Nonlinear Analysis: Modelling and Control, 2018, Vol. 23, No. 6, 830–850 ISSN 1392-5113
https://doi.org/10.15388/NA.2018.6.2

Eigenvalue problems for fractional differential equations with mixed derivatives and generalized p-Laplacian*

Yupin Wang[a], **Shutang Liu**[b,1], **Zhenlai Han**[c]

[a]Institute of Marine Science and Technology, Shandong University,
Jinan 250100, Shandong, China
wangyupin8252@163.com

[b]School of Control Science and Engineering, Shandong University,
Jinan 250061, Shandong, China
stliu@sdu.edu.cn

[c]School of Mathematical Sciences, University of Jinan,
Jinan 250022, Shandong, China
hanzhenlai@163.com

Received: December 1, 2017 / **Revised:** June 6, 2018 / **Published online:** October 31, 2018

Abstract. This paper reports the investigation of eigenvalue problems for two classes of nonlinear fractional differential equations with generalized p-Laplacian operator involving both Riemann–Liouville fractional derivatives and Caputo fractional derivatives. By means of fixed point theorem on cones, some sufficient conditions are derived for the existence, multiplicity and nonexistence of positive solutions to the boundary value problems. Finally, an example is presented to further verify the correctness of the main theoretical results and illustrate the wide range of their potential applications.

Keywords: fractional differential equation, two-point boundary value condition, positive solution, existence and nonexistence, Guo–Krasnosel'skii fixed point theorem.

1 Introduction

Eigenvalue problem is a class of homogeneous boundary value problems for differential equations with parameters. In the last few years, fractional differential equations have gained attentions due to their numerous applications in various aspects of science and technology. Eigenvalue problems and their applications are gradually beginning to be studied for fractional differential equations (see, e.g., [7–9,11,12,16,22]). In [22], Zhao et al. considered eigenvalue intervals of the following boundary value problem for nonlinear

*This research is supported by the Natural Science Foundation of China (No. 61533011).
[1]Corresponding author.

fractional differential equations:

$$\begin{aligned} &{}^C D^{\alpha}_{0+} u(t) = \lambda f\bigl(u(t)\bigr), \quad 0 < t < 1, \\ &u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \end{aligned}$$

where $1 < \alpha \leqslant 2$, ${}^C D^{\alpha}_{0+}$ is the Caputo fractional derivative. The authors studied the existence, multiplicity and nonexistence of positive solutions. In [8], Eloe et al. used the theory of u_0-positive operators to study boundary value problem for a class of linear fractional differential equations

$$D^{\alpha}_{0+} u(t) = \lambda h(t) u(t), \quad 0 < t < 1,$$

where $1 < \alpha \leqslant 2$. The existence and a comparison theorem of smallest positive eigenvalues were obtained. In [7], criteria were established to characterize the conjugate point to a kind of conjugate boundary value problems for linear fractional differential equations. In [12], eigenvalue comparison results of boundary value problems for linear fractional differential equations with the Caputo fractional derivative were obtained. In [9], Eloe et al. established the existence of smallest eigenvalues for the linear fractional differential equations with right focal boundary conditions.

In addition, p-Laplacian operator, which is a nonlinear generalization of the Laplace operator, is widely used in analyzing mathematical models about physical phenomena and many other related fields. For instance, when studying the steady-state turbulent flow with reaction, Bobisud (cf. [2]) introduced the differential equation

$$\bigl(\varphi_p\bigl(u'(t)\bigr)\bigr)' = f\bigl(t, u(t), u'(t)\bigr)$$

with an operator $\varphi_p(x) = |x|^{p-1}x$. This problem appears in the study of non-Newtonian fluids. It yields the usual problem for diffusion in a porous medium when $p = 1$, i.e., $\varphi_p(x) = x$. In 2003, Wang introduced a general assumption for this kind of operators:

(H1) $\phi : \mathbb{R} \to \mathbb{R}$ is an odd and increasing homeomorphism, and there exist two increasing homeomorphisms $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi_1(x)\phi(y) \leqslant \phi(xy) \leqslant \psi_2(x)\phi(y)$ for all $x, y > 0$.

For ϕ satisfying (H1), we call it a generalized p-Laplacian operator (cf. [19, 20]), which is satisfied by two important cases $\phi(x) = x$ and $\phi(x) = |x|^{p-2}x$, $p \geqslant 1$.

The research of boundary value problems for fractional differential equations with (generalized) p-Laplacian operator has begun in recent years. Lots of important results can be seen in [3, 4, 11, 13, 16–18] and references therein. In [18], Liu et al. studied a class of integral boundary value problems for nonlinear fractional differential equations. In [11], Han et al. investigated the existence of positive solutions to the following eigenvalue problem for nonlinear fractional differential equation with generalized p-Laplacian operator:

$$\begin{aligned} &D^{\beta}_{0+}\bigl(\phi\bigl(D^{\alpha}_{0+} u(t)\bigr)\bigr) = \lambda f\bigl(u(t)\bigr), \quad 0 < t < 1, \\ &u(0) = u'(0) = u'(1) = 0, \\ &\phi\bigl(D^{\alpha}_{0+} u(0)\bigr) = \bigl(\phi\bigl(D^{\alpha}_{0+} u(1)\bigr)\bigr)' = 0, \end{aligned}$$

where $2 < \alpha \leqslant 3$, $1 < \beta \leqslant 2$, D_{0+}^{α} and D_{0+}^{β} are the standard Riemann–Liouville fractional derivatives. By using a fixed-point theorem on cones, several existence and nonexistence results of positive solutions in terms of different eigenvalue intervals are obtained. In [16], Li et al. studied the eigenvalue problems for a class of nonlinear fractional q-difference equations with generalized p-Laplacian operator. In [17], Liu et al. considered four-point boundary value problem for autonomous fractional differential equation with mixed fractional derivatives

$$\begin{aligned}
&D_{0+}^{\alpha}\big(\varphi_p\big({}^{C}D_{0+}^{\beta}u(t)\big)\big) = f\big(t, u(t), {}^{C}D_{0+}^{\beta}u(t)\big), \quad 0 < t < 1,\\
&{}^{C}D_{0+}^{\beta}u(0) = u'(0) = 0,\\
&u(1) = r_1 u(\eta), \qquad {}^{C}D_{0+}^{\beta}u'(1) = {}^{C}D_{0+}^{\beta}r_2 u(\xi),
\end{aligned}$$

where $1 < \alpha, \beta \leqslant 2$ and $r_1, r_2 \geqslant 0$.

Since the fractional calculus, eigenvalue problem and p-Laplacian operator arise from many applied fields, it is worth studying the eigenvalue problems for fractional differential equations with p-Laplacian operator. However, to the best of our knowledge, there are relatively few results on this kind of problems with generalized p-Laplacian operator, and no paper is concerned with this kind of problems with both generalized p-Laplacian operator and mixed fractional derivatives. In this context, our purpose is to consider the following two kinds of eigenvalue problems:

$$\begin{aligned}
&{}^{C}D_{0+}^{\beta}\big(\phi\big(D_{0+}^{\alpha}u(t)\big)\big) = \lambda a(t) f\big(t, u(t)\big), \quad 0 < t < 1,\\
&u(0) = u'(0) = 0, \qquad u'(1) = \eta,\\
&\phi\big(D_{0+}^{\alpha}u(0)\big) = \big(\phi\big(D_{0+}^{\alpha}u(0)\big)\big)' = 0
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
&{}^{C}D_{0+}^{\beta}\big(\phi\big(D_{0+}^{\alpha}v(t)\big)\big) = \lambda a(t) f\big(t, v(t) + \varphi(t)\big), \quad 0 < t < 1,\\
&v(0) = v'(0) = v'(1) = 0,\\
&\phi\big(D_{0+}^{\alpha}v(0)\big) = \big(\phi\big(D_{0+}^{\alpha}v(0)\big)\big)' = 0,
\end{aligned} \tag{2}$$

here $\varphi(t) = \eta/(\alpha - 1) \cdot t^{\alpha-1}$ is the unique solution to

$$\begin{aligned}
&D_{0+}^{\alpha}\varphi(t) = 0, \quad 0 < t < 1,\\
&\varphi(0) = \varphi'(0) = 0, \qquad \varphi(1) = \eta,
\end{aligned}$$

where $2 < \alpha \leqslant 3$, $1 < \beta \leqslant 2$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, ${}^{C}D_{0+}^{\beta}$ is the standard Caputo fractional derivative and $\lambda, \eta > 0$. We always assume that:

- (H2) $a : [0, 1] \to [0, +\infty)$ is a continuing real-valued function and differentiable on $(0, 1)$;
- (H3) $f : [0, 1] \times \mathbb{R} \to [0, +\infty)$ is continuous and satisfied with the following condition: there exists $\ell > 0$ such that for all $t_1, t_2 \in [0, 1]$ and $u_1, u_2 \in \mathbb{R}$,

$$\big|f(t_2, u_2) - f(t_1, u_1)\big| \leqslant \ell \max\big\{|t_2 - t_1|, |u_2 - u_1|\big\}.$$

Motivated by [11, 16, 17, 22], we utilize Guo–Krasnosel'skii fixed point theorem to obtain the existence, multiplicity and nonexistence of solutions to problems (1) and (2).

Compared with the aforementioned references, problems considered here are novel. Specifically, the objects in this paper not only are eigenvalue problems with generalized p-Laplacian operator, but also involve two different types of derivatives, Riemann–Liouville type and Caputo type. This work fills the gap in the literature by discussing the existence, multiplicity and nonexistence of solutions to this kind of problems.

The remainder of the paper is organized as follows. In Section 2, some basic definitions and lemmas are collected and derived. In Section 3, conditions that boundary value problems transform into integral equations are discussed carefully; and then, the existence, multiplicity and nonexistence of solutions to problems (1) and (2) are investigated. In Section 4, an example is given to illustrate our main results. In Section 5, summary and prospect for our present work are made.

2 Preliminaries

For the convenience of the reader, we present here some necessary knowledge, which will be used in the following sections. More detailed information can be found in [5, 14].

Definition 1. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, \mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) \, \mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$, where $n = \lceil \alpha \rceil$ and $\lceil \alpha \rceil$ denotes the ceiling function of the number α.

Lemma 1. (*See* [1, *Lemma* 2.1].) *Let* $\alpha > 0$*. If we assume* $y \in C(0,1) \cap L(0,1)$*, then the fractional differential equation*

$$D_{0+}^{\alpha} y(t) = 0$$

has $y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$*,* $c_i \in \mathbb{R}$*,* $i = 1, 2, \ldots, n$*, as a general solution.*

Lemma 2. (*See* [1, *Lemma* 2.2].) *Assume that $y \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} y(t) = y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$.

Definition 3. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$^{C}D_{0+}^{\alpha} y(t) = I_{0+}^{n-\alpha} y^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\,\mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$, where $n = \lceil \alpha \rceil$.

For $n \in \mathbb{N}_+ := \{1, 2, \dots\}$, $AC^n[a,b]$ denotes the space of real-valued functions $f : [a,b] \to \mathbb{R}$, which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)} \in AC[a,b]$:

$$AC^n[a,b] = \bigl\{ f\colon\ [a,b] \to \mathbb{R} \text{ and } f^{(n-1)} \in AC[a,b] \bigr\},$$

where $AC[a,b]$ is the space of functions f, which are absolutely continuous on $[a,b]$.

Lemma 3. (*See* [22, *Lemma* 2.1].) *Let $\alpha > 0$. If we assume $y \in AC^n[0,1]$, then the fractional differential equation*

$$^{C}D_{0+}^{\alpha} y(t) = 0$$

has $y(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 2, \dots, n-1$, as a general solution.

Lemma 4. (*See* [22, *Lemma* 2.2].) *Let $\alpha > 0$. Assume that $y \in AC^n[0,1]$. Then*

$$I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha} y(t) = y(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 2, \dots, n-1$.

Lemma 5. *Given $h \in AC[0,1]$. Then the boundary value problem*

$$\begin{aligned} &^{C}D_{0+}^{\beta} \bigl(\phi\bigl(D_{0+}^{\alpha} u(t)\bigr)\bigr) = h(t), \quad 0 < t < 1, \\ &u(0) = u'(0) = 0, \qquad u'(1) = 0, \\ &\phi\bigl(D_{0+}^{\alpha} u(0)\bigr) = \bigl(\phi\bigl(D_{0+}^{\alpha} u(0)\bigr)\bigr)' = 0, \end{aligned} \tag{3}$$

where $2 < \alpha \leqslant 3$, $1 < \beta \leqslant 2$, has a unique solution, which is expressed by

$$u(t) = \int_0^1 G(t,s) \phi^{-1}\bigl(I_{0+}^{\beta} h(s)\bigr)\,\mathrm{d}s, \tag{4}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}\tag{5}$$

is called the Green's function of problems (3)*; operator* ϕ^{-1} *denotes the inverse of generalized* p*-Laplacian operator* ϕ*.*

Proof. Let $w=\phi(D_{0+}^{\alpha}u)\in AC^2[0,1]$. Then from problem (3) one gets

$$\begin{aligned}&{}^{C}D_{0+}^{\beta}w(t)=h(t),\quad 0<t<1,\\ &w(0)=w'(0)=0.\end{aligned}\tag{6}$$

For the sake of clarity, the proof can subsequently be separated into two steps.

Step 1. Boundary value problem (6) is equivalent to $w(t)=I_{0+}^{\beta}h(t)$. From Lemma 4 and $1<\beta\leqslant 2$ we have

$$w(t)=I_{0+}^{\beta}h(t)+c_0+c_1t.$$

Noticing $w(0)=w'(0)=0$, we get $c_0=c_1=0$. It follows that

$$w(t)=I_{0+}^{\beta}h(t).\tag{7}$$

Conversely, due to $h\in AC[0,1]\subset L^{\infty}(0,1)$, we know that $\int_0^1(1-s)^{\beta-2}h(s)\,\mathrm{d}s<+\infty$. So,

$$w'(t)=I_{0+}^{\beta-1}h(t)=\frac{1}{\Gamma(\beta-1)}\int_0^t(t-s)^{\beta-2}h(s)\,\mathrm{d}s$$

is well defined on $[0,1]$. According to [15, Lemma 2.3(i)] and in view of $h\in AC[0,1]$, we get $w'=I_{0+}^{\beta-1}h\in AC[0,1]$, which means that $w\in AC^2[0,1]$. Hence, w satisfies boundary value problem (6).

Step 2. Boundary value problem (3) is equivalent to equation (4). Taking action from both sides of equation (7) by ϕ^{-1}, we have $D_{0+}^{\alpha}u(t)=\phi^{-1}(w(t))=\phi^{-1}(I_{0+}^{\beta}h(t))$. Now we consider boundary value problem

$$\begin{aligned}&D_{0+}^{\alpha}u(t)=\phi^{-1}\bigl(I_{0+}^{\beta}h(t)\bigr),\\ &u(0)=u'(0)=u'(1)=0.\end{aligned}$$

From Lemma 2 and $2<\alpha\leqslant 3$ we have

$$u(t)=I_{0+}^{\alpha}\bigl(\phi^{-1}\bigl(I_{0+}^{\beta}h(t)\bigr)\bigr)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}.$$

Noticing $u(0) = u'(0) = u'(1) = 0$, we get $c_1 = -\int_0^1 (1-s)^{\alpha-2}\phi^{-1}(I_{0+}^{\beta}h(s))\,\mathrm{d}s/\Gamma(\alpha)$ and $c_2 = c_3 = 0$. Therefore,

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi^{-1}\big(I_{0+}^{\beta}h(s)\big)\,\mathrm{d}s - \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}\phi^{-1}\big(I_{0+}^{\beta}h(s)\big)\,\mathrm{d}s$$
$$= \int_0^1 G(t,s)\phi^{-1}\big(I_{0+}^{\beta}h(s)\big)\,\mathrm{d}s.$$

Conversely, equation (4) obviously satisfies boundary value problem (3). The proof is completed. □

Lemma 6. (*See* [6, *Lemma* 2.8].) *Function G defined as* (5) *satisfies the following properties*:

(i) $G(t,s) \geqslant 0$, $G(1,s) \geqslant G(t,s)$ *for* $0 \leqslant t, s \leqslant 1$.
(ii) $G(t,s) \geqslant k(t)G(1,s)$ *for* $0 \leqslant t, s \leqslant 1$, *where* $k(t) = t^{\alpha-1}$.

Lemma 7. *Boundary value problem* (2) *is equivalent to boundary value problem* (1), *i.e., if $v(t)$ is a solution to* (2), *then $u(t) = v(t) + \varphi(t)$ is a solution to* (1) *and vice versa.*

Proof. Since $D_{0+}^{\alpha}u(t) = D_{0+}^{\alpha}(v(t) + \varphi(t)) = D_{0+}^{\alpha}v(t) + D_{0+}^{\alpha}\varphi(t) = D_{0+}^{\alpha}v(t)$, then $\phi(D_{0+}^{\alpha}u(t)) = \phi(D_{0+}^{\alpha}v(t))$.

Let $v(t)$ be a solution to problem (2). Then we have

$$ {}^{C}D_{0+}^{\beta}\big(\phi\big(D_{0+}^{\alpha}u(t)\big)\big) = {}^{C}D_{0+}^{\beta}\big(\phi\big(D_{0+}^{\alpha}v(t)\big)\big) = \lambda a(t) f\big(t, v(t) + \varphi(t)\big) = \lambda a(t) f\big(t, u(t)\big).$$

Consider the boundary condition. Since $u(t) = v(t) + \varphi(t)$, we have

$$u(0) = v(0) + \varphi(0) = 0, \qquad u'(0) = v'(0) + \varphi'(0) = 0,$$
$$u'(1) = v'(1) + \varphi'(0) = \eta,$$
$$\phi\big(D_{0+}^{\alpha}u(0)\big) = \phi\big(D_{0+}^{\alpha}v(0)\big) = 0 \quad \text{and} \quad \big(\phi\big(D_{0+}^{\alpha}u(0)\big)\big)' = \big(\phi\big(D_{0+}^{\alpha}v(0)\big)\big)' = 0.$$

On the contrary, let $u(t)$ be a solution to (1). Then we have

$$ {}^{C}D_{0+}^{\beta}\big(\phi\big(D_{0+}^{\alpha}v(t)\big)\big) = {}^{C}D_{0+}^{\beta}\big(\phi\big(D_{0+}^{\alpha}u(t)\big)\big) = \lambda a(t) f\big(t, u(t)\big) = \lambda a(t) f\big(t, v(t) + \varphi(t)\big).$$

Consider the boundary condition. Since $v(t) = u(t) - \varphi(t)$, we have

$$v(0) = u(0) - \varphi(0) = 0, \qquad v'(0) = u'(0) - \varphi'(0) = 0,$$
$$v'(1) = u'(1) - \varphi'(0) = \eta,$$
$$\phi\big(D_{0+}^{\alpha}v(0)\big) = \phi\big(D_{0+}^{\alpha}u(0)\big) = 0 \quad \text{and} \quad \big(\phi\big(D_{0+}^{\alpha}v(0)\big)\big)' = \big(\phi\big(D_{0+}^{\alpha}u(0)\big)\big)' = 0.$$

The proof is completed. □

Lemma 8. (*See* [19, *Lemma* 2.6].) *Assume that* (H1) *holds. Then, for all* $x, y > 0$, $\psi_2^{-1}(x)y \leqslant \phi^{-1}(x\phi(y)) \leqslant \psi_1^{-1}(x)y$.

Lemma 9 [Guo–Krasnosel'skii fixed point theorem]. (*See* [10, *Thm.* 2.6].) *Let* Ω_1 *and* Ω_2 *be two bounded open sets in Banach space* E *such that* $\theta \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *Let operator* $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *be completely continuous, where* θ *denotes the zero element of* E, *and* P *is a cone of* E. *Suppose that one of the two conditions*:

(i) $\|Ax\| \leqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_2$;
(ii) $\|Ax\| \geqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_2$.

Then A *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

To begin with, we establish an important lemma, which transforms boundary value problem (2) into a certain integral equation. It is necessary to discuss relations between the solutions to differential equations and the solutions to the corresponding integral equations because the continuity assumptions on nonlinearities used previously are not sufficient when it comes to Caputo fractional derivative. However, few people pay enough attention to this issue.

Lemma 10. *Suppose that conditions* (H1)–(H3) *hold. Function* v *is a solution to boundary value problem* (2) *if* v *is a continuous solution to integral equation*

$$v(t) = \int_0^1 G(t,s)\phi^{-1}\bigl(\lambda I_{0+}^{\beta} a(s) f\bigl(s, v(s) + \varphi(s)\bigr)\bigr)\,\mathrm{d}s, \quad t \in [0,1]. \tag{8}$$

Proof. Assume that v is a continuous solution to (8) on $[0,1]$. Let $y(s) = a(s)f(s, v(s) + \varphi(s))$ for $s \in [0,1]$. From conditions (H2) and (H3) we know that $y \in AC[0,1]$. In fact, from hypothesis (H2) there exists

$$\xi_1 \in J := \bigl(\min\{s_1, s_2\},\, \max\{s_1, s_2\}\bigr) \subseteq [0,1]$$

such that $a(s_2) - a(s_1) = a'(\xi_1)(s_2 - s_1)$ for any $s_1, s_2 \in [0,1]$. Similarly, there exists $\xi_3 \in J$ such that $\varphi(s_2) - \varphi(s_1) = (\eta/(\alpha-1))(s_2^{\alpha-1} - s_1^{\alpha-1}) = \eta\xi_3^{\alpha-2}(s_2 - s_1)$ for any $s_1, s_2 \in [0,1]$. From the hypothesis that v is continuous on $[0,1]$ we know v is bounded on $[0,1]$, which is follows that $f(t, v(t) + \varphi(t))$ is bounded for all $t \in [0,1]$. So, noticing $2 < \alpha \leqslant 3$, $1 < \beta \leqslant 2$ and according to Lemma 8, we get

$$\begin{aligned}
\bigl|v'(t)\bigr| &= \left| \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\phi^{-1}\bigl(I_{0+}^{\beta} y(s)\bigr)\,\mathrm{d}s - \int_0^1 \frac{t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\phi^{-1}\bigl(I_{0+}^{\beta} y(s)\bigr)\,\mathrm{d}s \right| \\
&\leqslant \frac{2}{\Gamma(\alpha-1)} \int_0^1 \phi^{-1}\bigl(I_{0+}^{\beta} a(s) f\bigl(s, v(s) + \varphi(s)\bigr)\bigr)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{2}{\Gamma(\alpha-1)} \int\limits_0^1 \phi^{-1}\left(\frac{M_a}{\Gamma(\beta)} \int\limits_0^s f\bigl(\tau, v(\tau)+\varphi(\tau)\bigr)\,\mathrm{d}\tau\right) \mathrm{d}s \\
&\leqslant \frac{2}{\Gamma(\alpha-1)} \int\limits_0^1 \phi^{-1}\left(\frac{M_a M_f}{\Gamma(\beta)}\right) \mathrm{d}s \leqslant +\infty
\end{aligned}
$$

for any $t \in [0,1]$, where $M_a := \max_{t\in[0,1]} a(t)$ and $M_f := \max_{t\in[0,1]} f(t, v(t)+\varphi(t))$. Hence, there exists $\xi_2 \in J$ such that $v(s_2) - v(s_1) = v'(\xi_2)(s_2 - s_1)$ for any $s_1, s_2 \in [0,1]$. These, together with hypothesis (H3), imply that for $s_1, s_2 \in [0,1]$,

$$
\begin{aligned}
&\bigl|y(s_2) - y(s_1)\bigr| \\
&\quad \leqslant \bigl|a(s_2) - a(s_1)\bigr| f\bigl(s_2, v(s_2)+\varphi(s_2)\bigr) \\
&\qquad + a(s_1)\bigl|f\bigl(s_2, v(s_2)+\varphi(s_2)\bigr) - f\bigl(s_1, v(s_1)+\varphi(s_1)\bigr)\bigr| \\
&\quad \leqslant \bigl|a(s_2) - a(s_1)\bigr| f\bigl(s_2, v(s_2)+\varphi(s_2)\bigr) \\
&\qquad + \ell a(s_1) \max\bigl\{|s_2 - s_1|,\ \bigl|v(s_2) - v(s_1)\bigr| + \bigl|\varphi(s_2) - \varphi(s_1)\bigr|\bigr\} \\
&\quad = \bigl[f\bigl(s_2, v(s_2)+\varphi(s_2)\bigr)\bigl|a'(\xi_1)\bigr| \\
&\qquad + \ell a(s_1) \max\bigl\{1,\ \bigl|v'(\xi_2)\bigr| + \eta\xi_3^{\alpha-2}\bigr\}\bigr] \cdot |s_2 - s_1| \\
&\quad \leqslant \bigl[M_f\bigl|a'(\xi_1)\bigr| + \ell M_a \max\bigl\{1,\ |v'(\xi_2)| + \eta\xi_3^{\alpha-2}\bigr\}\bigr] \cdot |s_2 - s_1|.
\end{aligned}
$$

It follows that $y \in AC[0,1]$.

Set $w = \phi(D_{0+}^{\alpha} u) \in AC^2[0,1]$. Then from problem (2) one gets

$$
\begin{aligned}
&{}^{C}D_{0+}^{\beta} w(t) = \lambda a(t) f\bigl(t, v(t)+\varphi(t)\bigr), \quad 0<t<1, \\
&w(0) = w'(0) = 0.
\end{aligned} \tag{9}
$$

From condition (H3) an argument similar to Step 1 in the proof of Lemma 5 shows that

$$
w(t) = \lambda I_{0+}^{\beta} y(t) = \frac{\lambda}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} y(s)\,\mathrm{d}s
$$

and

$$
w'(t) = \lambda I_{0+}^{\beta-1} y(t) = \frac{\lambda}{\Gamma(\beta-1)} \int\limits_0^t (t-s)^{\beta-2} y(s)\,\mathrm{d}s
$$

are well defined on $[0,1]$. According to [15, Lemma 2.3(i)] and in view of $y \in AC[0,1]$ again, we get $w' = I_{0+}^{\beta-1} y \in AC[0,1]$, which means that $w \in AC^2[0,1]$. Hence, w satisfies boundary value problem (9). The remainder of the argument is analogous to Step 2 in the proof of Lemma 5 and so is omitted. The proof is completed. $\square$

Throughout the rest of this part, Banach space is taken as $E = C[0,1]$ equipped with the maximum norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. Set $P = \{x \in E\colon x(t) \geqslant k(t)\|x\| \text{ for all } t \in [0,1]\}$. Then P is a cone in E. From Lemma 10 define an operator $T_\lambda : P \to E$ as

$$(T_\lambda v)(t) = \int_0^1 G(t,s)\phi^{-1}\big(\lambda I_{0+}^{\beta} a(s) f\big(s, v(s) + \varphi(s)\big)\big)\,\mathrm{d}s, \quad t \in [0,1].$$

Lemma 11. *Suppose that conditions* (H1)–(H3) *hold. Then the operator $T_\lambda : P \to P$ is completely continuous.*

Proof. For the sake of clarity, the proof can subsequently be separated into three steps.

Step 1. $T_\lambda(P) \subseteq P$. First, according to the definition of the operator T_λ, the continuity and nonnegativity of functions a, f and G, it is easy to know $T_\lambda v \in C[0,1]$ for all $v \in P$. Second, in view of Lemma 6(i), we have

$$\begin{aligned}(T_\lambda v)(t) &= \int_0^1 G(t,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} a(\tau) f\big(\tau, v(\tau) + \varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant \int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} a(\tau) f\big(\tau, v(\tau) + \varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s,\end{aligned}$$

which means that

$$\|T_\lambda v\| = \int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} a(\tau) f\big(\tau, v(\tau) + \varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Then from Lemma 6(ii), for $t \in [0,1]$, we get

$$\begin{aligned}(T_\lambda v)(t) &\geqslant \int_0^1 t^{\alpha-1} G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} a(\tau) f\big(\tau, v(\tau) + \varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\geqslant k(t)\|T_\lambda v\|.\end{aligned}$$

Therefore, $T_\lambda(P) \subseteq P$.

Let $\Omega \in P$ be bounded, i.e., there exists a constant $M > 0$ such that $\|x\| \leqslant M$ for all $x \in \Omega$.

Step 2. $T_\lambda(\Omega)$ is bounded. From continuity of f, denote

$$L =: \max_{t\in[0,1],\, v\in[0,M]} \big|f(t, v+\varphi)\big| + 1.$$

Then from Lemma 8, for $t \in [0,1]$ and $v \in \Omega$, we have

$$\begin{aligned}
(T_\lambda v)(t) &\leqslant \int_0^1 G(t,s)\phi^{-1}\left(\frac{\lambda L}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \psi_1^{-1}\left(\frac{\lambda L}{\Gamma(\beta)}\right)\int_0^1 G(t,s)\phi^{-1}\left(\int_0^s (s-\tau)^{\beta-1}a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \psi_1^{-1}\left(\frac{\lambda L}{\Gamma(\beta)}\right)\int_0^1 G(1,s)\phi^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \frac{1}{(\alpha-1)\Gamma(\alpha+1)}\psi_1^{-1}\left(\frac{\lambda L}{\Gamma(\beta)}\right)\phi^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau\right),
\end{aligned}$$

which means that

$$\|T_\lambda v\| \leqslant \frac{1}{(\alpha-1)\Gamma(\alpha+1)}\psi_1^{-1}\left(\frac{\lambda L}{\Gamma(\beta)}\right)\phi^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau\right) < +\infty.$$

So, $T_\lambda(\Omega)$ is bounded.

Step 3. $T_\lambda(\Omega)(t)$ is equicontinuous on $[0,1]$. Since $G(t,s)$ is continuous on $[0,1] \times [0,1]$, it is uniformly continuous on $[0,1] \times [0,1]$. Thus, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $t_1, t_2 \in [0,1]$ with $|t_2 - t_1| < \delta$ imply

$$\left|G(t_2,s) - G(t_1,s)\right| \leqslant \frac{\varepsilon}{\psi_1^{-1}(\frac{\lambda L}{\Gamma(\beta)})\phi^{-1}(\int_0^1 a(\tau)\,\mathrm{d}\tau)}$$

for all $s \in [0,1]$. Hence, for any $v \in \Omega$, one gets

$$\begin{aligned}
&\left|T_\lambda(v)(t_2) - T_\lambda(v)(t_1)\right| \\
&\quad\leqslant \int_0^1 \left|G(t_2,s) - G(t_1,s)\right|\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)f\big(\tau, v(\tau)+\varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad\leqslant \psi_1^{-1}\left(\frac{\lambda L}{\Gamma(\beta)}\right)\int_0^1 \left|G(t_2,s) - G(t_1,s)\right|\phi^{-1}\left(\int_0^s a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad\leqslant \psi_1^{-1}\left(\frac{\lambda L}{\Gamma(\beta)}\right)\phi^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau\right)\int_0^1 \left|G(t_2,s) - G(t_1,s)\right|\mathrm{d}s < \varepsilon.
\end{aligned}$$

Therefore, $T_\lambda(\Omega)(t)$ is equicontinuous on $[0,1]$.

By the Arzelá–Ascoli theorem, we conclude that T_λ is a compact operator. Additionally, it is easy to see that the operator T_λ is continuous in view of continuity of G, ϕ and f. Therefore, we obtain that $T_\lambda : P \to P$ is completely continuous. The proof is completed. □

Now, we will establish some conditions for the existence, multiplicity and nonexistence of solutions to boundary value problems (1) and (2), and we obtain some new results. For notational convenience, denote

$$F_0 = \overline{\lim_{v\to 0^+}} \sup_{t\in[0,1]} \frac{f(t, v+\varphi(t))}{\phi(v)}, \qquad F_\infty = \overline{\lim_{v\to +\infty}} \sup_{t\in[0,1]} \frac{f(t, v+\varphi(t))}{\phi(v)},$$

$$f_0 = \underline{\lim_{v\to 0^+}} \inf_{t\in[0,1]} \frac{f(t, v+\varphi(t))}{\phi(v)}, \qquad f_\infty = \underline{\lim_{v\to +\infty}} \inf_{t\in[0,1]} \frac{f(t, v+\varphi(t))}{\phi(v)},$$

$$A_1 = \frac{1}{(\alpha-1)\Gamma(\alpha+1)} \psi_1^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau\right),$$

$$A_2 = k(\delta)\int_0^1 G(1,s)\psi_2^{-1}\left(\int_0^s (s-\tau)^{\beta-1} a(\tau)\psi_1\bigl(k(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$A_3 = k(\delta)\int_0^1 G(1,s)\psi_2^{-1}\left(\int_0^s (s-\tau)^{\beta-1} a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

where δ is a constant which are defined later in this section.

3.1 Existence

Theorem 1. *Suppose that conditions* (H1)–(H3) *hold. If there exists* $\delta \in (0,1)$ *such that* $F_0\psi_2(A_2^{-1}) < f_\infty\psi_1(A_1^{-1})$ *holds, then for each*

$$\lambda \in \bigl(\Gamma(\beta)\psi_2\bigl(A_2^{-1}\bigr)f_\infty^{-1},\, \Gamma(\beta)\psi_1\bigl(A_1^{-1}\bigr)F_0^{-1}\bigr), \tag{10}$$

boundary value problems (1) *and* (2) *have at least one positive solution, respectively. Here we impose* $f_\infty^{-1} = 0$ *if* $f_\infty = +\infty$ *and* $F_0^{-1} = +\infty$ *if* $F_0 = 0$.

Proof. From (10) there exists $\varepsilon > 0$ such that

$$\Gamma(\beta)\psi_2\bigl(A_2^{-1}\bigr)(f_\infty - \varepsilon)^{-1} \leqslant \lambda \leqslant \Gamma(\beta)\psi_1\bigl(A_1^{-1}\bigr)(F_0+\varepsilon)^{-1}. \tag{11}$$

In order to utilize Lemma 9, the proof can subsequently be separated into two steps.

Step 1. By the definition of F_0, there exists $r_1 > 0$ such that

$$f\bigl(t,\, v+\varphi(t)\bigr) \leqslant \phi(v)(F_0+\varepsilon) \tag{12}$$

for $t \in [0,1]$ and $v \in (0, r_1]$. So, if $v \in P$ with $\|v\| = r_1$, then from (11) and (12) one gets

$$\begin{aligned}
(T_\lambda v)(t) &\leqslant \int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)f\big(\tau, v(\tau)+\varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)\phi\big(v(\tau)\big)(F_0+\varepsilon)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)\phi(r_1)(F_0+\varepsilon)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda(F_0+\varepsilon)}{\Gamma(\beta)}\int_0^1 \phi(r_1)a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \psi_1^{-1}\left(\frac{\lambda(F_0+\varepsilon)}{\Gamma(\beta)}\right)\int_0^1 G(1,s)\phi^{-1}\left(\phi(r_1)\int_0^1 a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \psi_1^{-1}\left(\frac{\lambda(F_0+\varepsilon)}{\Gamma(\beta)}\right)\int_0^1 G(1,s)\psi_1^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\cdot r_1 \\
&= A_1\cdot\psi_1^{-1}\left(\frac{\lambda(F_0+\varepsilon)}{\Gamma(\beta)}\right)r_1 \leqslant r_1 = \|v\|.
\end{aligned}$$

Hence, if we choose $\Omega_1 = \{x \in E\colon \|x\| < r_1\}$, then

$$\|T_\lambda v\| \leqslant \|v\| \quad \text{for } v \in P \cap \partial\Omega_1. \tag{13}$$

Step 2. By the definition of f_∞, there exists $r_3 > 0$ such that

$$f\big(t, v+\varphi(t)\big) \geqslant \phi(v)(f_\infty - \varepsilon) \tag{14}$$

for $t \in [0,1]$ and $v \in [r_3, +\infty)$. So, if $v \in P$ with $\|v\| = r_2 := \max\{2r_1, r_3\}$, then from (11) and (14) one has

$$\begin{aligned}
\big\|(T_\lambda v)\big\| &\geqslant (T_\lambda v)(\delta) \\
&= \int_0^1 G(\delta,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)f\big(\tau, v(\tau)+\varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geqslant \int_0^1 k(\delta)G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)\phi\big(v(\tau)\big)(f_\infty-\varepsilon)\,\mathrm{d}\tau\right)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\geqslant k(\delta)\int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)\phi\big(k(\tau)\|v\|\big)(f_\infty-\varepsilon)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\geqslant k(\delta)\int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)\psi_1\big(k(\tau)\big)\phi\big(\|v\|\big)(f_\infty-\varepsilon)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\geqslant k(\delta)\psi_2^{-1}\left(\frac{\lambda(f_\infty-\varepsilon)}{\Gamma(\beta)}\right)\int_0^1 G(1,s)\phi^{-1}\left(\int_0^s (s-\tau)^{\beta-1}a(\tau)\psi_1\big(k(\tau)\big)\,\mathrm{d}\tau\cdot\phi(r_2)\right)\mathrm{d}s\\
&\geqslant k(\delta)\psi_2^{-1}\left(\frac{\lambda(f_\infty-\varepsilon)}{\Gamma(\beta)}\right)\int_0^1 G(1,s)\psi_2^{-1}\left(\int_0^s (s-\tau)^{\beta-1}a(\tau)\psi_1\big(k(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\cdot r_2\\
&= A_2\cdot\psi_2^{-1}\left(\frac{\lambda(f_\infty-\varepsilon)}{\Gamma(\beta)}\right)r_2\geqslant r_2=\|v\|.
\end{aligned}
$$

Hence, if we choose $\Omega_2=\{x\in E\colon\ \|x\|<r_2\}$, then

$$\|T_\lambda v\|\geqslant\|v\|\quad\text{for } v\in P\cap\partial\Omega_2. \tag{15}$$

Now, from Lemma 9(i) and inequalities (13) and (15) we conclude that operator T_λ has a fixed point $v^*\in P\cap(\overline{\Omega}_2\setminus\Omega_1)$ with $r_1\leqslant\|v^*\|\leqslant r_2$. It is clear that v^* is a solution to boundary value problem (2). By Lemma 7, $u^*(t)=v^*(t)+\varphi(t)$ is a solution to boundary value problem (1). The proof is completed. □

Theorem 2. *Suppose that conditions* (H1)–(H3) *hold. If there exists* $\delta\in(0,1)$ *such that* $F_\infty\psi_2(A_2^{-1})<f_0\psi_1(A_1^{-1})$ *holds, then for each*

$$\lambda\in\big(\Gamma(\beta)\psi_2\big(A_2^{-1}\big)f_0^{-1},\ \Gamma(\beta)\psi_1\big(A_1^{-1}\big)F_\infty^{-1}\big), \tag{16}$$

boundary value problems (1) *and* (2) *have at least one positive solution, respectively. Here we impose* $f_0^{-1}=0$ *if* $f_0=+\infty$ *and* $F_\infty^{-1}=+\infty$ *if* $F_\infty=0$.

Proof. From (16) there exists $\varepsilon>0$ such that

$$\Gamma(\beta)\psi_2\big(A_2^{-1}\big)(f_0-\varepsilon)^{-1}\leqslant\lambda\leqslant\Gamma(\beta)\psi_1\big(A_1^{-1}\big)(F_\infty+\varepsilon)^{-1}. \tag{17}$$

In order to utilize Lemma 9, the proof can subsequently be separated into two steps.

Step 1. By the definition of f_0, there exists $r_1>0$ such that

$$f\big(t,v+\varphi(t)\big)\geqslant\phi(v)(f_0-\varepsilon)$$

for $t\in[0,1]$ and $v\in(0,r_1]$. So, if $v\in P$ with $\|v\|=r_1$, we choose $\Omega_1=\{x\in E\colon\ \|x\|<r_1\}$, then an argument similar to Step 2 in the proof of Theorem 1 shows that

$$\|T_\lambda v\|\geqslant\|v\|\quad\text{for } v\in P\cap\partial\Omega_1. \tag{18}$$

Step 2. By the definition of F_∞, there exists $r_5 > 0$ such that

$$f\bigl(t, v + \varphi(t)\bigr) \leqslant \phi(v)(F_\infty + \varepsilon) \tag{19}$$

for $t \in [0,1]$ and $v \in [r_5, +\infty)$. Next, it is considered in two cases:

Case 1. Function f is bounded. Then there exists some $N > 0$ such that $f(t, v + \varphi(t)) \leqslant N$ for $t \in [0,1]$ and $v \in (0, +\infty)$. So, choosing $v \in P$ with $\|v\| = r_3 := \max\{2r_1, r_5, A_1\phi^{-1}(\lambda N/\Gamma(\beta))\}$, then from (17) and (19) one has

$$\begin{aligned}
(T_\lambda v)(t) &\leqslant \int_0^1 G(1,s)\phi^{-1}\Biggl(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)f\bigl(\tau, v(\tau)+\varphi(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\leqslant \int_0^1 G(1,s)\phi^{-1}\Biggl(\frac{\lambda N}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\leqslant \phi^{-1}\biggl(\frac{\lambda N}{\Gamma(\beta)}\biggr)\int_0^1 G(1,s)\psi_1^{-1}\Biggl(\int_0^s (s-\tau)^{\beta-1}a(\tau)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\leqslant \phi^{-1}\biggl(\frac{\lambda N}{\Gamma(\beta)}\biggr)\int_0^1 G(1,s)\psi_1^{-1}\Biggl(\int_0^1 a(\tau)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&= A_1 \cdot \phi^{-1}\biggl(\frac{\lambda N}{\Gamma(\beta)}\biggr) \leqslant r_3 = \|v\|.
\end{aligned}$$

Hence, if we choose $\Omega_3 = \{x \in E\colon \|x\| < r_3\}$, then $\|T_\lambda v\| \leqslant \|v\|$ for $v \in P \cap \partial\Omega_3$.

Case 2. Function f is unbounded. Then there exists some $r_4 > \max\{2r_1, r_5\}$ such that $f(t, v+\varphi(t)) \leqslant f(t, r_4 + \varphi(t))$ for $t \in [0,1]$ and $v \in (0, r_4]$. So, if $v \in P$ with $\|v\| = r_4$, we choose $\Omega_4 = \{x \in E\colon \|x\| < r_4\}$, then an argument similar to Step 1 in the proof of Theorem 1 shows that $\|T_\lambda v\| \leqslant \|v\|$ for $v \in P \cap \partial\Omega_4$.

Let $r_2 := \max\{r_3, r_4\}$. On account of Cases 1 and 2, setting $\Omega_2 = \{x \in E\colon \|x\| < r_2\}$, we have

$$\|T_\lambda v\| \leqslant \|v\| \quad \text{for } v \in P \cap \partial\Omega_2. \tag{20}$$

Now, from Lemma 9(ii) and inequalities (18) and (20) we conclude that operator T_λ has a fixed point $v^* \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leqslant \|v^*\| \leqslant r_2$. It is clear that v^* is a solution to boundary value problem (2), and $u^*(t) = v^*(t) + \varphi(t)$ is a solution to boundary value problem (1). The proof is completed. □

Theorem 3. *Suppose that conditions* (H1)–(H3) *hold. If there exist $r_2 > r_1 > 0$ such that*

$$\begin{aligned}
&\lambda \min_{t\in[0,1],\, v\in[0,r_1]} f\bigl(t, v+\varphi(t)\bigr) \geqslant \Gamma(\beta)\phi\biggl(\frac{r_1}{A_3}\biggr), \\
&\lambda \max_{t\in[0,1],\, v\in[0,r_2]} f\bigl(t, v+\varphi(t)\bigr) \leqslant \Gamma(\beta)\phi\biggl(\frac{r_2}{A_1}\biggr)
\end{aligned}$$

hold, then boundary value problems (1) *and* (2) *have at least one positive solution, respectively.*

Proof. First, choose $\Omega_1 = \{x \in E\colon \|x\| < r_1\}$. Then, for $t \in [0,1]$ and $v \in P \cap \partial\Omega_1$, we get

$$\begin{aligned}\big\|(T_\lambda v)\big\| &\geqslant \int_0^1 k(\delta)G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)f\big(\tau, v(\tau)+\varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\geqslant k(\delta)\int_0^1 G(1,s)\phi^{-1}\left(\phi\left(\frac{r_1}{A_3}\right)\int_0^s (s-\tau)^{\beta-1}a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\geqslant k(\delta)\,\frac{r_1}{A_3}\int_0^1 G(1,s)\psi_2^{-1}\left(\int_0^s (s-\tau)^{\beta-1}a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= r_1 = \|v\|.\end{aligned}$$

Second, choose $\Omega_2 = \{x \in E\colon \|x\| < r_2\}$. Then, for $t \in [0,1]$ and $v \in P \cap \partial\Omega_2$, we get

$$\begin{aligned}(T_\lambda v)(t) &\leqslant \int_0^1 G(1,s)\phi^{-1}\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}a(\tau)f\big(\tau, v(\tau)+\varphi(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant \int_0^1 G(1,s)\phi^{-1}\left(\phi\left(\frac{r_2}{A_1}\right)\int_0^s (s-\tau)^{\beta-1}a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant \frac{r_2}{A_1}\int_0^1 G(1,s)\psi_1^{-1}\left(\int_0^s (s-\tau)^{\beta-1}a(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant \frac{r_2}{(\alpha-1)\Gamma(\alpha+1)A_1}\,\psi_1^{-1}\left(\int_0^1 a(\tau)\,\mathrm{d}\tau\right) \\ &= r_2 = \|v\|.\end{aligned}$$

Therefore, according to Lemma 9(ii), boundary value problems (1) and (2) have a positive solution, respectively. The proof is completed. □

Theorem 4. *Suppose that conditions* (H1)–(H3) *hold. Set*

$$\lambda^* = \sup_{r>0}\frac{\Gamma(\beta)\phi(r)}{\psi_2(A_1)\max_{t\in[0,1],v\in[0,r]} f(t, v+\varphi(t))}.$$

If $f_0 = +\infty$ *and* $f_\infty = +\infty$, *then boundary value problems* (1) *and* (2) *have at least two positive solutions for each* $\lambda \in (0, \lambda^*)$, *respectively.*

Proof. Define

$$x(r) = \frac{\Gamma(\beta)\phi(r)}{\psi_2(A_1)\max_{t\in[0,1],v\in[0,r]} f(t, v+\varphi(t))}. \tag{21}$$

On the one hand, from $f_0 = +\infty$, $f_\infty = +\infty$ and the continuity of f we know that $x : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $\lim_{r\to 0^+} x(r) = \lim_{r\to+\infty} x(r) = 0$. Hence, there exists $r_0 \in (0, +\infty)$ such that

$$x(r_0) = \sup_{r>0} x(r) = \lambda^*.$$

So, for $\lambda \in (0, \lambda^*)$, there exist two constants a_1, a_2 with $0 < a_1 < r_0 < a_2 < +\infty$ and $x(a_1) = x(a_2) = \lambda$. From (21) we have

$$x(r) \max_{t\in[0,1],\, v\in[0,r]} f\bigl(t, v+\varphi(t)\bigr) = \frac{\Gamma(\beta)\phi(r)}{\psi_2(A_1)}.$$

Therefore,

$$\lambda f\bigl(t, v+\varphi(t)\bigr) \leqslant \frac{\Gamma(\beta)\phi(a_1)}{\psi_2(A_1)} \leqslant \Gamma(\beta)\phi\left(\frac{a_1}{A_1}\right) \tag{22}$$

for $t \in [0, 1]$ and $v \in [0, a_1]$;

$$\lambda f\bigl(t, v+\varphi(t)\bigr) \leqslant \frac{\Gamma(\beta)\phi(a_2)}{\psi_2(A_1)} \leqslant \Gamma(\beta)\phi\left(\frac{a_2}{A_1}\right) \tag{23}$$

for $t \in [0, 1]$ and $v \in [0, a_2]$.

On the other hand, in view of $f_0 = +\infty$ and $f_\infty = +\infty$ again, there exist two constants b_1, b_2 with $0 < b_1 < a_1 < r_0 < a_2 < b_2 < +\infty$ such that

$$\frac{f(t, v+\varphi(t))}{\phi(v)} \geqslant \frac{\Gamma(\beta)}{\lambda\psi_1(k(\delta))\psi_1(A_3)}$$

for $t \in [0, 1]$ and $v \in (0, b_1) \cup (k(\delta)b_2, +\infty)$. Therefore,

$$\lambda \min_{\substack{t\in[0,1]\\ v\in[k(\delta)b_1,b_1]}} f\bigl(t, v+\varphi(t)\bigr) \geqslant \Gamma(\beta)\phi\left(\frac{b_1}{A_3}\right), \tag{24}$$

$$\lambda \min_{\substack{t\in[0,1]\\ v\in[k(\delta)b_2,b_2]}} f\bigl(t, v+\varphi(t)\bigr) \geqslant \Gamma(\beta)\phi\left(\frac{b_2}{A_3}\right). \tag{25}$$

From inequalities (22) and (24), inequalities (23) and (25), combining with Theorem 3, boundary value problems (1) and (2) have at least two positive solutions for each $\lambda \in (0, \lambda^*)$, respectively. The proof is completed. □

3.2 Nonexistence of positive solutions

Next, we derive some sufficient conditions for nonexistence of positive solutions to problems (1) and (2).

Theorem 5. *If $F_0 < +\infty$ and $F_\infty < +\infty$, then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, boundary value problem* (2) *has no positive solution.*

Proof. From $F_0 < +\infty$ and $F_\infty < +\infty$ we know that there exist four constants $M_1, M_2, r_1, r_2 > 0$ with $r_1 < r_2$ such that

$$f\big(t, v+\varphi(t)\big) \leqslant M_1\phi(v), \quad t \in [0,1],\ v \in [0, r_1],$$
$$f\big(t, v+\varphi(t)\big) \leqslant M_2\phi(v), \quad t \in [0,1],\ v \in [r_2, +\infty).$$

Let

$$M_0 = \max\left\{M_1,\ M_2,\ \max_{t\in[0,1],\, v\in[r_1,r_2]} \frac{f(t, v+\varphi(t))}{\phi(v)}\right\}.$$

Then $f(t, v+\varphi(t)) \leqslant M_0\phi(v)$ for $t \in [0,1]$ and $v \in [0, +\infty)$.

Suppose $\mu(t)$ is a solution to boundary value problem (2). Take $\lambda_0 = M_0^{-1}\Gamma(\beta) \times \psi_1(A_1^{-1}) > \lambda > 0$. An argument similar to Step 1 in the proof of Theorem 1 shows that

$$(T_\lambda\mu)(t) \leqslant A_1\psi_1^{-1}\left(\frac{\lambda M_0}{\Gamma(\beta)}\right)\|\mu\| < A_1\psi_1^{-1}\left(\frac{\lambda_0 M_0}{\Gamma(\beta)}\right)\|\mu\| = \|\mu\|.$$

From $T_\lambda\mu(t) = \mu(t)$ on $[0,1]$ we have $\|\mu\| = \|T_\lambda\mu\| < \|\mu\|$. This is a contradiction. The proof is completed. □

Theorem 6. *If $f_0 > 0$ and $f_\infty > 0$, then there exists $\lambda_0 > 0$ such that for all $\lambda \in (\lambda_0, +\infty)$, boundary value problem* (2) *has no positive solution.*

Proof. From $f_0 > 0$ and $f_\infty > 0$ we know that there exist four constants $m_1, m_2, r_1, r_2 > 0$ with $r_1 < r_2$ such that

$$f\big(t, v+\varphi(t)\big) \geqslant m_1\phi(v), \quad t \in [0,1],\ v \in [0, r_1],$$
$$f\big(t, v+\varphi(t)\big) \geqslant m_2\phi(v), \quad t \in [0,1],\ v \in [r_2, +\infty).$$

Let

$$m_0 = \min\left\{m_1,\ m_2,\ \min_{t\in[0,1],\, v\in[r_1,r_2]} \frac{f(t, v+\varphi(t))}{\phi(v)}\right\}.$$

Then $f(t, v+\varphi(t)) \geqslant m_0\phi(v)$ for $t \in [0,1]$ and $v \in [0, +\infty)$.

Suppose $\mu(t)$ is a solution to boundary value problem (2). Take $\lambda_0 = m_0^{-1}\Gamma(\beta) \times \psi_2(A_2^{-1}) < \lambda$. From $T_\lambda\mu(t) = \mu(t)$ on $[0,1]$ an argument similar to Step 2 in the proof of Theorem 1 shows that

$$\|\mu\| = \big\|(T_\lambda\mu)\big\| \geqslant A_2\psi_2^{-1}\left(\frac{\lambda m_0}{\Gamma(\beta)}\right)\|\mu\| > A_2\psi_2^{-1}\left(\frac{\lambda_0 m_0}{\Gamma(\beta)}\right)\|\mu\| = \|\mu\|.$$

This is a contradiction. The proof is completed. □

4 An example

In this section, we will present an example to illustrate our main results. Consider the following problem:

$$\begin{aligned}
&{}^{C}D_{0^+}^{1.5}\big(D_{0^+}^{2.5}u(t)\big) = \lambda\left|120u(t) - 117\sin\left(u(t) - \frac{2}{3}t^{1.5}\right) - 80t^{1.5}\right|, \quad 0 < t < 1,\\
&u(0) = u'(0) = 0, \qquad u'(1) = 1, \qquad \phi\big(D_{0^+}^{\alpha}u(0)\big) = \big(\phi\big(D_{0^+}^{\alpha}u(0)\big)\big)' = 0,
\end{aligned} \tag{26}$$

where $\alpha = 2.5$, $\beta = 1.5$, $\phi(x) = x$, $a(t) \equiv 1$, $f(t,u) = |120u - 117\sin(u - (2/3)t^{1.5}) - 80t^{1.5}|$ and $\eta = 1$.

Notice $\phi(x) = x$, $a(t) \equiv 1$ on $[0,1]$. For any $t_1, t_2 \in [0,1]$ and $u_1, u_2 \in \mathbb{R}$, one gets

$$\begin{aligned}
&\big|f(t_2,u_2) - f(t_1,u_1)\big|\\
&\quad\leqslant 120|u_2 - u_1| + 117\left|\sin\left(u_2 - \frac{2}{3}t_2^{1.5}\right) - \sin\left(u_1 - \frac{2}{3}t_1^{1.5}\right)\right| + 80\big|t_2^{1.5} - t_1^{1.5}\big|\\
&\quad\leqslant 120|u_2 - u_1| + 117|u_2 - u_1| + 78\big|t_2^{1.5} - t_1^{1.5}\big| + 80\big|t_2^{1.5} - t_1^{1.5}\big|\\
&\quad\leqslant 237|u_2 - u_1| + 158|t_2 - t_1| \leqslant 395\max\big\{|t_2 - t_1|,\, |u_2 - u_1|\big\}.
\end{aligned}$$

From the above analysis it is shown that conditions (H1)–(H3) hold.

Take $\psi_1(x) = \psi_2(x) = x$ and $\delta = 0.9$. By a simple calculation, we get

$$\begin{aligned}
F_0 &= \overline{\lim_{v\to 0^+}}\sup_{t\in[0,1]}\frac{|120v - 117\sin v|}{v} = \lim_{v\to 0^+}\frac{120v - 117\sin v}{v} = 3,\\
f_\infty &= \underline{\lim_{v\to +\infty}}\inf_{t\in[0,1]}\frac{|120v - 117\sin v|}{v} = \lim_{v\to +\infty}\frac{120v - 117\sin v}{v} = 120,\\
A_1 &= \frac{1}{(2.5-1)\Gamma(2.5+1)} \approx 0.2006,\\
A_2 &= 0.9^{1.5}\int_0^1 \frac{(1-s)^{0.5} - (1-s)^{1.5}}{\Gamma(2.5)}\int_0^s (s-\tau)^{0.5}\tau^{1.5}\,\mathrm{d}\tau\,\mathrm{d}s \approx 0.0093,
\end{aligned}$$

where $\varphi(t) = \eta/(\alpha-1)\cdot t^{\alpha-1} = (2/3)t^{1.5}$. Then $F_0\psi_2(A_2^{-1}) < f_\infty\psi_1(A_1^{-1})$ holds. Therefore, by Theorem 1, problem (26) has at least one positive solution for each $\lambda \in (0.7926, 1.4726)$.

On the other hand, from $f(t, v+\varphi(t))'_v = 120 - 117\cos v > 0$ and

$$f_0 = \underline{\lim_{v\to 0^+}}\inf_{t\in[0,1]}\frac{|120v - 117\sin v|}{v} = \lim_{v\to 0^+}\frac{120v - 117\sin v}{v} = 3,$$

then we have $3v < f(t, v+\varphi(t)) < 237v$ for $t \in [0,1]$, $v \in (0,+\infty)$. Therefore,

(i) by Theorem 5, problem (26) has no positive solution for all $\lambda \in (0, 0.0186)$;
(ii) by Theorem 6, problem (26) has no positive solution for all $\lambda \in (31.7051, +\infty)$.

5 Conclusion

By using Guo–Krasnosel'skii fixed point theorem, we have discussed the eigenvalue problems for a new class of nonautonomous fractional differential equations that involve both mixed fractional derivatives and generalized p-Laplacian operator. This work is a good complement to [8, 11] due to the introduction of Caputo fractional derivatives and nonlinear time-varying forcing terms. Through analysis, it is not difficult to find that continuity assumptions on nonlinearities are not sufficient when one studies the fractional differential equations involving Caputo fractional derivatives. Our present work builds up Lipschitzian-type conditions for nonlinearities to makes up for a shortcoming of the continuity hypotheses that are widely used. This provides theoretical support for applications in engineering and technology better. Moreover, singularity of function a, as we know, can be taken into consideration with $\int_0^1 a(s)\,\mathrm{d}s < +\infty$ under continuity assumption but with Riemann–Liouville fractional derivatives (see, e.g., [21]). So, a natural and interesting problem is how to introduce singularity of function a into our problems here. Predictably, there is a lot of work that can be extended in these aspects.

Acknowledgment. The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript. The first author is also indebted to Dr. H. Li for numerous helpful discussions.

References

1. Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311**(2):495–505, 2005.
2. L. Bobisud, Steady-state turbulent flow with reaction, *Rocky Mt. J. Math.*, **21**(3):993–1007, 1991.
3. T. Chen, W. Liu, An anti-periodic boundary value problem for the fractional differential equation with a p-Laplacian operator, *Appl. Math. Lett.*, **25**(11):1671–1675, 2012.
4. T. Chen, W. Liu, Z. Hu, A boundary value problem for fractional differential equation with p-Laplacian operator at resonance, *Nonlinear Anal., Theory Methods Appl.*, **75**(6):3210–3217, 2012.
5. K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, Berlin, Heidelberg, 2010.
6. M. El-Shahed, Positive solutions for boundary value problem of nonlinear fractional differential equation, *Abstr. Appl. Anal.*, **2007**:10368, 2007.
7. P. Eloe, J. Neugebauer, Conjugate points for a fractional differential equation, *Fract. Calc. Appl. Anal.*, **17**(3):855–871, 2014.
8. P. Eloe, J. Neugebauer, Existence and comparison of smallest eigenvalues for a fractional boundary-value problem, *Electron. J. Differ. Equ.*, **2014**(43):1–10, 2014.
9. P. Eloe, J. Neugebauer, Smallest eigenvalues for a right focal boundary value problem, *Fract. Calc. Appl. Anal.*, **19**(1):11–18, 2016.

10. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
11. Z. Han, H. Lu, C. Zhang, Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian, *Appl. Math. Comput.*, **257**:526–536, 2015.
12. J. Henderson, N. Kosmatov, Eigenvalue comparison for fractional boundary value problems with the Caputo derivative, *Fract. Calc. Appl. Anal.*, **17**(3):872–880, 2014.
13. W. Jiang, Solvability of fractional differential equations with p-Laplacian at resonance, *Appl. Math. Comput.*, **260**:48–56, 2015.
14. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
15. K. Lan, W. Lin, Positive solutions of systems of Caputo fractional differential equations, *Commun. Appl. Anal.*, **17**(1):61–86, 2013.
16. X. Li, Z. Han, S. Sun, L. Sun, Eigenvalue problems of fractional q-difference equations with generalized p-Laplacian, *Appl. Math. Lett.*, **57**:46–53, 2016.
17. X. Liu, M. Jia, W. Ge, The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator, *Appl. Math. Lett.*, **65**:56–62, 2017.
18. Z. Liu, L. Lu, A class of BVPs for nonlinear fractional differential equations with p-Laplacian operator, *Electron. J. Qual. Theory Differ. Equ.*, **2012**(70):1–16, 2012.
19. H. Wang, On the number of positive solutions of nonlinear systems, *J. Math. Anal. Appl.*, **281**(1):287–306, 2003.
20. H. Wang, On the structure of positive radial solutions for quasilinear equations in annular domains, *Adv. Differ. Equ.*, **8**(1):111–128, 2003.
21. Y. Wang, S. Sun, Solvability to infinite-point boundary value problems for singular fractional differential equations on the half-line, *J. Appl. Math. Comput.*, **57**(1–2):359–373, 2018.
22. Y. Zhao, S. Sun, Z. Han, M. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, *Appl. Math. Comput.*, **217**(16):6950–6958, 2011.