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On the possibility of remote detection of conductive layers

Aleksandr S. Barashkov

Department of Mathematics,
National Research University "Moscow Power Engineering Institute",
Krasnokazarmennaya str. 14, Moscow, 111250, Russia
barashkovas@mpei.ru

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Abstract. A two-dimensional medium is considered in which the fields are described by the Helmholtz equation. The linearized formulation of the problem of restoring the parameters of the medium (the inverse problem for the Helmholtz equation) is studied. The conditions for the uniqueness of detection of thin conducting layers are established. Examples are given of the multivaluedness of the solution of the inverse problem in information, which was initially thought to be even redundant for an unambiguous solution.

Keywords: two-dimensional medium, inverse problem for the Helmholtz equation, linearized formulation, infinite strip, uniqueness theorems, examples of the multivaluedness of the solution in the reconstruction of the medium, Fourier transform.

1 Introduction

We consider the Helmholtz equation

$$\Delta u + \mu\sigma(x,y)u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \mu\sigma(x,y)u = 0 \tag{1}$$

in the band $D = \{(x,y)\colon -\infty < x < \infty,\ 0 < y < 1\}$.

Suppose that the solution $u(x,y,\mu)$ of equation (1) satisfies the boundary conditions

$$u(x,\, y=1,\, \mu) = 0; \qquad \frac{\partial u}{\partial y}(x,\, y=0,\, \mu) = -1. \tag{2}$$

Let the parameter μ and the coefficient $\sigma(x,y)$ be such that the solution of the boundary value problem (1), (2) – a function $u(x,y,\mu)$ – exists and is unique. The exact conditions on μ and σ will be indicated below.

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Definition 1. A direct problem for system (1), (2) is the problem of finding a function

$$\varphi(x,\mu) = u(x,\, y = 0,\, mu). \tag{3}$$

Definition 2. The inverse problem for system (1), (2) is the problem of determination the coefficient $\sigma(x,y)$ of the function $\varphi(x,\mu)$ from (3).

These problems are modeled in the calculation of fields and in the interpretation of sounding data. In applications, for example, in electrical prospecting [6], the coefficient $\sigma(x,y)$ from (1) characterizes the structure of the Earth, the parameter μ has a sense of the sounding frequency, $u|_{y=0}$, $\partial u/\partial y|_{y=0}$ expressed in terms of the values of the electric and magnetic field strengths on the Earth's surface.

We note that for a one-dimensional analogue of equation (1), that is, for equation $\partial^2 u/\partial y^2 + \mu\sigma(y)u = 0$, the inverse problem is solved in an exhaustive manner [2, 8, 16].

In particular, the uniqueness theorem for the recovery of the coefficient $\sigma(y)$ is proved for a wide class of functions that includes piecewise-continuous functions $\sigma(y)$.

The results on the inverse multidimensional problem for equation (1) are much more modest – special cases were studied within the framework of simplifying assumptions [1, 4, 10, 12, 13].

In this paper, we study the linearized version of problem (1), (2). The procedure for linearization is as follows [14].

We consider the coefficient $\sigma(x,y) = \sigma_0 + \gamma(x,y)$, where $\|\gamma(x,y)\| \ll \|\sigma_0\|$. We introduce a new parameter ε into system (1), (2) and expand the solution $u(x,y,\mu,\varepsilon)$ in a series in this parameter: $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$.

$$\begin{aligned}
&\varepsilon\colon\ \Delta u + \mu\bigl(\sigma_0 + \varepsilon\gamma(x,y)\bigr)u = 0,\\
&\varepsilon^0\colon\ \Delta u_0 + \mu\sigma_0 u_0 = 0, \qquad u_0|_{y=1} = 0; \quad \left.\frac{\mathrm{d}u_0}{\mathrm{d}y}\right|_{y=0} = -1,\\
&\varepsilon^1\colon\ \Delta u_1 + \mu\sigma_0 u_1 = -\mu u_0\gamma(x,y), \qquad u_1|_{y=1} = 0; \quad \left.\frac{\partial u_1}{\partial y}\right|_{y=0} = 0,\\
&\quad \dots\\
&\varepsilon^n\colon\ \Delta u_n + \mu\sigma_0 u_n = -\mu u_{n-1}\gamma(x,y), \qquad u_n|_{y=1} = 0; \quad \left.\frac{\partial u_n}{\partial y}\right|_{y=0} = 0.
\end{aligned}$$

We confine ourselves to the approximation $u \approx u_0 + \varepsilon u_1$. In this case, we consider that ε equal to 1. To shorten the entries, we denote $w = u_0$, $v = u_1$. Then $u = w + v$, where

$$\frac{\mathrm{d}^2 w}{\mathrm{d}y^2} + \mu\sigma_0 w = 0, \qquad w|_{y=1} = 0; \quad \left.\frac{\mathrm{d}w}{\mathrm{d}y}\right|_{y=0} = -1, \tag{4}$$

$$\Delta v + \mu\sigma_0 v = -\mu w(y,\mu)\gamma(x,y), \qquad v|_{y=1} = 0; \quad \left.\frac{\partial v}{\partial y}\right|_{y=0} = 0. \tag{5}$$

Next, we set and study inverse problems for system (4), (5).

2 A single-layer medium

We assume in (4) that $\mu = -1$, $\gamma(x,y) = \delta(y-b)\alpha(x)$, where $\delta(t)$ is the Dirac delta function ($\int_{-\infty}^{\infty} \delta(t) f(t)\,\mathrm{d}t = f(0)$, $b \in (0;1)$) [17]. In the application to electrical prospecting, this kind of function $\gamma(x,y)$ means that a thin conductive layer with total conductivity $\alpha(x)$ lies at the depth b [6].

Concerning relations (5), it will be convenient for us to use another equivalent form, which obviously does not contain a delta function:

$$\begin{aligned} &\Delta v - \sigma_0 v = 0, \quad 0 < y < 1,\ y \neq b, \\ &v|_{y=1} = 0; \qquad \left.\frac{\partial v}{\partial y}\right|_{y=0} = 0, \qquad [v]|_{y=b} = 0, \qquad \left.\left[\frac{\partial v}{\partial y}\right]\right|_{y=b} = w(b)\alpha(x). \end{aligned} \tag{6}$$

We recall the notation: $[v]|_{y=b} = v(x,\, b+0) - v(x,\, b-0)$.

Thus, according to (6), the solution $v(x,y)$ satisfies the equation for $y \neq b$, $0 < y < 1$, is continuous on the inner boundary $y = b$, and the derivative of the solution suffers a discontinuity on this boundary.

We rewrite relation (4) for the case under consideration as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}y^2} - \sigma_0 w = 0, \quad w|_{y=1} = 0; \quad \left.\frac{\mathrm{d}w}{\mathrm{d}y}\right|_{y=0} = -1. \tag{7}$$

We study system (7), (6). The function $w(y)$ from (7) is written explicitly:

$$w(y) = \frac{\operatorname{sh}(\sqrt{\sigma_0}(1-y))}{\sqrt{\sigma_0}\operatorname{ch}\sqrt{\sigma_0}}. \tag{8}$$

The following result concerns the solution of system (6) [4, 15].

Theorem 1. *Let $\alpha(x)$ be a bounded, infinitely differentiable function with bounded derivatives. Then the bounded solution of problem* (6) *exists and is unique. In this case, the solution $v(x,y)$ is a continuous function in the domain $\bar{D} = \{(x,y)\colon -\infty < x < \infty,\ 0 \leqslant y \leqslant 1\}$ and an infinitely differentiable function in the domains $\bar{D}_1 = \{(x,y)\colon -\infty < x < \infty,\ 0 \leqslant y \leqslant b\}$, $\bar{D}_2 = \{(x,y)\colon -\infty < x < \infty,\ b \leqslant y \leqslant 1\}$.*

Definition 3. A direct problem for system (7), (6) is the problem of determining the function

$$\varphi(x) = v(x,\, y = 0). \tag{9}$$

In this case, the coefficient (number) σ_0 and the function $\alpha(x)$ in (6) are assumed to be known.

Comment. The function φ in Definitions 1, 3 differs by a term $w(0)$, where $w(y)$ is the function from (7).

Definition 4. The inverse problem for system (7), (6) is the problem of reconstructing the coefficient $\alpha(x)$ from (6) with respect to the function$\varphi(x)$ in (7). In this case, the numbers σ_0, b participating in (6) are assumed to be known.

The solution of the direct problem for system (7), (6) can be written out in explicit form. To this end, we represent the function $v(x,y)$ in (6) as the sum of a Fourier series:

$$v(x,y) = \sum_{n=0}^{\infty} a_n(x)\cos\lambda_n y, \tag{10}$$

where $\lambda_n = \pi/2 + \pi n$, $n = 0,1,2,\dots$.

Using the formula for calculating the coefficients of the Fourier series

$$a_n(x) = 2\int_0^1 v(x,y)\cos(\lambda_n y)\,\mathrm{d}y,$$

the form of the function $w(y)$ in (8), relation (6), we obtain equations for determining the coefficients $a_n(x)$:

$$\frac{\mathrm{d}^2 a_n}{\mathrm{d}x^2} - \left(\lambda_n^2 + \sigma_0\right)a_n = \frac{2\cos(\lambda_n b)\,\mathrm{sh}\left(\sqrt{\sigma_0}(1-b)\right)}{\sqrt{\sigma_0}\,\mathrm{ch}\sqrt{\sigma_0}}\alpha(x). \tag{11}$$

We recall that $\alpha(x)$ from (11) is an infinitely differentiable function bounded on the whole axis. For equation (11), the Green's function is easily written [9], so the solution of the equation, bounded on the entire numerical axis, is given by the following formula:

$$a_n(x) = -\frac{\cos(\lambda_n b)\,\mathrm{sh}\left(\sqrt{\sigma_0}(1-b)\right)}{\sqrt{\sigma_0}\,\mathrm{ch}\sqrt{\sigma_0}}\int_{-\infty}^{\infty}\exp\left(-\sqrt{\lambda_n^2+\sigma_0}|t-x|\right)\alpha(t)\,\mathrm{d}t \tag{12}$$

Taking into account (12), (10), (9), we obtain an explicit formula for the solution of the direct problem

$$\varphi(x) = \int_{-\infty}^{\infty} K(x-t)\alpha(t)\,\mathrm{d}t, \tag{13}$$

where

$$K(x) = -\frac{\mathrm{sh}((1-b)\sqrt{\sigma_0})}{\sqrt{\sigma_0}\,\mathrm{ch}\sqrt{\sigma_0}}\sum_{n=0}^{\infty}\frac{\cos(\lambda_n b)\exp(-\sqrt{\lambda_n^2+\sigma_0}|x|)}{\sqrt{\lambda_n^2+\sigma_0}}.$$

Result (13) can easily be generalized to a multilayer medium using the linearity of the problem.

In (4), (5), we take that $\mu \in \{-\eta_1, -\eta_2, \dots, -\eta_N\}$, where $\eta_N > \eta_{N-1} > \dots > \eta_1 > 0$, $\gamma(x,y) = \sum_{i=1}^{n}\alpha_i(x)\delta(y-b_i)$, where $\alpha_i(x) \in \tilde{L}_1^{(\infty)}(R)$, $0 < b_1 < b_2 \dots < b_n < 1$. In an application to electrical prospecting, this kind of function means that thin conductive layers with total conductivity $\alpha_i(x)$ lie at depths b_i, respectively.

Concerning relations (5), it will be convenient for us to use another equivalent form, which obviously does not contain a delta function:

$$\begin{aligned}
&\Delta v_k - \eta_k \sigma_0 v_k = 0, \quad 0 < y < 1,\ y \neq b_i,\ i = 1, 2, \dots, n, \\
&v_k|_{y=1} = 0; \quad \frac{\partial v_k}{\partial y}\bigg|_{y=0} = 0, \quad [v_k]|_{y=b_i} = 0, \quad \left[\frac{\partial v_k}{\partial y}\right]\bigg|_{y=b_i} = w_k(b_i)\alpha_i(x), \\
&w_k(y) = \frac{\operatorname{sh}(\sqrt{\sigma_0 \eta_k}(1-y))}{\sqrt{\sigma_0 \eta_k}\operatorname{ch}(\sqrt{\sigma_0 \eta_k})}, \quad k = 1, 2, \dots, N.
\end{aligned} \tag{14}$$

We will explain that, according to (14), the solution $v_k(x, y)$ satisfies the equation for $0 < y < 1$, $y \neq b_i$, is continuous on internal boundaries $y = b_i$, and the derivative of solutions suffers a discontinuity at these boundaries.

The following result is true about the solution of system (14) [3, 4].

Theorem 2. *Let $\alpha_i(x)$, $i = 1, 2, \dots, n$, be bounded and infinitely differentiable functions with bounded derivatives. Then the bounded solution $v_k(x, y)$, $k = 1, 2, \dots, N$, of problem* (14) *exists and is unique.*

Moreover, the solution $v_k(x, y)$ is a continuous function in a domain $\bar{D} = \{(x, y){:}\ -\infty < x < \infty,\ 0 \leqslant y \leqslant 1\}$ and an infinitely differentiable function in the domains $\bar{D}_1 = \{(x, y){:}\ -\infty < x < \infty,\ 0 \leqslant y \leqslant b_1\}$, $\bar{D}_i = \{(x, y){:}\ -\infty < x < \infty,\ b_{i-1} \leqslant y \leqslant b_i\}$, $i = 1, 2, \dots, n$, $\bar{D}_{n+1} = \{(x, y){:}\ -\infty < x < \infty,\ b_n \leqslant y \leqslant 1\}$.

Definition 5. A direct problem for system (14) is the problem of defining functions

$$\varphi_k(x) = v_k(x, y = 0), \quad k = 1, 2, \dots, N. \tag{15}$$

In this case, the coefficient (number) σ_0, functions $\alpha_i(x)$, numbers b_i, $i = 1, 2, \dots, n$, from (14) are considered known.

The solution of the direct problem for system (14), (15) can be written out in explicit form:

$$\varphi_m(x) = \int_{-\infty}^{\infty} \sum_{j=1}^{n} K_{mj}(x - t)\alpha_j(t)\,\mathrm{d}t, \quad m = 1, 2, \dots, N. \tag{16}$$

The functions $K_{mj}(x)$ are completely analogous to the function $K(x)$ in (13):

$$K_{mj}(x) = -\frac{\operatorname{sh}((1 - b_j)\sqrt{\eta_m \sigma_0})}{\sqrt{\eta_m \sigma_0}\operatorname{ch}\sqrt{\eta_m \sigma_0}} \sum_{n=0}^{\infty} \frac{\cos(\lambda_n b_j)\exp(-\sqrt{\lambda_n^2 + \eta_m \sigma_0}|x|)}{\sqrt{\lambda_n^2 + \eta_m \sigma_0}}.$$

Let us return to the problem with one layer.

We apply to equation (13) the Fourier transform [7], assuming in addition that $\alpha(x) \in L_1(-\infty,\infty)$. Let $\varPhi(\omega)$, $\tilde{K}(\omega)$, $A(\omega)$ be the Fourier images of the functions $\varphi(x)$, $K(x)$, $\alpha(x)$, respectively. In particular,

$$\tilde{K}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} K(x) \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}x = -\sqrt{\frac{2}{\pi}} \frac{\operatorname{sh}((1-b)\sqrt{\sigma_0})}{\sqrt{\sigma_0}\operatorname{ch}\sqrt{\sigma_0}} \sum_{n=0}^{\infty} \frac{\cos(\lambda_n b)}{\lambda_n^2 + \sigma_0 + \omega^2}. \quad (17)$$

For Fourier images, equation (11) can be written in the form

$$\sqrt{2\pi}\tilde{K}(\omega)A(\omega) = \varPhi(\omega). \quad (18)$$

The uniqueness of the solution of (13) depends on the presence of zeros of the function $\tilde{K}(\omega)$ in (18). If $\tilde{K}(\omega) \neq 0$, $\omega \in (-\infty,\infty)$, $\varphi(x) \equiv 0$ and therefore $\varPhi(\omega) \equiv 0$, then $A(\omega) \equiv 0$, and therefore $\alpha(\omega) \equiv 0$.

For $b = 0$, the series in (17) consists of positive terms, therefore $\tilde{K}(\omega) < 0$, $\omega \in (-\infty,\infty)$. In addition, it turned out that, in this case, the sum of the series was calculated [11, (1.421.2)].

For $b \in (0;1)$, the series in (17) is alternating, and the author has not succeeded in finding the final result for calculating the sum of such a series.

Independent calculations led to the following result:

$$\tilde{K}(\omega) = \frac{-\operatorname{sh}((1-b)\sqrt{\sigma_0})}{\sqrt{2\pi}\sqrt{\sigma_0}\operatorname{ch}\sqrt{\sigma_0}} \frac{\operatorname{sh}((1-b)\sqrt{\omega^2+\sigma_0})}{\sqrt{\omega^2+\sigma_0}\operatorname{ch}\sqrt{\omega^2+\sigma_0}}. \quad (19)$$

It follows from (19) that $\tilde{K}(\omega) \neq 0, \omega \in R$. This yields

Theorem 3. *The inverse problem* (6)*,* (7)*,* (9) *has at most one solution in the class of infinitely differentiable bounded functions $\alpha(x)$ belonging to $L_1(R)$.*

The class of functions indicated in the theorem can be extended without loss of uniqueness of the solution, for example, adding to $\alpha(x)$ the term of the form

$$\sum_{k=1}^{n} (a_k \cos\omega_k x + b_k \sin\omega_k x).$$

In this case, the function $\alpha(x)$ will have both a continuous and a discrete spectrum; its Fourier image – a function $A(\omega)$ – will have several delta functions in its record.

We introduce the notation for the extended class of functions $\tilde{L}_1^{(\infty)}(-\infty,\infty)$ – functions of the form $\alpha(x) = \beta(x) + T(x)$, where $\beta(x)$ is an infinitely differentiable function bounded together with the derivatives belonging $L_1(-\infty,\infty)$, that is, $\int_{-\infty}^{\infty} |\beta(x)|\, \mathrm{d}x < \infty$, and $T(x)$ is a trigonometric polynomial.

Note that for practical applications, this class is enough. They are usually interested in the conductivity of the form $\alpha(x) = A + \beta(x)$, where A is a number, and $\beta(x)$ is a finite function. Such functions are represented in $\tilde{L}_1^{(\infty)}(-\infty,\infty)$.

We state the uniqueness theorem for the introduced class of functions.

Theorem 4. *The inverse problem* (6)*,* (7)*,* (9) *has no more than one solution in the class of functions* $\tilde{L}_1^{(\infty)}(-\infty,\infty)$.

An attempt to extend the class in the other direction leads to a loss of the result about uniqueness. We are talking about the determination both the function $\alpha(x)$ and the number b in (6) with respect to the function $\varphi(x)$ in (9) (that is, from the information measured on the surface of the medium we are trying to determine the depth of the layer and the conductivity of the layer).

We indicate in the arguments of the function (19) the value b: $\tilde{K}(\omega,b)$. Suppose that two sets $(\alpha_1(x),b_1)$, $(\alpha_2(x),b_2)$ from (6) correspond to the function $\varphi(x)$ from (7). Then (13) can be written as

$$\sqrt{2\pi}\tilde{K}(\omega,b_1)A_1(\omega)=\varPhi(\omega)=\sqrt{2\pi}\tilde{K}(\omega,b_2)A_2(\omega). \tag{20}$$

It follows from the congruence (20), (19) that if $0<b_2<b_1<1$ and $A_1(\omega)$ is the Fourier image of a function $\alpha_1(x)$ from the class indicated in Theorem 2, then the function $A_2(\omega)=A_1(\omega)\tilde{K}(\omega,b_1)/\tilde{K}(\omega,b_2)$ is also an image of a function $\alpha_2(x)$ from the specified class.

We give concrete examples of the nonuniqueness of the solution of the inverse problem for system (7), (6), (9).

Example 1. Suppose that $\alpha_i(x)\in L_1(R)$ and, in (6),

$$\sigma_0=1,\qquad b_1=\frac{1}{2},\qquad \alpha_1(x)=\frac{1}{1+x^2},$$
$$b_2=\frac{1}{3},\qquad \alpha_2(x)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\operatorname{sh}(0.5)\operatorname{sh}(0.5\sqrt{\omega^2+1})}{\operatorname{sh}(2/3)\operatorname{sh}(2\sqrt{\omega^2+1}/3)}\mathrm{e}^{-|\omega|}\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega.$$

These two sets $(b_1;\alpha_1)$, $(b_2;\alpha_2)$ correspond to the same function $\varphi(x)$ from (9):

$$\varphi(x)=-\frac{1}{2}\int_{-\infty}^{\infty}\frac{\operatorname{sh}(0.5)\operatorname{sh}(0.5\sqrt{\omega^2+1})}{\operatorname{ch}(1)\sqrt{\omega^2+1}\operatorname{ch}(\sqrt{\omega^2+1})}\mathrm{e}^{-|\omega|}\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega.$$

Example 2. Suppose that $\alpha_i(x)\in\tilde{L}_1^{(\infty)}(R)$ and, in (6),

$$\sigma_0=1,\qquad b_1=\frac{1}{2},\qquad \alpha_1(x)=\sin x,$$
$$b_2=\frac{1}{3},\qquad \alpha_2(x)=\sin x\frac{\operatorname{sh}(1/2)\operatorname{sh}(\sqrt{2}/2)}{\operatorname{sh}(2/3)\operatorname{sh}(2\sqrt{2}/3)}\approx 0.5122\sin x.$$

These two sets $(b_1;\alpha_1)$, $(b_2;\alpha_2)$ correspond to the same function $\varphi(x)$ from (9):

$$\begin{aligned}\varphi(x)&=-\sin x\frac{\operatorname{sh}(1/2)\operatorname{sh}(\sqrt{2}/2)}{\operatorname{ch}(1)\sqrt{2}\operatorname{ch}(\sqrt{2})}\approx-0.08414\sin x\varphi(x)\\&=-\sin x\frac{\operatorname{sh}(1/2)\operatorname{sh}(\sqrt{2}/2)}{\operatorname{ch}(1)\sqrt{2}\operatorname{ch}(\sqrt{2})}\approx-0.08414\sin x.\end{aligned}$$

3 Two-layered medium

We assume that in (14), $n = 2$, $N = 2$, that is, $\mu \in \{-\eta_1, -\eta_2\}$. Here $\eta_1, \eta_2 > 0$, $\gamma(x,y) = \sum_{i=1}^{2} \alpha_i(x)\delta(y - b_i)$, where $\alpha_i(x) \in \tilde{L}_1^{(\infty)}(R)$, $b_1, b_2 \in (0;1)$, $b_1 < b_2$.

As applied to electrical prospecting, this kind of function $\gamma(x,y)$ means that, at depths b_1, b_2, there are thin conductive layers with total conductivity $\alpha_1(x)$, $\alpha_2(x)$, respectively.

Definition 6. The inverse problem #1 for system (14), (15) for $n = 2$, $N = 2$ is called the problem of reconstructing the coefficients $\alpha_1(x)$, $\alpha_2(x)$ from (14) with respect to the functions $\varphi_1(x)$, $\varphi_2(x)$ in (15). In this case, the numbers σ_0, b_1, b_2, η_1, η_2 participating in (14) are assumed to be known.

The solution of the direct problem for system (14), (15) for $n = 2$, $N = 2$, according to (16), can be written out in explicit form

$$\varphi_m(x) = \int_{-\infty}^{\infty} \sum_{j=1}^{2} K_{mj}(x - t)\alpha_j(t)\,\mathrm{d}t, \quad m = 1, 2, \tag{21}$$

where

$$K_{mj}(x) = -\frac{\operatorname{sh}((1 - b_j)\sqrt{\eta_m\sigma_0})}{\sqrt{\eta_m\sigma_0}\operatorname{ch}\sqrt{\eta_m\sigma_0}} \sum_{n=0}^{\infty} \frac{\cos(\lambda_n b_j)\exp(-\sqrt{\lambda_n^2 + \eta_m\sigma_0}|x|)}{\sqrt{\lambda_n^2 + \eta_m\sigma_0}}.$$

The following assertion is true for the inverse problem.

Theorem 5. *The inverse problem* #1 *for system* (14)*,* (15) *for* $n = 2$*,* $N = 2$ *has no more than one solution* (*i.e., a set* $\alpha_1(x)$*,* $\alpha_2(x)$) *in the class of functions* $\tilde{L}_1^{(\infty)}(-\infty, \infty)$*.*

Proof. We use the fact that the functions are connected by equations (21).

Let $\Phi_m(\omega)$, $\tilde{K}_{mj}(\omega)$, $A_j(\omega)$ be the Fourier images of the functions $\varphi_m(x)$, $K_{mj}(x)$, $\alpha_j(x)$, respectively. For Fourier transforms, equations (21) go over into equations

$$\sqrt{2\pi} \sum_{j=1}^{2} \tilde{K}_{mj}(\omega)A_j(\omega) = \Phi_m(\omega), \quad m = 1, 2. \tag{22}$$

It follows from the calculation (19) that

$$\tilde{K}_{mj}(\omega) = \frac{-\operatorname{sh}((1 - b_j)\sqrt{\eta_m\sigma_0})}{\sqrt{2\pi}\sqrt{\eta_m\sigma_0}\operatorname{ch}\sqrt{\eta_m\sigma_0}} \frac{\operatorname{sh}((1 - b_j)\sqrt{\omega^2 + \eta_m\sigma_0})}{\sqrt{\omega^2 + \eta_m\sigma_0}\operatorname{ch}\sqrt{\omega^2 + \eta_m\sigma_0}}. \tag{23}$$

The uniqueness of the solution of (21) this time depends on the determinant [5]

$$\Delta(\omega) = \begin{vmatrix} \tilde{K}_{11}\tilde{K}_{12} \\ \tilde{K}_{21}\tilde{K}_{22} \end{vmatrix}. \tag{24}$$

If $\Delta(\omega) \neq 0, \omega \in (-\infty, \infty)$, then system (21) has no more than one solution.

We show that the determinant (24) is not equal to zero.

Suppose the contrary. Suppose that for some ω, $\Delta(\omega) = 0$. Then the columns of the matrix generating the determinant (24) are proportional:

$$\frac{\tilde{K}_{11}(\omega)}{\tilde{K}_{12}(\omega)} = \frac{\tilde{K}_{21}(\omega)}{\tilde{K}_{22}(\omega)}. \tag{25}$$

Taking (23) into account, we rewrite (25) in the form

$$\begin{aligned} &\frac{\operatorname{sh}((1-b_1)\sqrt{\eta_1\sigma_0})}{\operatorname{sh}((1-b_2)\sqrt{\eta_1\sigma_0})} \frac{\operatorname{sh}((1-b_1)\sqrt{\omega^2+\eta_1\sigma_0})}{\operatorname{sh}((1-b_2)\sqrt{\omega^2+\eta_1\sigma_0})} \\ &\quad = \frac{\operatorname{sh}((1-b_1)\sqrt{\eta_2\sigma_0})}{\operatorname{sh}((1-b_2)\sqrt{\eta_2\sigma_0})} \frac{\operatorname{sh}((1-b_1)\sqrt{\omega^2+\eta_2\sigma_0})}{\operatorname{sh}((1-b_2)\sqrt{\omega^2+\eta_2\sigma_0})}. \end{aligned} \tag{26}$$

We introduce the function

$$F(\eta) = \frac{\operatorname{sh}((1-b_1)\sqrt{\eta\sigma_0})}{\operatorname{sh}((1-b_2)\sqrt{\eta\sigma_0})} \frac{\operatorname{sh}((1-b_1)\sqrt{\omega^2+\eta\sigma_0})}{\operatorname{sh}((1-b_2)\sqrt{\omega^2+\eta\sigma_0})}. \tag{27}$$

Equality (26) means that $F(\eta_1) = F(\eta_2)$, but this cannot be since $F(\eta)$ from (27) is a monotone function. This follows from the fact that the function $G(x_1, x_2, t) = \operatorname{sh}(x_1 t)/\operatorname{sh}(x_2 t)$ is monotonic for $t > 0$ (increasing with the condition $x_1 > x_2 > 0$ and decreasing with the condition $0 < x_1 < x_2$).

Thus, it is shown that $\Delta(\omega) \neq 0$, $\omega \in R$. Theorem 5 is proved. □

We return to the inverse problem with one layer.

We will try to restore both the depth of the layer b and the "conductivity" $\alpha(x)$. Since information on one frequency was not enough (Examples 1, 2), we assume that information is available on two frequencies $\mu = -\eta_1$, $\mu = -\eta_2$. Instead of equations (14), for $n = 2$, $N = 2$, we have equations

$$\begin{aligned} &\Delta v_k - \eta_k\sigma_0 v_k = 0, \quad 0 < y < 1,\ y \neq b, \\ &v_k|_{y=1} = 0; \quad \frac{\partial v_k}{\partial y}\bigg|_{y=0} = 0, \qquad [v_k]|_{y=b} = 0, \quad \left[\frac{\partial v_k}{\partial y}\right]\bigg|_{y=b} = w_k(b)\alpha(x), \\ &w_k(y) = \frac{\operatorname{sh}(\sqrt{\sigma_0\eta_k}(1-y))}{\sqrt{\sigma_0\eta_k}\operatorname{ch}(\sqrt{\sigma_0\eta_k})}, \quad k = 1, 2. \end{aligned} \tag{28}$$

The inverse problem for equations (28) is the problem of reconstructing the coefficient $\alpha(x)$ and the number b with respect to the functions $\varphi_1(x)$, $\varphi_2(x)$ from (15). Number σ_0 of (28) is assumed to be known. For Fourier images of functions $\alpha(x)$, $\varphi_1(x)$, $\varphi_2(x)$, we have a system of equations analogous to system (22)

$$\sqrt{2\pi}\tilde{K}_{1m}(\omega)A(\omega) = \Phi_m(\omega), \quad m = 1, 2. \tag{29}$$

If $\Phi_1(\omega) \equiv 0$, then $A(\omega) \equiv 0$, $\alpha(x) \equiv 0$. At the same time, $\Phi_2(\omega) \equiv 0$, it is inevitable. In this case, any value $b \in (0;1)$ can be taken as a value b. Thus, there is no uniqueness in the definition of a pair $(\alpha(x), b)$.

Suppose that $\Phi_1(\omega)$ is not identically zero, i.e., there is ω_0 such that $\Phi_1(\omega_0) \neq 0$. Then by (29), $A(\omega_0) \neq 0$, $\Phi_2(\omega_0) \neq 0$.

Let there be two sets $(b_1, A_1(\omega_0))$, $(b_2, A_2(\omega_0))$ that satisfy system (29) for $\omega = \omega_0$, wherein $A_1(\omega_0) \neq 0$, $A_2(\omega_0) \neq 0$. Then

$$\tilde{K}_{1m}(\omega_0)A_1(\omega_0) = \tilde{K}_{1m}(\omega_0)A_2(\omega_0), \quad m = 1, 2,$$

or

$$\tilde{K}_{1m}(\omega_0)A_1(\omega_0) + \tilde{K}_{1m}(\omega_0)\bigl(-A_2(\omega_0)\bigr) = 0, \quad m = 1, 2. \tag{30}$$

Relations (30) can be regarded as a homogeneous system of linear equations by definition $A_1(\omega_0)$, $-A_2(\omega_0)$. The determinant of this system coincides with $\Delta(\omega_0)$ from (24). Since, by what has been proved, $\Delta(\omega_0) \neq 0$ for $b_1 \neq b_2$, then system (30) cannot have nontrivial solutions.

Thus, it is shown that, according to information at two frequencies, the "depth" of the layer and the "conductivity" of the layer are uniquely determined.

We now state the exact result.

Definition 7. The inverse problem #2 for system (28) is the problem of reconstructing the coefficient $\alpha(x)$ and the number b from (28) with respect to the functions $\varphi_1(x)$, $\varphi_2(x)$ from (19). The numbers σ_0, η_1, η_2, participating in (28), are assumed to be known.

Theorem 6. *The inverse problem* #2 *for system* (28) *has no more than one solution* (*that is, a set* $(b, \alpha(x))$) *if* $\varphi_1(x)$ *is not identically equal to zero,* $b \in (0;1)$*,* $\alpha(x) \in \tilde{L}_1^{(\infty)}(-\infty, \infty)$.

Proof. The proof is given in the neighborhood of formula (30)). □

We continue the study of the problem with two layers. We are trying to determine now the functions $\alpha_1(x)$, $\alpha_2(x)$ (the conductivity of the layers) and the numbers b_1, b_2 (the depth of the layers). It is quite obvious that, according to information on two frequencies functions $\varphi_1(x)$, $\varphi_2(x)$ from (15), it is impossible to determine $\alpha_1(x)$, $\alpha_2(x)$, b_1, b_2 uniquely.

The result of Theorem 6 allows us to hope that the addition of information at the third frequency will provide a single-valued recovery $\alpha_1(x)$, $\alpha_2(x)$, b_1, b_2. In this case, the obvious necessary condition for such a recovery: $\alpha_1(x)$, $\alpha_2(x)$ are not identically zero.

Thus, we formulate the following inverse problem for system (14), (15), where $n = 2$, $N = 3$: by functions $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ from (15) we reconstruct the functions $\alpha_1(x)$, $\alpha_2(x)$ (conductivity of the layers) and the numbers b_1, b_2 (the depth of the layers) from (14). In this case, of course, we assume that the frequencies η_1, η_2, η_3 from (14) do not coincide.

Example 3. In equation (14), we take $\sigma_0 = 1$, $\eta_1 = 1$, $\eta_2 = 4$, $\eta_3 = 9$. We give an example of functions $\varphi_i(x)$, $i = 1, 2, 3$, in (15), for which the inverse problem of finding the depths of the layers (numbers b_1, b_2) and conductivities (functions $\alpha_1(x)$, $\alpha_2(x)$) has a nonunique solution.

We take the first solution in the following form: $(b_1^{(1)}, \alpha_1^{(1)}(x), b_2^{(1)}, \alpha_2^{(1)}(x))$, where $b_1^{(1)} = 1/3$, $b_2^{(1)} = 2/3$,

$$\alpha_i^{(1)}(x) = A_{i1}\sin x + A_{i2}\sin 2x + C_{i1}\cos x + C_{i2}\cos 2x, \quad i = 1, 2.$$

We will indicate the numbers A_{ij}, C_{ij} later.

The second solution is taken in the similar form: $(b_1^{(2)}, \alpha_1^{(2)}(x), b_2^{(2)}, \alpha_2^{(2)}(x))$, where $b_1^{(2)} = 1/4$, $b_2^{(2)} = 1/2$,

$$\alpha_i^{(1)}(x) = B_{i1}\sin x + B_{i2}\sin 2x + D_{i1}\cos x + D_{i2}\cos 2x, \quad i = 1, 2.$$

We will indicate the numbers B_{ij}, D_{ij} later.

We introduce the numbers $d_1 = 1 - b_1^{(1)} = 2/3$, $d_2 = 1 - b_2^{(1)} = 1/3$, $d_3 = 1 - b_1^{(2)} = 3/4$, $d_4 = 1 - b_2^{(2)} = 1/2$. We introduce the numbers f_{ij}andg_{ij}:

$$f_{ij} = \operatorname{sh}(\sqrt{\eta_i}d_j)\operatorname{sh}(\sqrt{1+\eta_i}d_j), \qquad g_{ij} = \operatorname{sh}(\sqrt{\eta_i}d_j)\operatorname{sh}(\sqrt{4+\eta_i}d_j).$$

Consider matrices $F = \{f_{ij}\}$, $G = \{g_{ij}\}$ and vectors $M = \{f_{i4}\}$, $N = \{g_{i4}\}$, $1 \leqslant i, j \leqslant 3$.

Let $X = (x_1, x_2, x_3)^{\mathrm{T}}$, $Y = (y_1, y_2, y_3)^{\mathrm{T}}$ be solutions of linear systems $FX = M$, $GY = N$. Then the numbers A_{ij}, B_{ij} from the record of the two solutions of the inverse problem are expressed as follows:

$$\begin{aligned} &A_{11} = C_{11} = x_1, && A_{21} = C_{21} = x_2, && B_{11} = D_{11} = -x_3, && B_{21} = D_{21} = 1,\\ &A_{12} = C_{12} = y_1, && A_{22} = C_{22} = y_2, && B_{12} = D_{12} = -y_3, && B_{22} = D_{22} = 1. \end{aligned}$$

The corresponding functions $\varphi_i(x)$ in (19) are given by the formulas

$$\varphi_i(x) = H_i\sin x + Q_i\cos x + L_i\sin(2x) + R_i\cos(2x), \quad 1 \leqslant i \leqslant 3.$$

To determine the numbers H_i, L_i, we introduce the coefficients u_i, v_i:

$$u_i = \frac{\sqrt{\eta_i}}{\sqrt{1+\eta_i}\operatorname{ch}(\sqrt{\eta_i})\operatorname{ch}(\sqrt{1+\eta_i})}, \qquad v_i = \frac{\sqrt{\eta_i}}{\sqrt{4+\eta_i}\operatorname{ch}(\sqrt{\eta_i})\operatorname{ch}(\sqrt{4+\eta_i})}.$$

Then

$$\begin{aligned} H_i &= u_i(f_{i1}A_{11} + f_{i2}A_{21}), & Q_i &= u_i(f_{i1}C_{11} + f_{i2}C_{21}),\\ L_i &= v_i(g_{i1}B_{11} + g_{i2}B_{21}), & R_i &= v_i(g_{i1}D_{11} + g_{i2}D_{21}). \end{aligned}$$

The calculations yield the following result:

$$\begin{aligned} &x_1 = 0.506844122, && x_2 = 1.049457908, && x_3 = -0.162736559,\\ &y_1 = 0.478780482, && y_2 = 1.097927044, && y_3 = 0.148940562, \end{aligned}$$

$$H_1 = Q_1 = 0.119924845, \quad H_2 = Q_2 = 0.125599487, \quad H_3 = Q_3 = 0.071750573,$$
$$L_1 = R_1 = 0.062963089, \quad L_2 = R_2 = 0.079216346, \quad L_3 = R_3 = 0.051521584.$$

We rewrite the result more compactly by writing down numbers with four valid significant digits.

Information for solving the inverse problem (functions (15)):

$$\varphi_1(x) \approx -0.1199\sin x - 0.1199\cos x - 0.06296\sin(2x) - 0.06296\cos(2x)$$

for frequency $\eta \to 1$,

$$\varphi_2(x) \approx -0.1256\sin x - 0.1256\cos x - 0.07922\sin(2x) - 0.07922\cos(2x)$$

for frequency $\eta \to 4$,

$$\varphi_3(x) \approx -0.07175\sin x - 0.07175\cos x - 0.05152\sin(2x) - 0.05152\cos(2x)$$

for frequency $\eta \to 9$.

To these functions there correspond two solutions of the inverse problem:

$$b_1 = \frac{1}{3}, \qquad b_2 = \frac{2}{3},$$
$$\alpha_1(x) \approx 0.5068\sin x + 0.5068\cos x + 0.4788\sin 2x + 0.4788\cos 2x,$$
$$\alpha_2(x) \approx 1.049\sin x + 1.049\cos x + 1.098\sin 2x + 1.098\cos 2x$$

and

$$b_1 = \frac{1}{4}, \qquad b_2 = \frac{1}{2},$$
$$\alpha_1(x) \approx 0.1627\sin x + 0.1627\cos x + 0.1489\sin 2x + 0.1489\cos 2x,$$
$$\alpha_2(x) \approx \sin x + \cos x + \sin 2x + \cos 2x.$$

We note one circumstance that is not obvious to the author. In the given example, the information for solving the inverse problem – functions $\varphi_i(x)$ – depends on 12 coefficients. On these twelve numbers, it is necessary to determine 10 values: 2 depths b_1, b_2 and 8 coefficients of trigonometric sums for $\alpha_1(x)$, $\alpha_2(x)$. It was assumed that, in such situation, the solution of the inverse problem will be uniquely determined. But it turned out that this is not so.

Therefore, the following definition is reasonable. Suppose that in the formulas (14), (15) the frequency η takes 4 values: η_1, η_2, η_3, η_4.

Definition 8. The inverse problem # 2 for system (14), (15) for $n = 2$, $N = 4$ is the problem of determining the coefficients $\alpha_1(x)$, $\alpha_2(x)$ and numbers b_1, b_2 in (14) with respect to the functions $\varphi_k(x)$, $1 \leqslant k \leqslant 4$, in (15). The coefficient σ_0 in (14), (15) is assumed to be known.

We introduce the matrices

$$F_m(\omega) = \{f_{ij}\}, \quad 1 \leqslant i, j \leqslant m, \ m = 2, 3, 4, \tag{31}$$

where $f_{ij} = \operatorname{sh}(\sqrt{\eta_i\sigma_0}d_j)\operatorname{sh}(\sqrt{\omega^2 + \eta_i\sigma_0}d_j)$.

Theorem 7. *Let the frequencies η_i, $i = 1,2,3,4$, be such that* $\det F_m(\omega) \neq 0$, $m = 2,3,4$, *if* $\omega \in [0;\infty)$, $1 - b_i = d_i \in (0;1)$, $i = 1,2,3,4$, *and* $d_i \neq d_j$ *when* $i \neq j$. *Suppose that at least one function* $\varphi_k(x)$, $1 \leqslant k \leqslant 4$, *in* (15) *does not vanish identically. Then the inverse problem # 2 for system* (14), (15) *has no more than one solution* (*that is, a set* $\alpha_1(x), \alpha_2(x), b_1, b_2$), *where* $0 < b_1, b_2 < 1$, $\alpha_1(x), \alpha_2(x) \in \tilde{L}_1^{(\infty)}(-\infty,\infty)$.

Proof. Suppose the contrary. Let there be two sets $b_1^{(1)}$, $b_2^{(1)}$, $\alpha_1^{(1)}(x)$, $\alpha_2^{(1)}(x)$, $b_1^{(2)}$, $b_2^{(2)}$, $\alpha_1^{(2)}(x)$, $\alpha_2^{(2)}(x)$, which, at four frequencies η_i, give the same solutions to the direct problems (15).

We first consider the case when among the numbers $b_1^{(1)}$, $b_2^{(1)}$, $b_1^{(2)}$, $b_2^{(2)}$ are no equal. Analogously to (22), for the Fourier images of the corresponding functions, the following relations hold for $m = 1,2,3,4$:

$$\sqrt{2\pi}\sum_{j=1}^{2}\tilde{K}_{mj}^{(1)}(\omega)A_j^{(1)}(\omega) = \varPhi_m(\omega) = \sqrt{2\pi}\sum_{j=1}^{2}\tilde{K}_{mj}^{(2)}(\omega)A_j^{(2)}(\omega). \tag{32}$$

The functions $\tilde{K}_{mj}^{(1)}(\omega)$, $A_j^{(1)}(\omega)$ correspond to the first solution of the inverse problem $b_1^{(1)}$, $b_2^{(1)}$, $\alpha_1^{(1)}(x)$, $\alpha_2^{(1)}(x)$, and functions $\tilde{K}_{mj}^{(2)}(\omega)$, $A_j^{(2)}(\omega)$ – to the second. Since by the hypothesis of the theorem at least one of the functions $\varphi_k(x)$ does not vanish identically, it can be found ω_0, for which $(\varPhi_1(\omega_0), \varPhi_2(\omega_0), \varPhi_3(\omega_0), \varPhi_4(\omega_0)) \neq (0,0,0,0)$. Consequently, $(A_1^{(1)}(\omega_0), A_2^{(1)}(\omega_0)) \neq (0,0)$; $(A_1^{(2)}(\omega_0), A_2^{(2)}(\omega_0)) \neq (0,0)$.

From (32) there follow relations that can be regarded as a homogeneous system of linear equations with respect to $A_1^{(1)}(\omega_0), A_2^{(1)}(\omega_0), -A_1^{(2)}(\omega_0), -A_2^{(2)}(\omega_0)$

$$\sum_{j=1}^{2}\tilde{K}_{mj}^{(1)}(\omega)A_j^{(1)}(\omega) + \sum_{j=1}^{2}\tilde{K}_{mj}^{(2)}(\omega)\bigl(-A_j^{(2)}(\omega)\bigr) = 0, \quad m = 1,2,3,4. \tag{33}$$

The matrix of system (33) is obtained from matrix (31) by multiplying the rows by non-zero factors. Since by the hypothesis of the theorem $\det F_4(\omega_0) \neq 0$, system (33) has only a trivial solution, which contradicts the previously obtained relation $(A_1^{(1)}(\omega_0), A_2^{(1)}(\omega_0)) \neq (0,0)$. The case when among the numbers $b_1^{(1)}$, $b_2^{(1)}$, $b_1^{(2)}$, $b_2^{(2)}$ are equal numbers, is treated similarly.

Theorem 7 is proved. □

4 Multilayered medium

We return to system (14), (15): the medium contains n thin layers, measurements are known at N frequencies.

Definition 9. The inverse problem #1 for system (14), (15) is called the problem of restoring the coefficients $\alpha_i(x)$, $i = 1,2,\dots,n$, from (14) with respect to the functions $\varphi_k(x)$, $k = 1,2,\dots,N$, from (15). The numbers σ_0, b_i, η_k participating in (14) are assumed to be known.

Definition 10. The inverse problem #2 for system (14), (15) is called the problem of determining the coefficients $\alpha_i(x)$ and numbers b_i (depths of submergence of layers), $i = 1, 2, \dots, n$, from (14) with respect to the functions $\varphi_k(x)$, $1 \leqslant k \leqslant N$, in (15). In this case, the coefficient σ_0 and frequencies η_k are considered known.

We recall that the functions $\varphi_k(x)$, $\alpha_i(t)$ are connected by equations (16).

Let $\varPhi_m(\omega)$, $\tilde{K}_{mj}(\omega)$, $A_j(\omega)$ be the Fourier images of the functions $\varphi_m(x)$, $K_{mj}(x)$, $\alpha_j(x)$, respectively. For Fourier transforms, equations (16) go over into equations

$$\sqrt{2\pi}\sum_{j=1}^{n}\tilde{K}_{mj}(\omega)A_j(\omega) = \varPhi_m(\omega), \quad m = 1, 2, \dots, N. \tag{34}$$

The functions $\tilde{K}_{mj}(\omega)$ are defined in (23).

We introduce the matrix

$$F_{nN}(\omega) = \{f_{ki}\}, \quad 1 \leqslant k \leqslant N,\ 1 \leqslant i \leqslant n, \tag{35}$$

where $f_{ki} = \operatorname{sh}(\sqrt{\eta_k\sigma_0}d_i)\operatorname{sh}(\sqrt{\omega^2 + \eta_k\sigma_0}d_i)$. This matrix is analogous to the matrix in (31) and differs from it by the number of rows and columns.

Theorem 8. *Let $n = N$, and let the frequencies η_k and $d_i = 1 - b_i$ be such that* $\det F_{nn}(\omega) \neq 0$, $\omega \in [0; \infty)$. *Suppose that at least one of the functions $\varphi_m(x)$ in* (15) *does not vanish identically. Then the inverse problem* #1 *for system* (14), (15) *has no more than one solution* (*i.e., a set $\alpha_i(x)$, $i = 1, 2, \dots, n$*), *where $\alpha_i(x) \in \tilde{L}_1^{(\infty)}(-\infty, \infty)$.*

Proof. In the case under consideration, system (34) has a square matrix of coefficients $\{\sqrt{2\pi}\tilde{K}_{mj}(\omega)\}$, $1 \leqslant m, j \leqslant n$. If the determinant of this matrix is not equal to zero, then the inverse problem #1 has a unique solution. But this matrix is obtained from the matrix $F_{nn}(\omega)$ from (35) by multiplying the rows by nonzero factors. Hence, its determinant is not equal to zero, just like the determinant $\det F_{nn}(\omega)$.

Theorem 8 is proved. □

Now we indicate the situation when there is no uniqueness of the solution of the inverse problem.

Theorem 9. *Let $N = 2n - 1$. Let the frequencies η_k, $k = 1, 2, \dots, N$, be such that there are arguments $\omega_1, \omega_2, \dots, \omega_l$ for which* $\det F_{NN}(\omega_k) \neq 0$, *$k = 1, \dots, l$, $d_i = 1 - b^{(2)}_{i-n}$,* $i = n + 1, \dots, N$ (*the matrix F_{NN} is defined in* (35)). *Then there exists more than one solution of the inverse problem* #2, *namely, a solution of this type*:

$$\text{(i)} \quad \alpha_i^{(1)}(x) = \sum_{k=1}^{l} A_{ik}\sin(\omega_k x), \quad b_i^{(1)}, \quad i = 1, \dots, n,$$

$$\text{(ii)} \quad \alpha_i^{(2)}(x) = \sum_{k=1}^{l} B_{ik}\sin(\omega_k x), \quad b_i^{(2)}, \quad i = 1, \dots, n,$$

and among the numbers $b_i^{(1)}$, $b_i^{(2)}$ there are no identical.

Proof. We write system (34) for two supposed solutions:

$$\sqrt{2\pi}\sum_{j=1}^{n}\tilde{K}_{mj}^{(1)}(\omega_k)A_{jk} = \Phi_m(\omega_0) = \sqrt{2\pi}\sum_{j=1}^{n}\tilde{K}_{mj}^{(2)}(\omega_k)B_{jk},$$

$m = 1, 2, \dots, N$, $k = 1, 2, \dots, l$. We rewrite these relations in the form

$$\sum_{j=1}^{n}\tilde{K}_{mj}^{(1)}(\omega_k)A_{jk} + \sum_{j=1}^{n-1}\tilde{K}_{mj}^{(2)}(\omega_k)(-B_{jk}) = \tilde{K}_{mn}^{(2)}(\omega_k)B_{nk}. \tag{36}$$

Let us put it $B_{nk} = 1$. We consider (36) systems of linear equations for the determination of numbers A_{jk}, $1 \leqslant j \leqslant n$, B_{jk}, $1 \leqslant j \leqslant n-1$, $1 \leqslant k \leqslant l$. The matrix of coefficients of these systems is obtained from matrices $F_{NN}(\omega_k)$ by multiplying rows by non-zero factors. Consequently, systems (36) have a unique solution, which give two solutions of the inverse problem.

Theorem 9 is proved. □

Let us give a concrete example.

Example 4. The actions indicated in Theorem 9 are carried out for $n = 3$, $N = 5$, $l = 2$. Let $\sigma_0 = 1$, $\omega_0 = 1$, $\eta_1 = 1$, $\eta_2 = 4$, $\eta_3 = 9$, $\eta_4 = 16$, $\eta_5 = 25$. We indicate two solutions of the inverse problem #2:

$$\text{(i)}\quad \begin{aligned} b_1 &= 0.3, & \alpha_1(x) &= A_{11}\sin x + A_{12}\sin 2x,\\ b_2 &= 0.5, & \alpha_2(x) &= A_{21}\sin x + A_{22}\sin 2x,\\ b_3 &= 0.7, & \alpha_3(x) &= A_{31}\sin x + A_{32}\sin 2x; \end{aligned}$$

$$\text{(ii)}\quad \begin{aligned} b_1 &= 0.2, & \alpha_1(x) &= B_{11}\sin x + B_{12}\sin 2x,\\ b_2 &= 0.4, & \alpha_2(x) &= B_{21}\sin x + B_{22}\sin 2x,\\ b_3 &= 0.6, & \alpha_3(x) &= B_{31}\sin x + B_{32}\sin 2x, \end{aligned}$$

where (the results are given with four valid significant digits)

$$\begin{aligned} A_{11} &\approx 0.06164, & A_{21} &\approx 0.7440, & A_{31} &\approx 0.5978,\\ A_{12} &\approx 0.05742, & A_{22} &\approx 0.7287, & A_{32} &\approx 0.6081,\\ B_{11} &\approx 0.005284, & B_{21} &\approx 0.2962, & B_{31} &= 1.0000,\\ B_{12} &\approx 0.004785, & B_{22} &\approx 0.2832, & B_{32} &= 1.0000. \end{aligned}$$

Both these solutions correspond to functions $\varphi_i(x)$, $i = 1, 2, 3, 4, 5$, in (14) of the form

$$\varphi_i(x) = C_{i1}\sin x + C_{i2}\sin 2x$$

(for frequencies $\eta = 1, 4, 9, 16, 25$, respectively), where

$$\begin{aligned} C_{11} &\approx -0.09075, & C_{21} &\approx -0.02183, & C_{31} &\approx -0.004850,\\ C_{41} &\approx -0.001177, & C_{51} &\approx -0.0003194, \end{aligned}$$

$$C_{12} \approx -0.04606, \qquad C_{22} \approx -0.01336, \qquad C_{32} \approx -0.003403,$$
$$C_{42} \approx -0.0008992, \qquad C_{52} \approx -0.0002575.$$

We note an interesting circumstance. In the given example, the information for solving the inverse problem – functions $\varphi_i(x)$ – depends on 10 coefficients. For these ten numbers it is necessary to determine 9 values: 3 depths b_1, b_2, b_3 and 6 coefficients of trigonometric sums for $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$. A priori, it seemed that the information was superfluous, and the uniqueness of the solution to the reverse problem must be.

Theorem 9 and Example 4 show that this is not so.

Comment. It is possible to give an example of the nonuniqueness of the solution of the inverse problem #2 when the functions $\alpha_i(x) \in L_1(R)$. But then the functions will be more difficult to express in quadratures as in the example of nonuniqueness 1, which is less obvious and more difficult to verify.

Theorem 10. *Let $N = 2n$. Let the frequencies η_i, $i = 1, 2, \dots, N$, be such that* $\det F_{mm}(\omega) \neq 0$, *$m = 2, 3, \dots, N$, if $\omega \in [0; \infty)$, $1 - b_i = d_i \in (0; 1)$, $i = 1, 2, \dots, N$, and $d_i \neq d_j$ when $i \neq j$. Suppose that at least one function $\varphi_k(x)$, $1 \leqslant k \leqslant N$, in* (15) *does not vanish identically. Then the inverse problem #2 for system* (14), (15) *has no more than one solution* (*that is, a set $\alpha_i(x)$, b_i, $i = 1, 2, \dots, n$*), *where $0 < b_i < 1$, $\alpha_i(x) \in \tilde{L}_1^{(\infty)}(-\infty, \infty)$.*

Proof. The proof of Theorem 10 repeats word for word the proof of Theorem 7 when the dimension 2 is replaced by dimension n. □

5 Conclusion

To restore the conductivity of layers $\alpha_i(x)$, $i = 1, 2, \dots, n$, at known depths b_i, it is necessary to have measurements at n frequencies – functions $\varphi_i(x)$, $i = 1, 2, \dots, n$.

In order to restore the conductivity of the layers $\alpha_i(x)$, $i = 1, 2, \dots, n$, and the depths of submergence of layers b_i (i.e., additionally, to determine n numbers), it is necessary to have measurements at $N = 2n$ frequencies – functions $\varphi_i(x)$, $i = 1, 2, \dots, N$. A smaller number $N - 1$ of functions $\varphi_1(x)$, $\varphi_2(x)$, $\dots$, $\varphi_{N-1}(x)$, even if they are given by hundreds of parameters, are not enough to determine n depths b_i and n conductivities of the layers $\alpha_i(x)$.

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