, X. Zhang, Positive solutions for a class of fractional differential equation multi-point boundary value problems with changing sign nonlinearity, *J. Appl. Math. Comput.*, **47**(1–2): 15–31, 2015.
12. J. Jiang, L. Liu, Y. Wu, Positive solutions to singular fractional differential system with coupled boundary conditions, *Commun. Nonlinear Sci. Numer. Simul.*, **18**(11):3061–3074, 2013.
13. J. Jiang, L. Liu, Y. Wu, Symmetric positive solutions to singular system with multi-point coupled boundary conditions, *Appl. Math. Comput.*, **220**(4):536–548, 2013.
14. R. Leggett, L. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28**(4):673–688, 1979.

15. L. Liu, X. Hao, Y. Wu, Positive solutions for singular second order differential equations with integral boundary conditions, *Math. Comput. Modelling*, **57**(3–4):836–847, 2013.
16. L. Liu, F. Sun, X. Zhang, Y. Wu, Bifurcation analysis for a singular differential system with two parameters via to degree theory, *Nonlinear Anal. Model. Control*, **22**(1):31–50, 2017.
17. L. Liu, X. Zhang, J. Jiang, Y. Wu, The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems, *J. Nonlinear Sci. Appl.*, **9**(5):2943–2958, 2016.
18. X. Liu, L. Liu, Y. Wu, Existence of positive solution for a singular nonlinear fractional differential equation with integral boundary conditions involving fractional derivatives, *Bound. Value Probl.*, **2018**:24, 2018.
19. D. Ma, Positive solutions of multi-point boundary value problem of fractional differential equation, *Arab J. Math. Sci.*, **21**(2):225–236, 2015.
20. D. Min, L. Liu, Y. Wu, Uniqueness of positive solutions for the singular fractional differential equations involving integral boundary value conditions, *Bound. Value Probl.*, **2018**:23, 2018.
21. R. Pu, X. Zhang, Y. Cui, P. Li, W. Wang, Positive solutions for singular semipositone fractional differential equation subject to multipoint boundary conditions, *J. Funct. Spaces*, **2017**:5892616, 2017.
22. X. Qiu, J. Xu, D. O'Regan, Y. Cui, Positive solutions for a system of nonlinear semipositone boundary value problems with Riemann–Liouville fractional derivatives, *J. Funct. Spaces*, **2018**:7351653, 2018.
23. H.A.H. Salem, On the fractional order m -point boundary value problem in reflexive Banach spaces and weak topologies, *J. Comput. Appl. Math.*, **224**(2):565–572, 2009.
24. L. Wang, X. Zhang, Positive solutions of m -point boundary value problems for a class of nonlinear fractional differential equations, *J. Appl. Math. Comput.*, **42**(1–2):387–399, 2013.
25. L. Wang, X. Zhang, Existence of positive solutions for a class of higher-order nonlinear fractional differential equations with integral boundary conditions and a parameter, *Appl. Math. Comput.*, **226**(1–2):708–718, 2014.
26. Y. Wang, L. Liu, Y. Wu, Positive solutions for a nonlocal fractional differential equation, *Nonlinear Anal., Theory Methods Appl.*, **74**(11):3599–3605, 2011.
27. Y. Wang, L. Liu, Y. Wu, Positive solutions of a fractional boundary value problem with changing sign nonlinearity, *Nonlinear Anal., Theory Methods Appl.*, **74**(17):6434–6441, 2011.
28. C. Zhai, L. Xu, Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter, *Nonlinear Sci. Numer. Simul.*, **19**(8):2820–2827, 2014.
29. X. Zhang, Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions, *Appl. Math. Lett.*, **39**:22–27, 2015.
30. X. Zhang, M. Feng, W. Ge, Symmetric positive solutions for p -Laplacian fourth-order differential equations with integral boundary conditions, *J. Comput. Appl. Math.*, **222**(2):561–573, 2008.
31. X. Zhang, L. Liu, Y. Wu, Existence results for multiple positive solutions of nonlinear higher order perturbed fractional differential equations with derivatives, *Appl. Math. Comput.*, **219**(4):1420–1433, 2012.

32. X. Zhang, L. Liu, Y. Wu, The uniqueness of positive solution for a singular fractional differential system involving derivatives, *Commun. Nonlinear Sci. Numer. Simul.*, **18**(6):1400–1409, 2013.
33. X. Zhang, Z. Shao, Q. Zhong, Positive solutions for semipositone $(k, n - k)$ conjugate boundary value problems with singularities on space variables, *Appl. Math. Lett.*, **72**:50–57, 2017.
34. X. Zhang, Y. Wu, L. Caccetta, Nonlocal fractional order differential equations with changing-sign singular perturbation, *Appl. Math. Modelling*, **39**(21):6543–6552, 2015.
35. X. Zhang, Q. Zhong, Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions, *Fract. Calc. Appl. Anal.*, **20**(6):1471–1484, 2017.
36. X. Zhang, Q. Zhong, Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables, *Appl. Math. Lett.*, **80**:12–19, 2018.
37. Y. Zhao, S. Sun, Z. Han, Q. Li, The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **16**(4):2086–2097, 2012.
38. Q. Zhong, X. Zhang, Multiple positive solutions to some second-order integral boundary value problems with singularity on space variable, *Bound. Value Probl.*, **2017**:90, 2017.
39. B. Zhu, L. Liu, Y. Wu, Local and global existence of mild solutions for a class of nonlinear fractional reaction–diffusion equations with delay, *Appl. Math. Lett.*, **61**:73–79, 2016.
40. Y. Zou, G. He, On the uniqueness of solutions for a class of fractional differential equations, *Appl. Math. Lett.*, **74**:68–73, 2017.