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Maximal and minimal iterative positive solutions for singular infinite-point p-Laplacian fractional differential equations*

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Abstract. The existence of maximal and minimal positive solutions for singular infinite-point p-Laplacian fractional differential equation is investigated in this paper. Green's function is derived, and some properties of Green's function are obtained. Based upon these properties of Green's function, the existence of maximal and minimal positive solutions is obtained, and iterative schemes are established for approximating the maximal and minimal positive solutions.

Keywords: fractional differential equation, Green's function, infinite-point, maximal and minimal positive solutions.

1 Introduction

In this paper, we consider the following singular infinite-point p-Laplacian fractional differential equations:

$$\begin{aligned}
&\phi_p\big(D_{0+}^{\alpha}u(t)\big)+f\big(t,u(t),D_{0+}^{\mu}u(t)\big)=0, \quad 0<t<1,\\
&u^{(i)}(0)=0, \quad i=0,1,2,\dots,n-2,\\
&D_{0^+}^{p_1}u(1)=\sum_{j=1}^{\infty}\eta_j D_{0^+}^{p_2}u(\xi_j),
\end{aligned} \tag{1}$$

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where $\alpha, \mu, p_1, p_2 \in \mathbb{R}^+$ ($\mathbb{R}^+ = [0, +\infty)$), $n - 1 < \alpha \leqslant n$ ($n > 3$, $n \in \mathbb{N}$), $0 \leqslant \mu \leqslant n - 2$, $\eta_j \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ ($j = 1, 2 \dots$), $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$, $p_1, p_2 \in [2, n-2]$, $p_2 \leqslant p_1$, $f(t, x, y)$ may be singular at $t = 0$, and D^{α}_{0+}, D^{μ}_{0+}, $D^{p_1}_{0+}$, $D^{p_2}_{0+}$ are the standard Riemann–Liouville derivative. The existence of maximal and minimal positive solutions is obtained by iterative sequence for the boundary value problem (1) under certain conditions.

During the last decades, boundary value problems of nonlinear fractional differential equations constitutes a new and important branch of differential equation theory and has attracted great research efforts worldwide, and it is a valuable tool for simulating many phenomena in various fields such as fluid flows, electrical networks, rheology, biology, chemical physics, and so on. In order to solve practical problems, the existence of positive solutions for many types of fractional differential equations is investigated. For more details, the reader is referred to [1–5, 7, 8, 10–18, 21–30] and the references therein. For some differential equation in which fractional derivatives are involved in the nonlinear terms, reader can refer to [2, 7, 8], and when values at infinite points are involved in the boundary conditions, we refer the reader to [7, 8, 24] and the references therein. Later, due to the need of practical problems, the p-Laplacian operator is introduced into some boundary value problems, and about p-Laplacian fractional differential equation we refer the reader to [5, 17, 18, 25] for some relevant work. In [24], the author considered the following fractional differential equation:

$$\begin{aligned} &D^{\alpha}_{0+}u(t) + g(t)f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ &u^{(i)}(1) = \sum_{i=1}^{\infty} \alpha_j u(\xi_j), \end{aligned}$$

where $\alpha \in \mathbb{R}^+$, $n - 1 < \alpha \leqslant n$, $n > 3$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ ($j = 1, 2, \dots$), f is allowed to have singularities with respect to both time and space variables. Various theorems were established for the existence and multiplicity of positive solutions. In [19], the author discussed the existence and multiplicity of positive solutions of the following problem:

$$\begin{aligned} &D^{\alpha}_{0+}u(t) = a(t)f\bigl(t, u(t)\bigr), \quad t \in (0, 1), \\ &u(0) = u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m} \beta_i u(\xi_i), \end{aligned}$$

where $\alpha \in \mathbb{R}^+$, $2 < \alpha \leqslant 3$, $m \geqslant 1$ is integer, $\beta_i > 0$ for $1 \leqslant i \leqslant m$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $\sum_{i=1}^{m} \beta_i \xi_i^{\alpha-1} < 1$, $a(t) \in L[0, 1]$ is nonnegative and not identically zero on any compact subset of $(0, 1)$, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and D^{α}_{0+} is the Riemann–Liouville differential fractional derivative of order α. Some results on the existence and multiplicity of positive solutions were obtained by the fixed point theorem.

In [18], the authors considered the following fractional differential equation:

$$\begin{aligned}&\phi_p\bigl(D_{0+}^{\alpha}u(t)\bigr)+f\bigl(t,u(t)\bigr)=0,\quad 0<t<1,\\&u(0)=u'(0)=u'(1)=0,\end{aligned}$$

where $\alpha\in\mathbb{R}^+$, $2<\alpha\leqslant 3$, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $(\phi_p)^{-1}=\phi_q$, $1/p+1/q=1$, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous, and D_{0+}^{α} is the standard Riemann–Liouville derivative.

Motivated by the results above, in this paper, we investigate the existence of positive solutions for a class of infinite-point singular p-Laplacian fractional differential equations. p-Laplacian fractional differential equation is a type of equation that is very wide, and the general equation are special cases of p-Laplacian equation. Compared with [24, 29], the fractional-order derivatives are involved in the nonlinear term and boundary condition, and at the same time, iterative solutions are obtained by iterative sequences. Compared with [19], values at infinite points are involved in the boundary conditions of the boundary value problem (1), and the nonlinearity is singular, that is, $f(t,u,v)$ is allowed to be singular at $t=0$. Compared with [7], we do not only obtain the existence of positive solutions, but we also establish iterative sequences to approximate the maximal and minimal positive solutions.

2 Preliminaries and lemmas

Some basic definitions and lemmas, which will be used in the proof of our results and can also be found in the recent literature such as [9, 20], we omit some here.

Now we list a condition below to be used later in the paper.

(H0) $f:(0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, and there exists a constant $0<\sigma<1$ such that $t^{\sigma}\phi_q(f(t,x_0,x_1))$ is continuous on $[0,1]\times\mathbb{R}^+\times\mathbb{R}^+$.

Lemma 1. (*See* [9, 20].) *Assume that $u\in C^n[0,1]$, then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n},$$

where n is the least integer greater than or equal to α, $C_i\in\mathbb{R}$ $(i=1,2,\dots,n)$.

Lemma 2. (*See* [6, *Thm.* 1.2.7].) *Let $H\subset C^1[J,E]$, then H is a relatively compact set if and only if*

(i) *H' is equicontinuous, and $H'(t)$ is a relatively compact set for any $t\in J$ on E;*
(ii) *There exists $t_0\in J$ such that $H(t_0)$ is a relatively compact set on E.*

Lemma 3. *Given $y\in L^1[0,1]\cap C(0,1)$, then the solution of the BVP*

$$\begin{aligned}&\phi_p\bigl(D_{0+}^{\alpha}u(t)\bigr)+y(t)=0,\quad 0<t<1,\\&u^{(i)}(0)=0,\quad i=0,1,2,\dots,n-2,\\&D_{0+}^{p_1}u(1)=\sum_{j=1}^{\infty}\eta_jD_{0+}^{p_2}u(\xi_j)\end{aligned}\tag{2}$$

can be expressed by

$$u(t) = \int_0^1 G(t,s)\phi_q\bigl(y(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1], \tag{3}$$

where

$$G(t,s) = \frac{1}{\Delta\Gamma(\alpha)} \begin{cases} \Gamma(\alpha)t^{\alpha-1}P(s)(1-s)^{\alpha-p_1-1} - \Delta(t-s)^{\alpha-1}, \\ \quad 0 \leqslant s \leqslant t \leqslant 1, \\ \Gamma(\alpha)t^{\alpha-1}P(s)(1-s)^{\alpha-p_1-1}, \\ \quad 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{4}$$

in which

$$P(s) = \frac{1}{\Gamma(\alpha-p_1)} - \frac{1}{\Gamma(\alpha-p_2)} \sum_{s \leqslant \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-p_2-1} (1-s)^{p_1-p_2},$$

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)} \sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha-p_2-1},$$

and obviously, $G(t,s)$ is continuous on $[0,1] \times [0,1]$.

Proof. By means of Lemma 1, we reduce (2) to an equivalent integral equation

$$u(t) = -I_{0^+}^{\alpha}\phi_q\bigl(y(t)\bigr) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for $C_i \in \mathbb{R}$ ($i = 1,2,\dots,n$). From $u^{(i)}(0) = 0$ ($i = 0,1,2,\dots,n-2$) we have $C_i = 0$ ($i = 2,3,\dots,n$). Consequently, we get

$$u(t) = C_1 t^{\alpha-1} - I_{0^+}^{\alpha}\phi_q\bigl(y(t)\bigr).$$

By some properties of the fractional integrals and fractional derivatives, we have

$$\begin{aligned} D_{0^+}^{p_1} u(t) &= C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} t^{\alpha-p_1-1} - I_{0^+}^{\alpha-p_1}\phi_q\bigl(y(t)\bigr), \\ D_{0^+}^{p_2} u(t) &= C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)} t^{\alpha-p_2-1} - I_{0^+}^{\alpha-p_2}\phi_q\bigl(y(t)\bigr). \end{aligned} \tag{5}$$

On the other hand, $D_{0^+}^{p_1} u(1) = \sum_{j=1}^{\infty} \eta_j D_{0^+}^{p_2} u(\xi_j)$, and combining with (5), we get

$$\begin{aligned} C_1 &= \int_0^1 \frac{(1-s)^{\alpha-p_1-1}}{\Gamma(\alpha-p_1)\Delta}\phi_q\bigl(y(s)\bigr)\,\mathrm{d}s - \sum_{j=1}^{\infty} \eta_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)\Delta}\phi_q\bigl(y(s)\bigr)\,\mathrm{d}s \\ &= \int_0^1 \frac{(1-s)^{\alpha-p_1-1}P(s)}{\Delta}\phi_q\bigl(y(s)\bigr)\,\mathrm{d}s, \end{aligned}$$

where

$$P(s)=\frac{1}{\Gamma(\alpha-p_1)}-\frac{1}{\Gamma(\alpha-p_2)}\sum_{s\leqslant\xi_j}\eta_j\left(\frac{\xi_j-s}{1-s}\right)^{\alpha-p_2-1}(1-s)^{p_1-p_2},$$

$$\Delta=\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)}-\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{j=1}^{\infty}\eta_j\xi_j^{\alpha-p_2-1}.$$

Hence,

$$\begin{aligned}u(t)&=C_1t^{\alpha-1}-I_{0+}^{\alpha}\phi_q\bigl(y(t)\bigr)\\&=-\int_0^t\frac{\Delta(t-s)^{\alpha-1}}{\Gamma(\alpha)\Delta}\phi_q\bigl(y(s)\bigr)\,\mathrm{d}s+\int_0^1\frac{(1-s)^{\alpha-p_1-1}t^{\alpha-1}P(s)}{\Delta}\phi_q\bigl(y(s)\bigr)\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$G(t,s)=\frac{1}{\Delta\Gamma(\alpha)}\begin{cases}\Gamma(\alpha)t^{\alpha-1}P(s)(1-s)^{\alpha-p_1-1}-\Delta(t-s)^{\alpha-1}, & 0\leqslant s\leqslant t\leqslant 1,\\ \Gamma(\alpha)t^{\alpha-1}P(s)(1-s)^{\alpha-p_1-1}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}$$

and

$$D_{0+}^{\mu}G(t,s)=\frac{1}{\Delta\Gamma(\alpha-\mu)}\begin{cases}t^{\alpha-1-\mu}\Gamma(\alpha)P(s)(1-s)^{\alpha-p_1-1}-\Delta(t-s)^{\alpha-1-\mu},\\ \quad 0\leqslant s\leqslant t\leqslant 1,\\ t^{\alpha-1-\mu}\Gamma(\alpha)P(s)(1-s)^{\alpha-p_1-1}, \quad 0\leqslant t\leqslant s\leqslant 1.\end{cases} \tag{6}$$

It is easy to check that $G(t,s)$ and $D_{0+}^{\mu}G(t,s)$ are uniformly continuous on $[0,1]\times[0,1]$. □

Lemma 4. *Let $\Delta>0$, then the Green function* (4) *has the following properties*:

$$\begin{aligned}&\Delta t^{\alpha-1}(1-s)^{\alpha-p_1-1}\bigl[1-(1-s)^{p_1}\bigr]\\&\quad\leqslant\Delta\Gamma(\alpha)G(t,s)\leqslant\Gamma(\alpha)t^{\alpha-1}P(s)(1-s)^{\alpha-p_1-1},\end{aligned} \tag{7}$$

$$\begin{aligned}&\Delta t^{\alpha-1-\mu}(1-s)^{\alpha-p_1-1}\bigl[1-(1-s)^{p_1}\bigr]\\&\quad\leqslant\Delta\Gamma(\alpha-\mu)D_{0+}^{\mu}G(t,s)\leqslant\Gamma(\alpha-\mu)t^{\alpha-1-\mu}P(s)(1-s)^{\alpha-p_1-1}.\end{aligned} \tag{8}$$

Proof. Let

$$G_0(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}t^{\alpha-1}(1-s)^{\alpha-p_1-1}-(t-s)^{\alpha-1}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{\alpha-1}(1-s)^{\alpha-p_1-1}, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

From [10], for $p_1\in[2,n-2]$, we have

$$\begin{aligned}0&\leqslant t^{\alpha-1}(1-s)^{\alpha-p_1-1}\bigl[1-(1-s)^{p_1}\bigr]\leqslant\Gamma(\alpha)G_0(t,s)\\&\leqslant t^{\alpha-1}(1-s)^{\alpha-p_1-1}.\end{aligned} \tag{9}$$

By direct calculation, we get $P'(s) \geqslant 0$, $s \in [0,1]$, and so, $P(s)$ is nondecreasing with respect to s. For $p_2 \leqslant p_1$, $p_1, p_2 \in [2, n-2]$, $s \in [0,1]$, we get

$$\begin{aligned}\Gamma(\alpha)P(s) &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)} \sum_{s\leqslant\xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-p_2-1} (1-s)^{p_1-p_2} \\ &\geqslant \Gamma(\alpha)P(0) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)} \sum \eta_j \xi_j^{\alpha-p_2-1} = \Delta, \end{aligned} \tag{10}$$

by (4) and (10), we have

$$\Delta\Gamma(\alpha)G(t,s) \geqslant \begin{cases} \Delta t^{\alpha-1}(1-s)^{\alpha-p_1-1} - \Delta(t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \Delta t^{\alpha-1}(1-s)^{\alpha-p_1-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{11}$$

by (9) and (11), we have

$$\begin{aligned}\Delta\Gamma(\alpha)G(t,s) &\geqslant \Delta\Gamma(\alpha)G_0(t,s) \\ &\geqslant \Delta t^{\alpha-1}(1-s)^{\alpha-p_1-1}\left[1-(1-s)^{p_1}\right]. \end{aligned} \tag{12}$$

Clearly, $\Delta\Gamma(\alpha)G(t,s) \leqslant \Gamma(\alpha)t^{\alpha-1}P(s)(1-s)^{\alpha-p_1-1}$. So, the proof of (7) is completed. Similarly, (8) also holds.

Let $E = \{u(t)\colon u(t) \in C[0,1],\ D_{0+}^{\mu}u(t) \in C[0,1]\}$ be a Banach space with the norm

$$\left\|u(t)\right\| = \max\Big\{\max_{t\in[0,1]} \left|u(t)\right|,\ \max_{t\in[0,1]} D_{0+}^{\mu}\left|u(t)\right|\Big\},$$

and E is endowed with an order relation $u \leqslant v$ if $u(t) \leqslant v(t), D_{0+}^{\mu}u(t) \leqslant D_{0+}^{\mu}v(t)$. Moreover, we define a cone of E by

$$K = \left\{u \in E\colon u(t) \geqslant 0,\ D_{0+}^{\mu}u(t) \geqslant 0,\ t \in [0,1]\right\},$$

and define an operator

$$Au(t) = \int_0^1 G(t,s)\phi_q\big(f\big(s, u(s), D_{0+}^{\mu}u(s)\big)\big)\,\mathrm{d}s,\ u \in K.$$

Problems (1) has a positive solution if and only if u is a fixed point of A in K. $\square$

Lemma 5. *The operator $A : K \to E$ is continuous.*

Proof. First, for $u \in P$, by the continuity of $G(t,s)$, $s^{\sigma}\phi_q(f(s,u(s),D_{0+}^{\mu}u(s)))$, and the integrability of $s^{-\sigma}$,

$$Au(t) = \int_0^1 G(t,s)\phi_q\big(f\big(s, u(s), D_{0+}^{\mu}u(s)\big)\big)\,\mathrm{d}s, \quad u \in K,$$

is well defined on K. Thus, it follows from the uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$ and

$$\bigl|Au(t_2)-Au(t_1)\bigr|\leqslant\int_0^1\bigl|G(t_2,s)-G(t_1,s)\bigr|s^{-\sigma}s^{\sigma}\phi_q\bigl(f\bigl(s,u(s),D_{0+}^{\mu}u(s)\bigr)\bigr)\,\mathrm{d}s$$

that $Au\in C[0,1]$, $u\in K$. Furthermore, by the uniform continuity of $D_{0+}^{\mu}G(t,s)$, for $t,s\in[0,1]$, we get

$$D_{0+}^{\mu}(Au)(t)=\int_0^1 D_{0+}^{\mu}G(t,s)\phi_q\bigl(f\bigl(s,u(s),D_{0+}^{\mu}u(s)\bigr)\bigr)\,\mathrm{d}s\in C[0,1].$$

Let $u_n,u\in K$, $u_n\to u$ in E. Since $G(t,s), D_{0+}^{\mu}G(t,s)$ is uniformly continuous, there exists $M>0$ such that

$$\max\bigl\{G(t,s),D_{0+}^{\mu}G(t,s)\bigr\}\leqslant M,\quad t,s\in[0,1].$$

On the other hand, since $u_n\to u$ in $C^1[0,1]$, there exists $A>0$ such that $\|u_n\|\leqslant A$ $(n=1,2,\dots)$, and then $\|u\|\leqslant A$. Furthermore, $s^{\sigma}\phi_q(f(s,x_0,x_1))$ is continuous on $[0,1]\times\mathbb{R}^+\times\mathbb{R}^+$, so, $s^{\sigma}\phi_q(f(s,x_0,x_1))$ is uniformly continuous on $[0,1]\times[0,A]\times[0,A]$. Hence, for any $\varepsilon>0$, there exists $\delta>0$ such that for any $s_1,s_2\in[0,1]$, $x_0^1,x_0^2,x_1^1,x_1^2\in[0,A]$, $|s_1-s_2|<\delta$, $|x_0^1-x_0^2|<\delta$, $|x_1^1-x_1^2|<\delta$, we have

$$\bigl|s_1^{\sigma}\phi_q\bigl(f\bigl(s_1,x_0^1,x_1^1\bigr)\bigr)-s_2^{\sigma}\phi_q\bigl(f\bigl(s_2,x_0^2,x_1^2\bigr)\bigr)\bigr|<\varepsilon. \tag{13}$$

By $\|u_n-u\|\to 0$, for the above $\delta>0$, there exists N such that for all $n>N$, we get

$$\bigl|u_n(t)-u(t)\bigr|,\ \bigl|D_{0+}^{\mu}u_n(s)-D_{0+}^{\mu}u(s)\bigr|\leqslant\|u_n-u\|<\delta\quad\text{for any } t\in[0,1].$$

Hence, for any $t\in[0,1]$, $n>N$, by (13), we derive

$$\bigl|t^{\sigma}\phi_q\bigl(f\bigl(t,u_n(t),D_{0+}^{\mu}u_n(t)\bigr)-t^{\sigma}\phi_q\bigl(f\bigl(t,u(t),D_{0+}^{\mu}u(t)\bigr)\bigr)\bigr|<\varepsilon. \tag{14}$$

Thus, for $n>N$, $t\in[0,1]$, by (14), we have

$$\begin{aligned}
&\bigl|(Au_n)(t)-(Au)(t)\bigr|\\
&\quad=\left|\int_0^1 G(t,s)\phi_q\bigl(f\bigl(s,u_n(s),D_{0+}^{\mu}u_n(s)\bigr)\bigr)\,\mathrm{d}s\right.\\
&\qquad\left.-\int_0^1 G(t,s)\phi_q\bigl(f\bigl(s,u(s),D_{0+}^{\mu}u(s)\bigr)\bigr)\,\mathrm{d}s\right|\\
&\quad=\left|\int_0^1 G(t,s)s^{-\sigma}\bigl(s^{\sigma}\phi_q\bigl(f\bigl(s,u_n(s),D_{0+}^{\mu}u_n(s)\bigr)\bigr)\right.\\
&\qquad\left.-s^{\sigma}\phi_q\bigl(f\bigl(s,u(s),D_{0+}^{\mu}u(s)\bigr)\bigr)\bigr)\,\mathrm{d}s\right|
\end{aligned}$$

$$\begin{aligned}&\leqslant M\int_0^1 s^{-\sigma}\big(s^\sigma\phi_q\big(f\big(s,u_n(s),D_{0+}^\mu u_n(s)\big)\big)\\&\qquad - s^\sigma\phi_q\big(f\big(s,u(s),D_{0+}^\mu u(s)\big)\big)\big)\,\mathrm{d}s\\&\leqslant M\varepsilon\int_0^1 s^{-\sigma}\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\big|D_{0+}^\mu(Au_n)(t)-D_{0+}^\mu(Au)(t)\big|\\&\quad=\bigg|\int_0^1 D_{0+}^\mu G(t,s)\phi_q\big(f\big(s,u_n(s),D_{0+}^\mu u_n(s)\big)\big)\,\mathrm{d}s\\&\qquad-\int_0^1 D_{0+}^\mu G(t,s)\phi_q\big(f\big(s,u(s),D_{0+}^\mu u(s)\big)\big)\,\mathrm{d}s\bigg|\\&\quad=\bigg|\int_0^1 D_{0+}^\mu G(t,s)s^{-\sigma}\big(s^\sigma\phi_q\big(f\big(s,u_n(s),D_{0+}^\mu u_n(s)\big)\big)\\&\qquad- s^\sigma\phi_q\big(f\big(s,u(s),D_{0+}^\mu u(s)\big)\big)\big)\,\mathrm{d}s\bigg|\\&\quad\leqslant M\int_0^1 s^{-\sigma}\big(s^\sigma\phi_q\big(f\big(s,u_n(s),D_{0+}^\mu u_n(s)\big)\big)\\&\qquad- s^\sigma\phi_q\big(f\big(s,u(s),D_{0+}^\mu u(s)\big)\big)\big)\,\mathrm{d}s\\&\quad\leqslant M\varepsilon\int_0^1 s^{-\sigma}\,\mathrm{d}s,\end{aligned}$$

and hence, we get $\|Au_n - Au\|_0 \to 0$, $\|D_{0+}^\mu(Au_n) - D_{0+}^\mu(Au)\|_0 \to 0$ $(n\to\infty)$. That is, $\|Au_n - Au\| \to 0 (n \to \infty)$, namely, A is continuous in the space E. □

Lemma 6. *$A : K \to K$ is completely continuous.*

Proof. From Lemma 4 we have $(Au)(t) \geqslant 0, D_{0+}^\mu(Au)(t) \geqslant 0$, $t \in [0,1]$, hence $A(K) \subset K$. Now we will prove that AV is relatively compact for bounded $V \subset K$. Since V is bounded, there exists $D > 0$ such that for any $u \in V$, $\|u\| \leqslant D$, and by the continuity of $t^\sigma\phi_q(f(t, x_0, x_1))$ on $[0,1]\times[0,D]\times[0,D]$, there exists $C > 0$ such that

$|s^\sigma \phi_q(f(s,u(s),D_{0+}^\mu u(s)))| \leqslant C$ for $s \in [0,1]$, $u \in V$. Hence, for $t \in [0,1],\ u \in V$, we have

$$\begin{aligned}
\bigl|Au(t)\bigr| &= \int_0^1 G(t,s)\phi_q\bigl(f\bigl(s,u(s),D_{0+}^\mu u(s)\bigr)\bigr)\,\mathrm{d}s \\
&= \int_0^1 G(t,s)s^{-\sigma}s^\sigma\phi_q\bigl(f\bigl(s,u(s),D_{0+}^\mu u(s)\bigr)\bigr)\,\mathrm{d}s \\
&\leqslant C\int_0^1 \frac{1}{\varDelta}P(s)(1-s)^{\alpha-p_1-1}s^{-\sigma}\,\mathrm{d}s \\
&= \frac{CB_1}{\Gamma(\alpha-p_1)\varDelta},
\end{aligned}$$

where $B_1 = \int_0^1 (1-s)^{\alpha-p_1-1}s^{-\sigma}\,\mathrm{d}s$. Similarly, we derive

$$\bigl|D_{0+}^\mu (Au)(t)\bigr| \leqslant \frac{CB_1}{\Gamma(\alpha-p_1)\varDelta}, \quad t \in [0,1],\ u \in V,$$

which shows that AV is bounded in E. Next, we will verify that $D_{0+}^\mu(AV)$ is equicontinuous. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $u \in V$, we get

$$\begin{aligned}
&\bigl|D_{0+}^\mu (Au)(t_2) - D_{0+}^\mu (Au)(t_1)\bigr| \\
&\quad = \Biggl|t_2^{\alpha-1-\mu}\int_0^1 \frac{P(s)(1-s)^{\alpha-p_1-1}}{\varDelta}\phi_q\bigl(f\bigl(s,u(s),D_{0+}^\mu u(s)\bigr)\bigr)\,\mathrm{d}s \\
&\qquad - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1-\mu}}{\Gamma(\alpha)}\phi_q\bigl(f\bigl(s,u(s),D_{0+}^\mu u(s)\bigr)\bigr)\,\mathrm{d}s \\
&\qquad - t_1^{\alpha-1-\mu}\int_0^1 \frac{P(s)(1-s)^{\alpha-p_1-1}}{\varDelta}\phi_q\bigl(f\bigl(s,u(s),D_{0+}^\mu u(s)\bigr)\bigr)\,\mathrm{d}s \\
&\qquad + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1-\mu}}{\Gamma(\alpha)}\phi_q\bigl(f\bigl(s,u(s),D_{0+}^\mu u(s)\bigr)\bigr)\,\mathrm{d}s\Biggr| \\
&\quad \leqslant \bigl|\bigl(t_2^{\alpha-1-\mu} - t_1^{\alpha-1-\mu}\bigr)\bigr|\int_0^1 \frac{P(s)(1-s)^{\alpha-p_1-1}}{\varDelta}\phi_q\bigl(f\bigl(s,u(s),D_{0+}^\mu u(s)\bigr)\bigr)\,\mathrm{d}s \\
&\qquad + \Biggl|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1-\mu}s^{-\sigma}s^\sigma\phi_q\bigl(f\bigl(s,u(s),D_{0+}^\mu u(s)\bigr)\bigr)\,\mathrm{d}s
\end{aligned}$$

$$
\left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1-\mu} s^{-\sigma} s^{\sigma} \phi_q\big(f\big(s, u(s), D_{0+}^{\mu} u(s)\big)\big)\, \mathrm{d}s \right|
$$

$$
\leqslant \frac{C}{\Gamma(\alpha)} \left[\int_0^{t_2} (t_2 - s)^{\alpha-1-\mu} s^{-\sigma}\, \mathrm{d}s - \int_0^{t_1} (t_1 - s)^{\alpha-1-\mu} s^{-\sigma}\, \mathrm{d}s \right].
$$

Furthermore,

$$
\int_0^{t} (t - s)^{\alpha-1-\mu} s^{-\sigma}\, \mathrm{d}s = t^{\alpha-\mu-\sigma} \int_0^1 (1 - s)^{\alpha-1-\mu} s^{-\sigma}\, \mathrm{d}s.
$$

Thus, we obtain

$$
\begin{aligned}
&\left| D_{0+}^{\mu}(Au)(t_2) - D_{0+}^{\mu}(Au)(t_1) \right| \\
&\quad \leqslant \frac{C}{\Gamma(\alpha - p_1)\Delta} \big(t_2^{\alpha-\mu-1} - t_1^{\alpha-\mu-1}\big) + \frac{CB_2}{\Gamma(\alpha)} \big(t_2^{\alpha-\mu-\sigma} - t_1^{\alpha-\mu-\sigma}\big) \quad \forall u \in V,
\end{aligned}
$$

where $B_2 = \int_0^1 (1-s)^{\alpha-\mu-1} s^{-\sigma}\, \mathrm{d}s$. From above, the uniform continuity of $t^{\alpha-\mu-\sigma}$, $t^{\alpha-\mu-1}$, and together with Lemma 2, we can derive that AV is relatively compact in E, and so, we get that $A: K \to K$ is completely continuous. $\square$

3 Main results

For convenience, we denote

$$
\varpi = \left(\int_0^1 \frac{1}{\Delta} P(s)(1-s)^{\alpha-p_1-1} s^{-\sigma}\, \mathrm{d}s \right)^{-1}. \tag{15}
$$

Theorem 1. *Assume that* (H0) *holds, and*

(H2) $t^{\sigma}\phi_q(f(t, x_0, x_1))$ *is continuous and nondecreasing on* x_0, x_1;
(H3) *For any* $t \times x_0 \times x_1 \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$, *there exists* $d > 0$ *such that* $t^{\sigma}\phi_q(f(t, x_0, x_1)) \leqslant \varpi d$ *holds. Then the boundary value problem* (1) *has the maximal and minimal positive solutions* u^* *and* v^* *on* $[0,1]$, *respectively, such that* $0 < \|u^*\| \leqslant d$, $0 < \|v^*\| \leqslant d$. *Moreover, for initial values* $u_0(t) = dt^{\alpha-1}$, $v_0(t) = 0$, $t \in [0,1]$, *define the iterative sequences* $\{u_n\}$ *and* $\{v_n\}$ *by*

$$
u_n = Au_{n-1} = A^n u_0, \qquad v_n = Av_{n-1} = A^n v_0,
$$

then

$$
\lim_{n\to\infty} u_n = \lim_{n\to\infty} A^n u_0 = u^*, \qquad \lim_{n\to\infty} v_n = \lim_{n\to\infty} A^n v_0 = v^*.
$$

Proof. By Lemma 6, we know that $A: K \to K$ is completely continuous. Now we show that A is nondecreasing. For any $u_1, u_2, D^{\mu}_{0+}u_1, D^{\mu}_{0+}u_2 \in K$ and $u_1 \leqslant u_2$, $D^{\mu}_{0+}u_1 < D^{\mu}_{0+}u_2$, according to the definition of A and (H2), we know that $Au_1 \leqslant Au_2$. Let $\overline{K}_d = \{x \in K\colon \|x\| \leqslant d\}$. Next, we prove that $A: \overline{K}_d \to \overline{K}_d$. If $u \in \overline{P}_d$, then $\|u\| \leqslant d$, i.e., $\|u\|_0 \leqslant d$, $\|D^{\mu}_{0+}u\|_0 \leqslant d$, by Lemma 4 and (H1), (H2), we have

$$\begin{aligned}
(Au)(t) &= \int\limits_0^1 G(t,s)\phi_q\bigl(f\bigl(s,u(s),D^{\mu}_{0+}u(s)\bigr)\bigr)\,\mathrm{d}s \\
&\leqslant \int\limits_0^1 \frac{1}{\Delta}P(s)(1-s)^{\alpha-p_1-1}s^{-\sigma}s^{\sigma}\phi_q\bigl(f(s,d,d)\bigr)\,\mathrm{d}s \\
&\leqslant \varpi d\int\limits_0^1 \frac{1}{\Delta}P(s)(1-s)^{\alpha-p_1-1}s^{-\sigma}\,\mathrm{d}s = d, \quad t \in [0,1],
\end{aligned} \tag{16}$$

$$\begin{aligned}
D^{\mu}_{0+}(Au)(t) &= \int\limits_0^1 D^{\mu}_{0+}G(t,s)\phi_q\bigl(f\bigl(s,u(s),D^{\mu}_{0+}u(s)\bigr)\bigr)\,\mathrm{d}s \\
&\leqslant \int\limits_0^1 \frac{1}{\Delta}P(s)(1-s)^{\alpha-p_1-1}s^{-\sigma}s^{\sigma}\phi_q\bigl(f(s,d,d)\bigr)\,\mathrm{d}s \\
&\leqslant \varpi d\int\limits_0^1 \frac{1}{\Delta}P(s)(1-s)^{\alpha-p_1-1}s^{-\sigma}\,\mathrm{d}s = d, \quad t \in [0,1],
\end{aligned} \tag{17}$$

then (16), (17) show that $\|Au\| = \max\{\max_{t\in[0,1]}|Au(t)|,\ \max_{t\in[0,1]} D^{\mu}_{0+}|Au(t)|\} \leqslant d$, hence $A(K_d) \subseteq K_d$.

Let $u_0(t) = dt^{\alpha-1}$, $t \in [0,1]$, then $u_0(t) \in \overline{K}_d$. Let $u_1 = Au_0$, $u_2 = A^2u_0$, then we have $u_1, u_2 \in \overline{K}_d$. We denote $u_{n+1} = Au_n = A^n u_0$ ($n = 0,1,2,\dots$). In view of the fact that $A: K_d \to K_d$, it follows that $u_n \in A(K_d) \subseteq K_d$ ($n = 1,2,\dots$). Since A is completely continuous, we assert that the sequence $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} u_{n_k} = u^* \in K_d$.

Since $u_1 = Au_0 \in K_d$, by Lemma 3 and (H3), we get

$$\begin{aligned}
Au_0(t) &= \int\limits_0^1 G(t,s)s^{-\sigma}s^{\sigma}f\bigl(s,u_0(s),D^{\mu}_{0+}u_0(s)\bigr)\,\mathrm{d}s \\
&\leqslant \varpi dt^{\alpha-1}\int\limits_0^1 \frac{1}{\Delta}P(s)(1-s)^{\alpha-p_1-1}s^{-\sigma}s^{\sigma}\,\mathrm{d}s \\
&= dt^{\alpha-1} = u_0(t), \quad t \in [0,1],
\end{aligned} \tag{18}$$

which implies $u_1 \leqslant u_0$. Hence, by (H1),

$$\begin{aligned} u_2(t) = Au_1(t) &= \int_0^1 G(t,s)s^{-\sigma}s^{\sigma}f\big(s,u_1(s),D_{0+}^{\mu}u_1(s)\big)\,\mathrm{d}s \\ &\leqslant \int_0^1 G(t,s)s^{-\sigma}s^{\sigma}f\big(s,u_0(s),D_{0+}^{\mu}u_0(s)\big)\,\mathrm{d}s \\ &= Au_0(t) = u_1(t), \quad t \in [0,1]. \end{aligned}$$

By the induction, we have $u_{n+1} \leqslant u_n$ $(n = 0,1,2,\dots)$. Therefore, $\lim_{n\to\infty} u_n = u^*$. Using the continuity of A and taking the limit $n \to \infty$ in $u_{n+1} = Au_n$ yields $Au^* = u^*$.

Let $v_0(t) = 0$, $t \in [0,1]$, apparently $v_0(t) \in \overline{K}_d$. Let $v_1 = Av_0$, $v_2 = A^2v_0$, then we have $v_1 \in \overline{K}_d$, $v_2 \in \overline{K}_d$. Let $v_n = Av_{n-1} = A^n v_0$ $(n = 0,1,2,\dots)$, and since $A : \overline{K}_d \to \overline{K}_d$, we have $v_n \in A(\overline{K}_d) \subseteq \overline{K}_d$ $(n = 1,2,3,\dots)$. It follows from the complete continuity of A that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set. Since $v_1 = Av_0 \in \overline{K}_d$, we get

$$v_1(t) = Av_0(t) = (A0)(t) \geqslant 0, \quad 0 \leqslant t < 1.$$

Hence, we obtain

$$v_2(t) = Av_1(t) \geqslant (A0)(t) = v_1(t), \quad 0 \leqslant t < 1.$$

By induction, we have $v_{n+1} \geqslant v_n$ $(n = 0,1,2,\dots)$, $0 \leqslant t < 1$. Hence, there exists $v^* \in \overline{K}_d$ such that $v_n \to v^*$ as $n \to \infty$. Applying the continuity of A and $v_{n+1} = Av_n$, we have that $Av^* = v^*$.

If $f(t,0) \not\equiv 0$, $0 \leqslant t \leqslant 1$, then the zero function is not the solution of BVP (1). Hence, v^* is a positive solution of BVP (1).

Since each fixed point of A in K is a solution of BVP (1), by above proof, we get that u^* and v^* are positive solutions of the BVP (1) on $[0,1]$. □

Remark 1. The iterative sequences in Theorem 1 begins with a simple function, which is useful for computational purpose.

Remark 2. u^* and v^* are the maximal and minimal solutions of the BVP (1), respectively, but u^* and v^* may be coincident, and when u^* and v^* are coincident, the boundary value problem (1) will have a unique solution in $\overline{K}_d$.

4 An example

Consider the following infinite-point p-Laplacian fractional differential equations:

$$\begin{aligned} &\phi_p\big(D_{0+}^{11/2}u(t)\big) + f\big(t,u(t),D_{0+}^{1/2}u(t)\big) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = u''(0) = u'''(0) = u^{(4)}(0) = 0, \\ &D_{0+}^{7/2}u(1) = \sum_{j=1}^{\infty} \frac{1}{2j^2} D_{0+}^{5/2}u\left(\frac{1}{j^4}\right), \end{aligned} \tag{19}$$

where $\alpha = 11/2$, $\mu = 1/2$, $p_1 = 7/2$, $p_2 = 5/2$, $p = 3$, $q = 3/2$, $\eta_j = 1/2j^2$, $\xi_j = 1/j^4$, $\sigma = 1/2$,

$$f(t,x,y) = \begin{cases} \frac{2400}{\pi t}(x^2+y^2)^2, & (t,x,y) \in (0,1] \times [0,1] \times [0,1], \\ \frac{2400}{\pi t}, & (t,x,y) \in (0,1] \times [1,\infty) \times [1,\infty). \end{cases}$$

Clearly,

$$\sqrt{t}\phi_q\big(f(t,x,y)\big) = \big|f(t,x,y)\big|^{-1/2} f(t,x,y) = \sqrt{t}\big(f(t,x,y)\big)^{1/2},$$

and $\sqrt{t}\phi_q(f(t,x,y)) = \sqrt{t}[2400/(\pi t)(x^2+y^2)^2]^{1/2} = \sqrt{2400/\pi}(x^2+y^2)$ is continuous on $[0,1] \times R^+ \times R^+$.

By simple calculation, we have

$$\begin{aligned} \varDelta &= \frac{\Gamma(\alpha)}{\Gamma(\alpha - p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha - p_2)} \sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha - p_2 - 1} \\ &= \frac{\Gamma(\frac{11}{2})}{\Gamma(\frac{11}{2} - \frac{7}{2})} - \frac{\Gamma(\frac{11}{2})}{\Gamma(\frac{11}{2} - \frac{5}{2})} \sum_{j=1}^{\infty} \frac{1}{2j^2} \left(\frac{1}{j^4}\right)^2 \\ &\approx 38.18, \end{aligned} \tag{20}$$

$$\begin{aligned} P(s) &= \frac{1}{\Gamma(\alpha - p_1)} - \frac{1}{\Gamma(\alpha - p_2)} \sum_{s \leqslant \xi_j} \eta_j \left(\frac{\xi_j - s}{1 - s}\right)^{\alpha - p_2 - 1} (1-s)^{p_1 - p_2} \\ &= \frac{1}{\Gamma(\frac{11}{2} - \frac{7}{2})} - \frac{1}{\Gamma(\frac{11}{2} - \frac{5}{2})} \sum_{s \leqslant 1/j^4} \frac{1}{2j^2} \left(\frac{\frac{1}{j^4} - s}{1 - s}\right)^{11/2 - 5/2 - 1} (1-s)^{7/2 - 5/2} \\ &= 1 - \frac{1}{2} \sum_{s \leqslant \xi_j} \left(\frac{\frac{1}{j^4} - s}{1 - s}\right)(1-s), \end{aligned} \tag{21}$$

by (20) and (21), we have

$$\begin{aligned} \varpi &= \left(\int_0^1 \frac{1}{\varDelta} P(s)(1-s)^{\alpha - p_1 - 1} s^{-\sigma} \, \mathrm{d}s\right)^{-1} \\ &= \varDelta\left(\int_0^1 \left(1 - \frac{1}{2} \sum_{s \leqslant 1/j^4} \left(\frac{\frac{1}{j^4} - s}{1 - s}\right)(1-s)\right)(1-s)s^{-1/2} \, \mathrm{d}s\right)^{-1} \\ &= \varDelta\left(\int_0^1 (1-s)s^{-1/2} \, \mathrm{d}s - \frac{1}{2}\int_0^1 \sum_{s \leqslant 1/j^4} \left(\frac{\frac{1}{j^4} - s}{1 - s}\right)^2 (1-s)^2 s^{-1/2} \, \mathrm{d}s\right)^{-1} \\ &\geqslant \varDelta\left(B\left(\frac{1}{2}, 2\right)\right)^{-1} = \frac{3}{4}\varDelta \approx 28.64. \end{aligned}$$

Taking d=28, then $t^{\sigma}\phi_q(f(t,x_0,x_1)) \approx 2400/\pi \approx 764 \leqslant 28.64 \times 28 = \varpi d$, so, all condition of Theorem 1 hold, then boundary value problem (19) has the maximal and minimal positive solutions u^* and v^* on $[0,1]$.

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References

1. R.I. Avery, J. Henderson, Three positive fixed points of nonlinear operators on ordered Banach spaces, *Comput. Math. Appl.*, **42**(3):313–322, 2001.
2. Z. Bai, W. Sun, Existence and multiplicity of positive solutions for singular fractional boundary value problems, *Comput. Math. Anal.*, **63**:1369–1381, 2012.
3. S. Bhalekar, V. Daftardargejji, Antisynchronization of nonidentical fractional-order chaotic systems using active control, *Int. J. Differ. Equ.*, **2011**(8):1495–1508, 2011.
4. A. Cabada, Z. Hamdi, Nonlinear fractional differential equations with integral boundary value conditions, *Appl. Math. Comput.*, **228**(2012):251–257, 2014.
5. G. Chai, Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator, *Bound. Value Probl.*, **2012**:18, 2012.
6. D. Guo, Y. Cho, J. Zhu, *Partial Ordering Methods in Nonlinear Problems*, Nova Science, New York, 2004.
7. L. Guo, L. Liu, Y. Wu, Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions, *Nonlinear Anal. Model. Control*, **21**(5):635–650, 2016.
8. L. Guo, L. Liu, Y. Wu, Existence of positive solutions for singular higher-order fractional differential equations with infinite-point boundary conditions, *Bound. Value Probl.*, **2016**(1): 1–22, 2016.
9. P. Hentenryck, R. Bent, E. Upfal, *An introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
10. M. Jleli, B. Samet, Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method, *Nonlinear Anal. Model. Control*, **20**(3):367–376, 2015.
11. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
12. V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal., Theory Methods Appl.*, **69**(8):2677–2682, 2008.
13. R.W. Leggett, L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28**:673–505, 1979.
14. X. Li, S. Liu, W. Jiang, Positive solutions for boundary value problem of nonlinear fractional functional differential equations, *Appl. Math. Comput.*, **217**(22):9278–9285, 2011.
15. S. Liang, J. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equation, *Nonlinear Anal., Theory Methods Appl.*, **71**(11):5545–5550, 2009.

16. L. Liu, F. Sun, X.and Wu Y. Zhang, Bifurcation analysis for a singular differential system with two parameters via to degree theory, *Nonlinear Anal., Model. Control*, **22**(1):31–50, 2017.
17. Y. Liu, New existence results on nonhomogeneous Sturm–Liouville type BVPs for higher-order p-Laplacian differential equations, *Appl. Math.*, **38**(3):295–314, 2011.
18. H. Lu, Z. Han, C. Zhang, Y. Zhao, Positive solutions for boundary value problem of nonlinear fractional differential equation with p-Laplacian operator, in I. Dimov, I. Faragó, L. Vulkov (Eds.), *Finite Difference Methods, Theory and Applications. 6th International Conference, FDM 2014, Lozenetz, Bulgaria, June 18–23, 2014*, Springer, Cham, 2014, pp. 274–281.
19. D. Ma, Positive solutions of multi-point boundary value problem of fractional differential equation, *Arab. J. Math. Sci.*, **21**(2):225–236, 2015.
20. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
21. X. Su, S. Zhang, Unbounded solutions to a boundary value problem of fractional order on the half-line, *Comput. Math. Appl.*, **61**(4):1079–1087, 2011.
22. Y. Wang, L. Liu, Y. Wu, Positive solutions for a nonlocal fractional differential equation, *Nonlinear Anal., Theory Methods Appl.*, **74**(11):3599–3605, 2011.
23. G. Zhang, J. Sun, Positive solutions of m-point boundary value problems, *J. Math. Anal. Appl.*, **291**(2):406–418, 2004.
24. X. Zhang, Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions, *Appl. Math. Lett.*, **39**:22–27, 2015.
25. X. Zhang, M. Feng, W. Ge, Symmetric positive solutions for p-Laplacian fourth-order differential equations with integral boundary conditions, *J. Comput. Appl. Math.*, **222**:561–573, 2008.
26. X. Zhang, Q. Shao, Z.and Zhong, Positive solutions for semipositone $(k, n-k)$ conjugate boundary value problems with singularities on space variables, *Appl. Math. Lett.*, **217**(16):50–57, 2017.
27. X. Zhang, L. Wang, Q. Sun, Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, *Appl. Math. Comput.*, **226**:708–718, 2014.
28. X. Zhang, Q. Zhong, Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions, *Fract. Calc. Appl.Anal.*, **20**(6):1471–1484, 2017.
29. X. Zhang, Q. Zhong, Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables, *Appl. Math. Lett.*, **80**:12–19, 2018.
30. Y. Zhao, S. Sun, Z. Han, M. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, *Appl. Math. Comput.*, **217**(16):6950–6958, 2011.