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Integral estimates of the solutions of fractional-like equations of perturbed motion

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Abstract. In this paper, the results of the analysis of nonlinear systems with fractional-like derivatives of the state vector are presented. Using the method of integral inequalities, some estimates of the solutions are obtained, and criteria for Heyers–Ulam–Rassias stability of a fractional-like scalar equation are established.

Keywords: fractional-like system of equations, method of integral inequalities, solution estimate, Heyers–Ulam–Rassias stability.

1 Introduction

The increased interest in the last two decades to equations with fractional derivatives [6, 8, 11, 12, 18, 19], fractional-order hybrid systems [9, 22], and fractional-order equations on time scale [2, 5] has prompted many researchers to create qualitative methods for the dynamic analysis of the equations of perturbed motion with a fractional derivative of the state vector of the system. The main reason for such an interest is the possibility of a more accurate description of processes in some models of real-world phenomena.

The basic known methods in the qualitative analysis of systems of ordinary differential equations are the method of integral inequalities [13], the method of Lyapunov functions [14], the comparison method [3, 15, 16], and their combinations. These methods with their proper adaptation are suitable for building of the qualitative theory of fractional differential equations [21].

On the other side, fractional-like derivatives are also defined (see [1, 10, 17, 20]) as natural extensions of integer-order derivatives, and the theory of equations with fractional-like derivatives has been started quite recently. In the paper [17], some main theorems of the direct Lyapunov method and the comparison principle for scalar and vector Lyapunov functions on the basis of fractional-like derivatives of Lyapunov-type functions are given.

The role of integral inequalities in the study of various problems in the qualitative theory of ordinary differential equations is well known, for example, the Grönwall–Bellman theorem [4] and its generalizations (see [16] and the references therein). For differential equations with a fractional-like derivative, similar results have not yet been obtained. This is the main aim of the presented paper.

This paper is a continuation of our investigations on the theory of fractional-like equations of perturbed motion started in [17] in which some main theorems of the direct Lyapunov method have been proved and the comparison principle has been established using scalar and vector Lyapunov functions. In the current paper, using the method of integral inequalities, we establish estimates of the solutions and derive criteria for the Heyers–Ulam–Rassias-stability of a fractional-like scalar equation.

The paper is organized as follows. In Section 2, some results on the mathematical analysis of equations with fractional-like derivatives in the state vector are presented. In Section 3 the physical interpretation of a fractional-like derivative is given, which answered to the professor's T. Burton question formulated during the discussions on the paper [17]. In Section 4, for a fractional-like nonhomogeneous system, estimates of the norms of solutions and some consequences for specific systems are derived. Section 5 is devoted to the estimates of solutions of nonlinear fractional-like systems under some perturbations. In Section 6 the established estimates are applied to prove criteria for the Heyers–Ulam–Rassias-stability of a fractional-like scalar equation. In the concluding section, some general comments and conclusions on the results obtained are presented.

2 Preliminaries

Let $\mathbb{R}_+ = [0, \infty)$ and $t_0 \in \mathbb{R}_+$. Following [1] and [10], for $0 < q \leqslant 1$, we consider a continuous function $x(t) : [t_0, \infty) \to \mathbb{R}$.

We recall that [10] the notion of a fractional derivative of a continuous function appears in the mathematical analysis after l'Hôpital's question to Liouville in 1695: how to understand the expression $\mathrm{d}^n x/\mathrm{d}t^n$ for $n = 1/2$?

Definition 1. For any $t \geqslant t_0$, the *fractional-like derivative of order* q, $0 < q \leqslant 1$, with the lower limit t_0 for a continuous function $x(t)$ is defined as

$$\mathcal{D}^q_{t_0}\bigl(x(t)\bigr) = \lim_{\theta \to 0} \frac{x(t + \theta(t - t_0)^{1-q}) - x(t)}{\theta}.$$

In the case $t_0 = 0$, we have

$$\mathcal{D}^q_0\bigl(x(t)\bigr) = \mathcal{D}^q\bigl(x(t)\bigr) = \lim_{\theta \to 0} \frac{x(t + \theta t^{1-q}) - x(t)}{\theta}.$$

If $\mathcal{D}^q(x(t))$ exists on $(0, b)$, then $\mathcal{D}^q(x(0)) = \lim_{t \to 0^+} \mathcal{D}^q(x(t))$.

If the fractional-like derivative of order q for a function $x(t)$ exists on (t_0, ∞), $t_0 \in \mathbb{R}_+$, then we will say that the function $x(t)$ is q-differentiable on (t_0, ∞), i.e. $x \in C^q((t_0, \infty), \mathbb{R})$.

Remark 1. In contrast to numerous definitions of a fractional derivative of a continuous (or an absolutely differentiable) function, in Definition 1 a limit is used instead of an integral, which has significantly changed its properties.

The following properties can be proved directly by Definition 1.

Lemma 1. *Let $q \in (0, 1]$ and $x(t), y(t)$ be q-differentiable at a point $t > 0$. Then*:

(i) $\mathcal{D}^q_{t_0}(ax(t) + by(t)) = a\mathcal{D}^q_{t_0}(x(t)) + b\mathcal{D}^q_{t_0}(y(t))$ *for all* $a, b \in \mathbb{R}$;
(ii) $\mathcal{D}^q_{t_0}(t^p) = pt^{p-1}(t - t_0)^{1-q}$ *for any* $p \in \mathbb{R}$;
(iii) $\mathcal{D}^q_{t_0}(x(t)y(t)) = x(t)\mathcal{D}^q_{t_0}(y(t)) + y(t)\mathcal{D}^q_{t_0}(x(t))$;
(iv) $\mathcal{D}^q_{t_0}(x(t)/y(t)) = (y(t)\mathcal{D}^q_{t_0}(x(t)) - x(t)\mathcal{D}^q_{t_0}(y(t))/y^2(t)$;
(v) $\mathcal{D}^q_{t_0}(x(t)) = 0$ *for any* $x(t) = \lambda$, *where* λ *is an arbitrary constant.*

Remark 2. Lemma 1 is a generalization of Theorem 2.2 in [10]. Theorem 2.2 in [10] can be considered as a corollary of Lemma 1 for $t_0 = 0$.

Remark 3. For known fractional derivatives (cf. [7, 11, 19]), including the Riemann–Liouville fractional derivative of order q

$$D^r_{t_0}x(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{q-n+1}}\, \mathrm{d}s,$$

where $n - 1 < q < n$ and $\Gamma(z) = \int_0^\infty \mathrm{e}^{-t}t^{z-1}\, \mathrm{d}t$ is the Gamma function, and Caputo fractional derivative of order q

$$D^c_{t_0}x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{x^n(s)}{(t-s)^{q-n+1}}\, \mathrm{d}s,$$

properties (i)–(v) do not hold, except for the statement (v) for a fractional derivative of Caputo. The reason for this is the application of the integral in the known definitions of fractional derivatives.

The fractional-like integral of order $0 < q \leqslant 1$ with a lower limit t_0 is defined by (see [10, 17])

$$I^q_{t_0}x(t) = \int_{t_0}^{t} (s - t_0)^{q-1}x(s)\, \mathrm{d}s,$$

where the integration is considered in the Riemann's sense.

The following lemma holds [1, 10, 17].

Lemma 2. *Let the function $x(t) : (t_0, \infty) \to \mathbb{R}$ be q-differentiable for $0 < q \leqslant 1$. Then for all $t > t_0$,*

$$I^q_{t_0}\bigl(\mathcal{D}^q_{t_0} x(t)\bigr) = x(t) - x(t_0).$$

For more properties of fractional-like derivatives, we refer the reader to [1, 10, 17, 20].

3 Physical interpretation of a fractional-like derivative

The application of a limit in Definition 1 instead of an integral used in the Riemann–Liouville, Caputo, and some other classical definitions of fractional derivatives allows us to give the following physical interpretation of a fractional-like derivative.

Consider the movement of a point P on a line in $\mathbb{R}_+$ for the moments of time $t_1 = t$ and $t_2 = t + \theta(t - t_0)^{1-q}$, where $\theta > 0$ and $0 < q \leqslant 1$. Denote by $S(t_1)$ and $S(t_2)$ the distances at times t_1 and t_2, respectively. The expression

$$\frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{S(t + \theta(t - t_0)^{1-q}) - S(t)}{\theta(t - t_0)^{1-q}} = v_{\mathrm{avr}}(t)$$

is the q-average velocity of the point P between the instants t_1 and t_2, i.e. for the time $\theta(t - t_0)^{1-q}$.

Now, consider

$$\begin{aligned}
\mathcal{D}^q_{t_0}\bigl(S(t)\bigr) &= \lim_{\theta \to 0} \frac{S(t + \theta(t - t_0)^{1-q}) - S(t)}{\theta} \\
&= \lim_{\theta \to 0} \frac{S(t + \theta(t - t_0)^{1-q}) - S(t)}{\theta(t - t_0)^{1-q}} (t - t_0)^{1-q} \\
&= \frac{\mathrm{d}S}{\mathrm{d}t}(t - t_0)^{1-q} = v_{\mathrm{inst}}(t),
\end{aligned}$$

where $\mathrm{d}S/\mathrm{d}t$ is the ordinary instantaneous velocity of the point P.

For $q = 1$, the above limit is the ordinary (integer-order) instantaneous velocity of the point P at an arbitrary $t \in \mathbb{R}_+$. For $0 < q \leqslant 1$, the limit gives the q-instantaneous velocity of the point P for a $t \in \mathbb{R}_+$.

Therefore, the physical interpretation of a fractional-like derivative of order q, $0 < q \leqslant 1$, is the q-instantaneous velocity of the state vector of a mechanical or other nature system, i.e. the q-rate of change of $S(t)$ with respect to t.

4 Fractional-like nonhomogeneous systems

In this section, we consider a system of nonhomogeneous differential equations of the perturbed motion with fractional-like derivatives of the type

$$\mathcal{D}^q_{t_0} x(t) = g\bigl(t, x(t)\bigr) + f(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $g \in C^q(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f \in C^q(\mathbb{R}_+, \mathbb{R}^n)$, and $t > t_0$. Denote by $x(t) = x(t; t_0, 0)$ the solution of system (1), satisfying the initial condition

$$x(t_0) = 0. \tag{2}$$

We will further assume that the functions g and f are such that the solution $x(t, t_0, 0)$ of the initial value problem (IVP) (1), (2) exists on J, and $x(t, t_0, 0) \in C^q(J \times \mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ for $t \in J$, where $J \subset \mathbb{R}_+$ is an open interval.

Remark 4. For some efficient sufficient conditions about the existence of the solutions of an IVP of type (1), (2) with Caputo fractional derivatives, we refer the reader to [6].

The following lemma holds.

Lemma 3. *For the solution* $x(t) = x(t; t_0, 0)$ *of the fractional-like IVP* (1), (2)*, the following integral representation*

$$x(t) = \int_{t_0}^{t} \frac{f(s)}{(s - t_0)^{1-q}} \,\mathrm{d}s + \int_{t_0}^{t} \frac{g(s, x(s))}{(s - t_0)^{1-q}} \,\mathrm{d}s$$

holds for all $t \in J$*, and* $J = (t_0, \infty)$.

Proof. Since $x(t)$, $g(t, x(t))$, and $f(t)$ are continuous functions, then the integrals $I^q_{t_0}(g(t, x(t)))$ and $I^q_{t_0}(f(t))$ exist, and $I^q_{t_0}(\mathcal{D}^q_{t_0} x(t))$ is defined for all $t \in J$. From Lemma 2 we have

$$I^q_{t_0}\bigl(\mathcal{D}^q_{t_0} x(t)\bigr) = x(t) - x(t_0). \tag{3}$$

Applying the operator $I^q_{t_0}$ to the fractional-like system (1) and taking into account the initial condition (2), we obtain

$$x(t) = I^q_{t_0}\bigl(g\bigl(s, x(s)\bigr) + f(s)\bigr), \quad t > t_0.$$

This completes the proof of Lemma 3. □

For the IVP (1), (2), we introduce the following assumptions:

(H1) $F(t) = \int_{t_0}^{\infty} f(s)/(s - t_0)^{1-q} \,\mathrm{d}s < +\infty$ uniformly on $t_0 \in \mathbb{R}_+$;

(H2) There exists a positive constant $k > 0$ such that $\|g(t, x)\| \leqslant k\|x\|$ for $(t, x) \in J \times B_r$, where $B_r = \{x \in \mathbb{R}^n\colon \|x\| \leqslant r\}$, and $\|.\|$ is any norm in $\mathbb{R}^n$.

Using assumptions (H1) and (H2), we will prove our main result in this section.

Theorem 1. *Let, for the fractional-like system* (1)*, conditions* (H1)–(H2) *are met. Then there exists a positive constant* B *such that, for the solution of the IVP* (1), (2)*, the following estimate*

$$\bigl\|x(t)\bigr\| < B \exp\biggl(k \frac{(t - t_0)^q}{q}\biggr) \tag{4}$$

holds for all $t > t_0$.

Proof. From condition (H1) it follows that there exists a positive constant $B > 0$ such that

$$\|F(t)\| \leqslant B, \quad t > t_0. \tag{5}$$

Taking into account condition (H2), estimate (5), and also relation (3), we derive the integral inequality

$$\|x(t)\| \leqslant B + \int_{t_0}^{t} k(s-t_0)^{q-1}\|x(s)\|\,\mathrm{d}s.$$

Denote $v(s) = k(s-t_0)^{q-1}$. Then we have

$$\|x(t)\| \leqslant B + \int_{t_0}^{t} v(s)\|x(s)\|\,\mathrm{d}s.$$

Applying to the above inequality the Grönwall–Bellman lemma (see [4]), we obtain

$$\|x(t)\| \leqslant B\exp\left(\int_{t_0}^{t} v(s)\,\mathrm{d}s\right) = B\exp\left(k\frac{(t-t_0)^q}{q}\right), \quad t > t_0.$$

This completes the proof of Theorem 1. □

Corollary 1. *If, for the fractional-like system* (1)*, condition* (H1) *is satisfied and the vector-function $g(t,x) = A(t)x$, where $A(t)$ is an $(n \times n)$ matrix with continuous entries on any finite interval, then estimate* (4) *holds for the solution $x(t)$ with a constant $k = \sup_{t>t_0}\|A(t)\|$.*

Corollary 2. *If, for the fractional-like system* (1)*, condition* (H1) *is satisfied and the vector-function $g(t,x) = Cx$, where C is an $(n \times n)$ constant matrix, then the function*

$$\begin{aligned} x(t) &= W_C^q(t,t_0)x(t_0) + \int_{t_0}^{t} W_C^q(t,s)f(s)(s-t_0)^{q-1}\,\mathrm{d}s \\ &= \int_{t_0}^{t} W_C^q(t,s)f(s)(s-t_0)^{q-1}\,\mathrm{d}s, \end{aligned}$$

where

$$W_C^q(t,s) = \exp\frac{C(t-t_0)^q}{q}\exp\frac{-C(s-t_0)^q}{q}$$

is the solution of the IVP (1), (2).

Remark 5. In this case, we will call $W_C^q(t,s)$, $t, s \geqslant t_0$, $s < t$, the q-solving operator of the fractional-like system (1).

5 Fractional-like systems with permanent perturbations

In this section, we will present effective estimates of the solutions of systems of equations with fractional-like derivatives with permanent perturbations.

Consider the system of equations with fractional-like derivatives of the type

$$\mathcal{D}^q_{t_0} x(t) = g\bigl(t, x(t)\bigr) + r\bigl(t, x(t)\bigr), \tag{6}$$

where $x \in \mathbb{R}^n$, $g \in C^q(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $r \in C^q(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, and $r(t, 0) \neq 0$ for all $t \in \mathbb{R}_+$.

Let $x(t)$ denotes the solution of system (6), satisfying the initial condition (2), and assume that the solution of the IVP (6), (2) exists on J and is q-differentiable for $t \in J$.

The following lemma holds.

Lemma 4. *The fractional-like IVP* (6), (2) *inverts to the integral equation*

$$x(t) = \int\limits_{t_0}^{t} \frac{g(s, x(s))}{(s - t_0)^{1-q}}\,\mathrm{d}s + \int\limits_{t_0}^{t} \frac{r(s, x(s))}{(s - t_0)^{1-q}}\,\mathrm{d}s \tag{7}$$

for all $t \in J$*, and* $J = (t_0, \infty)$.

The proof of Lemma 4 is similar to the proof of Lemma 3.

Next, we make the following assumptions about the components of the right-hand side of the fractional-like system (6):

(H3) There exists a continuous function $h(t)$ such that $\|r(t, x)\| \leqslant h(t)$ for all $(t, x) \in J \times B_r$;

(H4) $H(t) = \int_{t_0}^{\infty} h(s)/(s - t_0)^{1-q}\,\mathrm{d}s < +\infty$ uniformly on $t_0 \in \mathbb{R}_+$;

(H5) There exists a continuous function $m(t)$ such that $\|g(t, x)\| \leqslant m(t)\|x\|$ for $(t, x) \in J \times B_r$.

We will present an estimate of the solutions of the fractional-like system (6) under the zero initial conditions, i.e. under conditions that the initial state of the motion is at the equilibrium state.

Theorem 2. *Assume that, for the fractional-like system* (6)*, conditions* (H3)–(H5) *are met. Then for the norm of the solution* $x(t)$ *of the IVP* (6), (2)*, the following estimate*

$$\bigl\|x(t)\bigr\| \leqslant H(t) + \int\limits_{t_0}^{t} m(r)H(r)\exp\Biggl(\int\limits_{r}^{t} m(s)(s - t_0)^{q-1}\,\mathrm{d}s\Biggr)\mathrm{d}r \tag{8}$$

holds for $t \in J$.

Proof. From (7), taking into account conditions (H3)–(H5), we obtain the integral inequality

$$\bigl\|x(t)\bigr\| \leqslant H(t) + \int\limits_{t_0}^{t} m(s)\bigl\|x(s)\bigr\|(s - t_0)^{q-1}\,\mathrm{d}s, \quad t \in J. \tag{9}$$

After we set in (9) $\upsilon(s) = m(s)(s - t_0)^{q-1}$, we derive the integral inequality

$$\|x(t)\| \leqslant H(t) + \int_{t_0}^{t} \upsilon(s)\|x(s)\| \, \mathrm{d}s, \quad t > t_0.$$

Applying to the above inequality Theorem 1.1.2 from [16], we obtain the estimate

$$\|x(t)\| \leqslant H(t) + \int_{t_0}^{t} \big[\upsilon(s)H(s)\big] \exp\left(\int_{s}^{t} \upsilon(\xi)\, \mathrm{d}\xi\right) \mathrm{d}s, \quad t > t_0. \tag{10}$$

This completes the proof of Theorem 2. □

Corollary 3. *If, in the fractional-like system* (6)*, the vector-function* $g(t, x) = A(t)x$*, where* $A(t)$ *is an* $(n \times n)$ *matrix with continuous entries on any finite interval, then estimate* (8) *is in the form*

$$\|x(t)\| \leqslant H(t) + k \int_{t_0}^{t} W_k^q(t, s) H(s)(s - t_0)^{q-1} \, \mathrm{d}s$$

for all $t \in J$*, where the constant* $k = \sup_{t > t_0} \|A(t)\|$.

Corollary 4. *If, in estimate* (10)*, the function* $H(t)$ *is nondecreasing, then estimate* (8) *is of the type*

$$\|x(t)\| \leqslant H(t) \exp\left(\int_{t_0}^{t} \upsilon(s)\, \mathrm{d}s\right) = H(t) \exp\left(\int_{t_0}^{t} m(s)(s - t_0)^{q-1} \, \mathrm{d}s\right)$$

for all $t > t_0$.

6 Heyers–Ulam–Rassias stability

In this section, we will apply the presented estimates to a scalar equation with a fractional-like derivative of the function of the state, and we will establish sufficient conditions for the Heyers–Ulam–Rassias stability of the solutions.

Consider the fractional-like equation of the type

$$\mathcal{D}_{t_0}^q x(t) = g(t, x) + r(t, x), \tag{11}$$

where $g \in C^q([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, $r \in C^q([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, $r(t, 0) \neq 0$ for all $t \in [t_0, \infty)$. Denote again by $x(t)$ the solution of (11), satisfying the initial condition

$$x(t_0) = 0,$$

and assume that it exists on J, and is q-differentiable.

We will use the following definition for the Heyers–Ulam–Rassias (HUR) stability of equation (11), which is a generalization of the definition in [23].

Definition 2. Equation (11) is said to be *HUR-stable* if, for any $\varepsilon > 0$ and any function $x_\varepsilon(t) : J \to \mathbb{R}_+$ that satisfy the inequality

$$\left|\mathcal{D}^q_{t_0} x_\varepsilon(t) - g\big(t, x_\varepsilon(t)\big) - r\big(t, x_\varepsilon(t)\big)\right| \leqslant \varepsilon, \tag{12}$$

there exists a solution $x(t)$ of (11) and a constant $Q = Q(\varepsilon) > 0$ such that, for all $t \in J$,

$$\left|x(t) - x_\varepsilon(t)\right| \leqslant Q\varepsilon.$$

Now, we will prove the main stability result in the section.

Theorem 3. *Assume that, for the fractional-like differential equation* (11):

(i) *There exists a constant* $L > 0$ *such that*

$$\left|g(t, x) - g(t, x_\varepsilon)\right| \leqslant L|x - x_\varepsilon|$$

for all $(t, x) \in D \subset \mathbb{R}_+ \times \mathbb{R}$, $(t, x_\varepsilon) \in D$;

(ii) *There exists a constant* $k > 0$ *such that*

$$\left|r(t, x) - r(t, x_\varepsilon)\right| \leqslant k$$

for all $(t, x) \in D$, $(t, x_\varepsilon) \in D$;

(iii) $$\sup_{t \geqslant t_0} \left(H(t) + \int_{t_0}^{t} \big[\upsilon(s)H(s)\big] \exp\left(\int_{s}^{t} \upsilon(\xi)\, \mathrm{d}\xi \right) \mathrm{d}s \right) < +\infty,$$

where

$$H(t) = \frac{\kappa - \varepsilon}{q}(t - t_0)^q, \quad 0 < \varepsilon < \kappa < +\infty;$$
$$\upsilon(s) = \frac{L}{(s - t_0)^{1-q}}, \quad 0 < q \leqslant 1.$$

Then the fractional-like differential equation (11) *is HUR-stable.*

Proof. From equation (11) and Lemma 2 it follows that (11) inverts to the integral equation

$$x(t) = \int_{t_0}^{t} \frac{g(s, x(s))}{(s - t_0)^{1-q}}\, \mathrm{d}s + \int_{t_0}^{t} \frac{r(s, x(s))}{(s - t_0)^{1-q}}\, \mathrm{d}s, \quad t \geqslant t_0.$$

Next, from (12) it follows that

$$\mathcal{D}^q_{t_0} x_\varepsilon(t) - g\big(t, x_\varepsilon(t)\big) - r\big(t, x_\varepsilon(t)\big) \leqslant \varepsilon \tag{13}$$

or

$$\mathcal{D}^q_{t_0} x_\varepsilon(t) - g\big(t, x_\varepsilon(t)\big) - r\big(t, x_\varepsilon(t)\big) \geqslant -\varepsilon. \tag{14}$$

From (13) we have

$$x_\varepsilon(t) \leqslant \int\limits_{t_0}^{t} \frac{g(s, x_\varepsilon(s))}{(s-t_0)^{1-q}}\,\mathrm{d}s + \int\limits_{t_0}^{t} \frac{r(s, x_\varepsilon(s))}{(s-t_0)^{1-q}}\,\mathrm{d}s + \int\limits_{t_0}^{t} \frac{\varepsilon}{(s-t_0)^{1-q}}\,\mathrm{d}s$$
$$= \int\limits_{t_0}^{t} \frac{g(s, x_\varepsilon(s))}{(s-t_0)^{1-q}}\,\mathrm{d}s + \int\limits_{t_0}^{t} \frac{r(s, x_\varepsilon(s))}{(s-t_0)^{1-q}}\,\mathrm{d}s + \frac{\varepsilon(t-t_0)^q}{q}. \tag{15}$$

It follows from (13)–(15) that

$$x(t) - x_\varepsilon(t) \leqslant \int\limits_{t_0}^{t} \frac{(g(s, x(s)) - g(s, x_\varepsilon(s)))}{(s-t_0)^{1-q}}\,\mathrm{d}s + \int\limits_{t_0}^{t} \frac{(r(s, x(s)) - r(s, x_\varepsilon(s)))}{(s-t_0)^{1-q}}\,\mathrm{d}s - \frac{\varepsilon(t-t_0)^q}{q}. \tag{16}$$

Hence, taking into account conditions (i) and (ii) of Theorem 3, from inequality (16) we obtain

$$\bigl|x(t) - x_\varepsilon(t)\bigr| \leqslant \int\limits_{t_0}^{t} \frac{L|x(s) - x_\varepsilon(s)|\,\mathrm{d}s}{(s-t_0)^{1-q}} + \int\limits_{t_0}^{t} \frac{k\,\mathrm{d}s}{(s-t_0)^{1-q}} - \frac{\varepsilon(t-t_0)^q}{q}.$$

From the above inequality we derive the estimate

$$\bigl|x(t) - x_\varepsilon(t)\bigr| \leqslant H(t) + \int\limits_{t_0}^{t} \upsilon(s)\bigl|x(s) - x_\varepsilon(s)\bigr|\,\mathrm{d}s. \tag{17}$$

Applying Theorem 1.1.2 from [16] to inequality (17) will lead to

$$\bigl|x(t) - x_\varepsilon(t)\bigr| \leqslant H(t) + \int\limits_{t_0}^{t} \bigl[\upsilon(s)H(s)\bigr] \exp\Biggl(\int\limits_{s}^{t} \upsilon(\xi)\,\mathrm{d}\xi\Biggr)\,\mathrm{d}s, \quad t \in J.$$

Condition (iii) of Theorem 3 implies the existence of a constant $Q = Q(\varepsilon) > 0$ such that $|x(t) - x_\varepsilon(t)| \leqslant Q\varepsilon$ for all $t \in J$. This completes the proof of Theorem 3. □

Remark 6. Since the function $H(t)$ is positive and nondecreasing for $t \in J$, then we have

$$\bigl|x(t) - x_\varepsilon(t)\bigr| \leqslant H(t) \exp\Biggl(\int\limits_{t_0}^{t} \upsilon(s)\,\mathrm{d}s\Biggr), \quad t \in J.$$

From the above estimate it follows that condition (iii) of Theorem 3 can be replaced by the condition

$$\sup_{t\geqslant t_0}\left(H(t)\exp\left(\frac{L}{q}(t-t_0)^q\right)\right) < +\infty.$$

Moreover, the assertion of Theorem 3 remains true.

7 Conclusions

The estimates of the solutions of equations with perturbations and fractional-like derivatives of the state vector are important fragments of the qualitative theory of this class of equations. The method of integral inequalities allow us to obtain simple and effective estimates of the norm of solutions of nonlinear fractional-like systems and sufficient conditions for Hyers–Ulam–Rassias stability of the equations. Since the theory of equations with fractional-like derivatives started quite recently, our results will be useful for further study of the qualitative behavior of their solutions.

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