Nonlinear Analysis: Modelling and Control, Vol. 24, No. 1, 95–120 ISSN 1392-5113
https://doi.org/10.15388/NA.2019.1.6

New unique existence criteria for higher-order nonlinear singular fractional differential equations*

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Received: January 14, 2018 / **Revised:** August 8, 2018 / **Published online:** December 14, 2018

Abstract. In this paper, a nonlinear three-point boundary value problem of higher-order singular fractional differential equations is discussed. By applying the properties of Green function and some fixed point theorems for sum-type operator on cone, some new criteria on the existence and uniqueness of solutions are obtained. Moreover, two iterative sequences are given for uniformly approximating the positive solution, which are important for practical application. At last, we give two examples to illustrate the main results.

Keywords: singular fractional differential equations, three-point boundary value problem, existence and uniqueness, fixed point theorem, sum-type operator.

1 Introduction

As we all know, fractional calculus describes many phenomena in various fields of science and engineering such as physics, biology, chemistry, control, economics, etc.; see [7, 17, 19–21]. Hence, the research on fractional differential equation plays a very important role in both theory and application. Especially, the higher-order fractional differential equations with a variety of boundary conditions have been of great interest recently. During the past decades, while the theory of fractional differential equations has developed in a variety of directions, we notice that the research are mainly focused on existence, uniqueness, and multiplicity of positive solutions for fractional differential equations under nonlinear boundary value conditions, see, for instances, [1, 2, 5, 6, 8–16, 18, 24, 26–28] and the references given there. These results are obtained by applying some efficient

*This research was supported by the Key Research and Development Plan of Shanxi Province, grant No. 201703D221031-1, and it is also supported by the Innovation Project of Shanxi Postgraduate Education, grant No. 2017BY039.

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tools such as Guo–Krasnosel'skii's fixed point theorem, Schauder fixed point theorem, upper and lower solution method, and topological degree theory.

In [12], Li et al. discussed the three point boundary value problem of fractional-order differential equations

$$
\begin{aligned}
&D_{0+}^{\alpha}u(t)+g\big(t,u(t)\big)=0, \quad 0<t<1,\\
&u(0)=0, \qquad D_{0+}^{\nu}u(1)=bD_{0+}^{\nu}u(\xi),
\end{aligned}
$$

where D_{0+}^{α}, D_{0+}^{ν} are the standard Riemann–Liouville fractional-order derivative of order $1<\alpha\leqslant 2$, $0\leqslant\nu\leqslant 1$. $0\leqslant b\leqslant 1$, $\xi\in(0,1)$, $0\leqslant\alpha-\nu-1$, $a\xi^{\alpha-\nu-2}\leqslant 1-\nu$, and $g\colon[0,1]\times[0,+\infty)\to[0,+\infty)$ satisfies Carathéodory-type conditions. This literature is focused on studying the existence and multiplicity results of positive solutions by means of Krasnosel'skii's fixed point theorem and other classical fixed point theorems, which required the operator to be completely continuous.

In [13], Liang and Zhang investigated the following nonlinear fractional three-point boundary value problem:

$$
\begin{aligned}
&D_{0+}^{\alpha}u(t)+g\big(t,u(t)\big)=0, \quad 0<t<1,\ 3<\alpha\leqslant 4,\\
&u(0)=u'(0)=u''(0)=0, \qquad u''(1)=bu''(\xi),
\end{aligned}
$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional-order derivative, and $0<\xi<1$ satisfy $0<b\xi^{\alpha-3}<1$. $g\colon[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. This paper proved the existence and uniqueness of a positive and nondecreasing solution by using a fixed point theorem in partially ordered sets.

In [10], Jleli and Samet studied the following arbitrary-order nonlinear fractional differential equation:

$$
\begin{aligned}
&-D_{0+}^{\alpha}u(t)=f\big(t,u(t),u(t)\big)+g\big(t,u(t)\big), \quad 0<t<1,\ n-1<\alpha\leqslant n,\\
&u^{(i)}(0)=0, \quad i=0,\dots,n-2, \qquad D_{0+}^{\nu}u(1)=0, \quad 2\leqslant\nu\leqslant n-2,
\end{aligned}
$$

where D^{α} is the standard Riemann–Liouville fractional derivative of order α. $n>3$ ($n\in\mathbb{N}$), and $f\colon[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ and $g\colon[0,1]\times[0,+\infty)\to[0,+\infty)$ are continuous functions. Using a mixed monotone operator method, authors determined sufficient conditions under which the above boundary value problem has a unique positive solution.

In a word, fractional differential equations of higher-order including singular and nonsingular cases are capable of describing memory and hereditary properties of certain important materials and processes. Note that the results dealing with the existence and uniqueness of positive solutions of multi-point boundary value problems for higher-order singular fractional differential equations are relatively scarce.

Motivated by the excellent results, in this article, we will discuss the following three-point boundary value problem for a class of higher-order singular fractional differential equations:

$$\begin{aligned}&-D_{0+}^{\alpha}u(t)=f\big(t,u(t),u(t)\big)+g\big(t,u(t)\big),\quad 0<t<1,\ n-1<\alpha\leqslant n,\\&u^{(i)}(0)=0,\quad i=0,\dots,n-2,\\&D_{0+}^{\nu}u(1)=bD_{0+}^{\nu}u(\xi),\quad n-2\leqslant\nu\leqslant n-1,\end{aligned}\tag{1}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of high order $n-1<\alpha\leqslant n$ ($n\in\mathbb{N}$, $n\geqslant 2$), D_{0+}^{ν} is the standard Riemann–Liouville fractional derivative of order $n-2<\nu\leqslant n-1$, and $x^{(i)}$ represents the ith (ordinary) derivative of x, $0\leqslant b\leqslant 1$, $0<\xi<1$, $\alpha-\nu-1\geqslant 0$, $0\leqslant b\xi^{\alpha-\nu-1}<1$. $f(t,u,v)$ may be singular at $t=0$ or 1, and $v=0$, $g(t,u)$ may be singular at $t=0$ or 1. By the properties of Green function and the two fixed point theorems for sum-type operator, we derived sufficient conditions for the existence and uniqueness of positive solutions to problem (1). The characteristic features presented in this paper are as follows. Firstly, the equations in this paper are the generalization of the equations studied in [12], where $n=2$ and $f(t,u,u)\equiv 0$. Other particular cases of our research was investigated in [13] (where $n=4$, $\nu=2$, $f(t,u,u)\equiv 0$) and in [10] (where $b=0$). Secondly, in our work, the nonlinearity is allowed to be singular in both time and space variables elements. Thirdly, we provide some alternative approaches to study equations (1) under different conditions. Our methods do not demand the existence of upper-lower solutions and compactness and continuity conditions for the operators. At last, we obtained the sufficient conditions, which guarantee the existence and uniqueness of the positive solution, and we also construct two iterative sequences to approximate the unique positive solution. Here we should point out that the obtained positive solution $u^*\in P_h$. That is, there exist $\mu\geqslant 1$ such that $t^{\alpha-1}/\mu\leqslant u^*(t)\leqslant \mu t^{\alpha-1}$, which makes the property of unique positive solution clearer. So, this paper extends and improves many known results including singular and nonsingular cases.

The outline of this paper is as follows. In Section 2, we review some of standard facts on definitions and summarize without proofs the relevant material on lemmas. In Section 3, we convert the boundary value problem (1) into an equivalent integral equation, we derive the corresponding fractional Green function and argue its properties. Then we provide some conditions under which our main results are stated and proved. In Section 4, we give two concrete examples to illustrate these results, which can be used in practice. Some conclusions are drawn in Section 5.

2 Preliminaries

For the convenience of the reader, we repeat the relevant definitions and lemmas from [1, 3, 4, 20, 22, 23, 25] without proofs, thus it makes our exposition self-contained.

Definition 1. (See [20].) The integral

$$I_{0+}^{\alpha}h(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{h(t)}{(x-t)^{1-\alpha}}\,\mathrm{d}t,\quad x>0,$$

is called the Riemann–Liouville fractional integral of order α of a function h, where $\alpha > 0$, $h\colon (0,+\infty) \to \mathbb{R}$, and $\Gamma(\alpha)$ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

Definition 2. (See [20].) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $h\colon (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} h(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_0^x (x-t)^{n-\alpha-1}h(t)\,\mathrm{d}t, \quad x > 0,$$

where $\Gamma(\cdot)$ is the gamma function, provided that the right-hand side is pointwise defined on $(0,+\infty)$, and $n = [\alpha] + 1$ with $[\alpha]$ standing for the largest integer less than α.

Lemma 1. (*See* [1].) *Assume that* $u \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$ $(i = 1, \dots, n)$, *where* $n = [\alpha] + 1$.

Lemma 2. (*See* [20].) *Let* $\alpha > -1$, $\nu > 0$, *and* $t > 0$. *Then*

$$D_{0+}^{\nu} t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\nu+1)} t^{\alpha-\nu}.$$

For more details on fractional calculus, we refer the reader to [1, 20].

In the sequel, we present some basic concepts in ordered Banach spaces and several fixed point theorems, which we will be used later. We suggest that one refers to [3, 4, 22, 23, 25] for more details.

Let $(E, \|\cdot\|)$ be a real Banach space with θ the zero element, which is partially ordered by a cone $P \subset E$, i.e., $x \leqslant y$ if and only if $y - x \in P$. If $x \leqslant y$ and $x \neq y$, then we denote $x < y$ or $y > x$. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies: (i) $x \in P$, $\lambda \geqslant 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$. Cone P is said to be solid if $\mathring{P} = \{x \in P\colon x \text{ is an interior point of } P\}$ is nonempty; P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$. In this case, N is called the normality constant of P. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda, \mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E\colon x \sim h\}$. It is easy to see that $P_h \subset P$. Moreover, operator $A\colon E \to E$ is increasing (decreasing) if $x \leqslant y$ implies $Ax \leqslant Ay$ $(Ax \geqslant Ay)$.

Definition 3. (See [4].) $A\colon P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i \in P$ $(i = 1, 2)$, $u_1 \leqslant u_2$, $v_1 \geqslant v_2$ imply $A(u_1, v_1) \leqslant A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 4. (See [4].) An operator $A\colon P \to P$ is said to be subhomogeneous if it satisfies

$$A(tx) \geqslant tA(x) \quad \forall t \in (0,1),\ x \in P.$$

Definition 5. (See [4].) Let $D = P$ and β be a real number with $0 \leqslant \beta < 1$. An operator $A\colon D \to D$ is said to be β-concave if it satisfies

$$A(tx) \geqslant t^{\beta}A(x) \quad \forall t \in (0,1),\ x \in D.$$

Lemma 3. (*See* [22, *Cor.* 2.7].) *Let $\beta \in (0,1)$, $A\colon P_h \times P_h \to P_h$ is a mixed monotone operator and satisfies*

$$A\bigl(tx, t^{-1}y\bigr) \geqslant t^{\beta}A(x,y) \quad \forall t \in (0,1),\ x,y \in P_h, \tag{2}$$

and $B\colon P_h \to P_h$ is an increasing subhomogeneous operator. Assume that there exists a constant $\delta_0 > 0$ such that

$$A(x,y) \geqslant \delta_0 Bx \quad \forall x,y \in P_h. \tag{3}$$

Then:

(i) *There exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leqslant u_0 < v_0$, $u_0 \leqslant A(u_0, v_0) + Bu_0 \leqslant A(v_0, u_0) + Bv_0 \leqslant v_0$.*
(ii) *The operator equation $A(x,x) + Bx = x$ has a unique solution x^* in P_h.*
(iii) *For any initial values $x_0, y_0 \in P_h$, constructing successively the sequences $x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}$, $y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}$, $n = 1, \dots,$ we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.*

Lemma 4. (*See* [22, *Cor.* 2.8].) *Let $\beta \in (0,1)$. $A\colon P_h \times P_h \to P_h$ is a mixed monotone operator and satisfies*

$$A\bigl(tx, t^{-1}y\bigr) \geqslant tA(x,y) \quad \forall t \in (0,1),\ x,y \in P_h. \tag{4}$$

$B\colon P_h \to P_h$ is an increasing β-concave operator. Assume that there exists a constant $\delta_0 > 0$ such that

$$A(x,y) \leqslant \delta_0 Bx \quad \forall x,y \in P_h. \tag{5}$$

Then conclusions (i)–(iii) *in Lemma 3 hold.*

Lemma 5. (*See* [25, *Cor.* 2.2].) *Let $\beta \in (0,1)$ and $A\colon P_h \times P_h \to P_h$ be a mixed monotone operator. Assume* (2) *holds. Then the operator A has a unique fixed point x^* in P_h. Moreover, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, \dots,$ we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.*

Lemma 6. (*See* [23, *Thm.* 2.7].) *Let $\beta \in (0,1)$ and $B\colon P_h \to P_h$ is an increasing β-concave operator. Then the operator B has a unique fixed point in P_h.*

3 Main results

In this section, we work in a Banach space $E = C[0,1]$, which is endowed with the norm $\|x\| = \max\{|x(t)|\colon t \in [0,1]\}$ and equipped with a partial order: for all $x, y \in E$, $x \preccurlyeq y \Leftrightarrow x(t) \leqslant y(t)$, where $t \in [0,1]$. Set

$$P = \{x \in C([0,1])\colon x(t) \geqslant 0,\ t \in [0,1]\}.$$

Clearly, P is a normal cone and $P \subset E$, the normality constant is 1. Let $h(t) = t^{\alpha-1}$, and we define

$$P_h = \left\{x \in P \;\middle|\; \exists \mu \geqslant 1\colon \frac{1}{\mu}h(t) \leqslant x(t) \leqslant \mu h(t),\ t \in [0,1]\right\}.$$

At first, we introduce the following lemmas, which are crucial in the proof of the main results.

Lemma 7. *Assume that $\zeta \in C[0,1] \cap L^1(0,1)$ is a continuous function. Then $u \in C[0,1]$ is a solution to the following boundary value problem of the fractional differential equation*:

$$\begin{aligned} &-D_{0+}^{\alpha} u(t) = \zeta(t), \quad 0 < t < 1,\ n-1 < \alpha \leqslant n, \\ &u^{(i)}(0) = 0, \quad i = 0, \dots, n-2, \\ &D_{0+}^{\nu} u(1) = b D_{0+}^{\nu} u(\xi), \quad n-2 < \nu \leqslant n-1, \end{aligned}$$

where $n \geqslant 2$, $0 \leqslant b \leqslant 1$, $0 < \xi < 1$, $\alpha - \nu - 1 \geqslant 0$, $0 \leqslant b\xi^{\alpha-\nu-1} < 1$, if and only if u satisfies the integral equation

$$u(t) = \int_0^1 G(t,s)\zeta(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{d\,\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\nu-1} - bt^{\alpha-1}(\xi-s)^{\alpha-\nu-1} & \\ \quad - d(t-s)^{\alpha-1}, & 0 \leqslant s \leqslant \min\{t,\xi\} < 1, \\ t^{\alpha-1}(1-s)^{\alpha-\nu-1} - d(t-s)^{\alpha-1}, & 0 < \xi \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-\nu-1} - bt^{\alpha-1}(\xi-s)^{\alpha-\nu-1}, & 0 \leqslant t \leqslant s \leqslant \xi < 1, \\ t^{\alpha-1}(1-s)^{\alpha-\nu-1}, & 0 \leqslant \max\{t,\xi\} \leqslant s \leqslant 1, \end{cases}$$

where $d = 1 - b\xi^{\alpha-\nu-1} > 0$. $G(t,s)$ is called the fractional Green function. It is easy to see that $G(t,s)$ is continuous on $[0,1] \times [0,1]$.

Proof. From Definition 1 and Lemma 1, $u \in C[0,1]$ is a solution to the boundary value problem if and only if

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\zeta(s)\,\mathrm{d}s$$

for some real constants $c_j, j = 1, \dots, n$. By using the boundary conditions $u^{(i)}(0) = 0$, $i = 0, \dots, n-2$, we get immediately $c_n = c_{n-1} = c_{n-2} = \cdots = c_2 = 0$. Furthermore, the boundary condition $D_{0+}^{\nu} u(1) = bD_{0+}^{\nu} u(\xi)$, $n-2 < \nu \leqslant n-1$, combining with

$$D_{0+}^{\nu} u(t) = \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha - \nu)} t^{\alpha-\nu-1} - \frac{1}{\Gamma(\alpha - \nu)} \int_0^t (t-s)^{\alpha-\nu-1} \zeta(s) \, \mathrm{d}s,$$

deduces

$$c_1 = \frac{1}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \zeta(s) \, \mathrm{d}s - \frac{b}{d\,\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-\nu-1} \zeta(s) \, \mathrm{d}s.$$

Hence,

$$\begin{aligned} u(t) = {}& -\int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \zeta(s) \, \mathrm{d}s + \frac{t^{\alpha-1}}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \zeta(s) \, \mathrm{d}s \\ & - \frac{bt^{\alpha-1}}{d\,\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-\nu-1} \zeta(s) \, \mathrm{d}s. \end{aligned}$$

When $t \leqslant \xi$,

$$\begin{aligned} u(t) = {}& -\int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \zeta(s) \, \mathrm{d}s + \frac{t^{\alpha-1}}{d\,\Gamma(\alpha)} \Biggl[\Biggl(\int_0^t + \int_t^\xi + \int_\xi^1 \Biggr) (1-s)^{\alpha-\nu-1} \zeta(s) \, \mathrm{d}s \Biggr] \\ & - \frac{bt^{\alpha-1}}{d\,\Gamma(\alpha)} \Biggl[\Biggl(\int_0^t + \int_t^\xi \Biggr) (\xi - s)^{\alpha-\nu-1} \zeta(s) \, \mathrm{d}s \Biggr] \\ = {}& \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1} - bt^{\alpha-1}(\xi - s)^{\alpha-\nu-1} - d(t-s)^{\alpha-1}}{d\,\Gamma(\alpha)} \zeta(s) \, \mathrm{d}s \\ & + \int_t^\xi \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1} - bt^{\alpha-1}(\xi - s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)} \zeta(s) \, \mathrm{d}s \\ & + \int_\xi^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)} \zeta(s) \, \mathrm{d}s \\ = {}& \int_0^1 G(t,s) \zeta(s) \, \mathrm{d}s. \end{aligned}$$

When $t \geqslant \xi$,

$$\begin{aligned}
u(t) &= -\left(\int_0^\xi + \int_\xi^t\right) \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\zeta(s)\,\mathrm{d}s \\
&\quad + \frac{t^{\alpha-1}}{d\,\Gamma(\alpha)}\left[\left(\int_0^\xi + \int_\xi^t + \int_t^1\right)(1-s)^{\alpha-\nu-1}\zeta(s)\,\mathrm{d}s\right] - \frac{bt^{\alpha-1}}{d\,\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-\nu-1}\zeta(s)\,\mathrm{d}s \\
&= \int_0^\xi \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1} - bt^{\alpha-1}(\xi-s)^{\alpha-\nu-1} - d(t-s)^{\alpha-1}}{d\,\Gamma(\alpha)}\zeta(s)\,\mathrm{d}s \\
&\quad + \int_\xi^t \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1} - d(t-s)^{\alpha-1}}{d\,\Gamma(\alpha)}\zeta(s)\,\mathrm{d}s + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)}\zeta(s)\,\mathrm{d}s \\
&= \int_0^1 G(t,s)\zeta(s)\,\mathrm{d}s.
\end{aligned}$$

The proof is completed. □

In what follows, we define two operators $A\colon P_h \times P_h \to P$, $B\colon P_h \to P$ by

$$A(u,v)(t) = \int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,\mathrm{d}s, \qquad (Bu)(t) = \int_0^1 G(t,s)g\big(s,u(s)\big)\,\mathrm{d}s.$$

Setting $\zeta(t) = f(t,u(t),u(t)) + g(t,u(t))$ in Lemma 7, it is easy to prove that u is the solution of problem (1) if and only if it is a fixed point of the operator equation $u = A(u,u) + Bu$.

Lemma 8. *The function* $G(t,s)$ *defined in Lemma* 7 *satisfies the following properties*:

(i) $G(t,s) > 0$ *for* $(t,s) \in (0,1)\times(0,1)$.
(ii) *For* $(t,s) \in [0,1]\times[0,1]$,

$$\frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}[1-(1-s)^\nu]}{\Gamma(\alpha)} \leqslant G(t,s) \leqslant \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)}.$$

Proof. For the first property, when $0 \leqslant s \leqslant \min\{t,\xi\} < 1$,

$$G(t,s) = \frac{1}{d\,\Gamma(\alpha)}\big[t^{\alpha-1}(1-s)^{\alpha-\nu-1} - bt^{\alpha-1}(\xi-s)^{\alpha-\nu-1} - d(t-s)^{\alpha-1}\big]$$

$$
\begin{aligned}
&= \frac{1}{d\,\Gamma(\alpha)}\biggl\{t^{\alpha-1}\bigl(d+b\xi^{\alpha-\nu-1}\bigr)(1-s)^{\alpha-\nu-1} \\
&\quad - bt^{\alpha-1}\biggl[\xi\biggl(1-\frac{s}{\xi}\biggr)\biggr]^{\alpha-\nu-1} - d\biggl[t\biggl(1-\frac{s}{t}\biggr)\biggr]^{\alpha-1}\biggr\} \\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\biggl[(1-s)^{\alpha-\nu-1} - \biggl(1-\frac{s}{t}\biggr)^{\alpha-1}\biggr] \\
&\quad + \frac{bt^{\alpha-1}\xi^{\alpha-\nu-1}}{d\,\Gamma(\alpha)}\biggl[(1-s)^{\alpha-\nu-1} - \biggl(1-\frac{s}{\xi}\biggr)^{\alpha-\nu-1}\biggr] \\
&> \frac{t^{\alpha-1}}{\Gamma(\alpha)}\bigl[(1-s)^{\alpha-1} - (1-s)^{\alpha-1}\bigr] \\
&\quad + \frac{bt^{\alpha-1}\xi^{\alpha-\nu-1}}{d\,\Gamma(\alpha)}\bigl[(1-s)^{\alpha-\nu-1} - (1-s)^{\alpha-\nu-1}\bigr] \\
&= 0.
\end{aligned}
$$

By a similar argument, when $0<\xi\leqslant s\leqslant t\leqslant 1$; $0\leqslant t\leqslant s\leqslant \xi<1$; $0\leqslant \max\{t,\xi\}\leqslant s\leqslant 1$, we can deduce $G(t,s)>0$. Therefore, we get that $G(t,s)>0$ for any $t,s\in(0,1)$.

Next, we will prove the second property. From the fact that $0\leqslant b\leqslant 1$, $0<d=1-b\xi^{\alpha-\nu-1}\leqslant 1$, we easily obtain

$$
G(t,s)\leqslant \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)} \quad \forall (t,s)\in[0,1]\times[0,1].
$$

Further, we need to prove $G(t,s)\geqslant t^{\alpha-1}(1-s)^{\alpha-\nu-1}[1-(1-s)^{\nu}]/\Gamma(\alpha)$.

When $0\leqslant s\leqslant \min\{t,\xi\}<1$,

$$
\begin{aligned}
G(t,s) &= \frac{1}{d\,\Gamma(\alpha)}\bigl[t^{\alpha-1}(1-s)^{\alpha-\nu-1} - bt^{\alpha-1}(\xi-s)^{\alpha-\nu-1} - d(t-s)^{\alpha-1}\bigr] \\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)} - \frac{t^{\alpha-1}b\xi^{\alpha-\nu-1}(1-\frac{s}{\xi})^{\alpha-\nu-1}}{d\,\Gamma(\alpha)} - \frac{t^{\alpha-1}(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)} \\
&\geqslant t^{\alpha-1}\biggl[\frac{(1-s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)} - \frac{b\xi^{\alpha-\nu-1}(1-s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\biggr] \\
&= t^{\alpha-1}\biggl[\frac{(1-s)^{\alpha-\nu-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\biggr] = \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}[1-(1-s)^{\nu}]}{\Gamma(\alpha)}.
\end{aligned}
$$

When $0<\xi\leqslant s\leqslant t\leqslant 1$, by using $0<d=1-b\xi^{\alpha-\nu-1}\leqslant 1$, we derive

$$
\begin{aligned}
G(t,s) &= \frac{1}{d\,\Gamma(\alpha)}\bigl[t^{\alpha-1}(1-s)^{\alpha-\nu-1} - d(t-s)^{\alpha-1}\bigr] \\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\biggl[\frac{(1-s)^{\alpha-\nu-1}}{d} - \biggl(1-\frac{s}{t}\biggr)^{\alpha-1}\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{(1-s)^{\alpha-\nu-1}}{d}-(1-s)^{\alpha-1}\right] \geqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-\nu-1}-(1-s)^{\alpha-1}\right]\\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}[1-(1-s)^{\nu}]}{\Gamma(\alpha)}.
\end{aligned}
$$

When $0 \leqslant t \leqslant s \leqslant \xi < 1$, by $G(t,s) > 0$, we have $0 < 1-(1-s)^{\nu} \leqslant 1$. Combining with $0 < d \leqslant 1$, we obtain

$$
\begin{aligned}
G(t,s) &= \frac{1}{d\,\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-\nu-1} - bt^{\alpha-1}(\xi-s)^{\alpha-\nu-1}\right]\\
&= \frac{t^{\alpha-1}}{d\,\Gamma(\alpha)}\left[(1-s)^{\alpha-\nu-1} - b\xi^{\alpha-\nu-1}\left(1-\frac{s}{\xi}\right)^{\alpha-\nu-1}\right]\\
&\geqslant \frac{t^{\alpha-1}}{d\,\Gamma(\alpha)}\left[(1-s)^{\alpha-\nu-1} - b\xi^{\alpha-\nu-1}(1-s)^{\alpha-\nu-1}\right]\\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha)}(1-s)^{\alpha-\nu-1} \geqslant \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}[1-(1-s)^{\nu}]}{\Gamma(\alpha)}.
\end{aligned}
$$

When $0 \leqslant \max\{t,\xi\} \leqslant s \leqslant 1$, by $0 < d \leqslant 1$ and $0 < 1-(1-s)^{\nu} \leqslant 1$,

$$
\begin{aligned}
G(t,s) &= \frac{1}{d\,\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-\nu-1}\right] \geqslant \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-\nu-1}\right]\\
&\geqslant \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}[1-(1-s)^{\nu}]}{\Gamma(\alpha)}.
\end{aligned}
$$

From above, for $(t,s) \in [0,1]\times[0,1]$,

$$
\frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}[1-(1-s)^{\nu}]}{\Gamma(\alpha)} \leqslant G(t,s) \leqslant \frac{t^{\alpha-1}(1-s)^{\alpha-\nu-1}}{d\,\Gamma(\alpha)}.
$$

The proof is completed. □

Now, we mention our first main theorem in this paper, which is concerned with the existence-uniqueness of positive solution to problem (1).

Theorem 1. *Assume that*:

(L1) $f\colon (0,1)\times[0,+\infty)\times(0,+\infty)\to[0,+\infty)$ *and* $g\colon (0,1)\times[0,+\infty)\to[0,+\infty)$ *are continuous, and* $f(t,u,v)$, $g(t,u)$ *may be singular at* $t=0$ *or* 1, *and* $f(t,u,v)$ *also may be singular at* $v=0$.

(L2) $f(t,u,v)$ *is increasing in* $u\in[0,+\infty)$ *for fixed* $t\in(0,1)$ *and* $v\in(0,+\infty)$, *decreasing in* $v\in(0,+\infty)$ *for fixed* $t\in(0,1)$ *and* $u\in[0,+\infty)$, *and* $g(t,u)$ *is increasing in* $u\in[0,+\infty)$ *for fixed* $t\in(0,1)$.

(L3) *There exists a constant* $\beta\in(0,1)$ *such that, for all* $\lambda\in(0,1)$, $t\in(0,1)$, $u\in[0,+\infty)$, $v\in(0,+\infty)$,

$$
f\bigl(t,\lambda u,\lambda^{-1}v\bigr) \geqslant \lambda^{\beta} f(t,u,v). \tag{6}
$$

For all $\lambda \in (0,1)$, $t \in (0,1)$, $u \in [0,+\infty)$,

$$g(t,\lambda u) \geqslant \lambda g(t,u). \tag{7}$$

(L4) $f(t,1,1) \not\equiv 0$, $g(t,1) \not\equiv 0$, *and*

$$\int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s < +\infty,$$

$$\int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\alpha-1}} g(s,1)\,\mathrm{d}s < +\infty.$$

(L5) *There exists a constant* $\delta_0 > 0$ *such that, for* $t \in (0,1)$, $u \in [0,+\infty)$, $v \in (0,+\infty)$,

$$f(t,u,v) \geqslant \delta_0 g(t,u).$$

Then:

(i) *The three-point boundary value problem* (1) *has a unique positive solution* u^*, *which satisfies* $1/\mu t^{\alpha-1} \leqslant u^* \leqslant \mu t^{\alpha-1}$, *where*

$$\begin{aligned}\mu > \max\Biggl\{&\eta, \frac{\eta^\beta}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s,\\ &\left(\frac{1}{\eta^\beta \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1}[1-(1-s)^\nu] s^{\beta(\alpha-1)} f(s,1,1)\,\mathrm{d}s\right)^{-1},\\ &\frac{\eta}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\alpha-1}} g(s,1)\,\mathrm{d}s,\\ &\left(\frac{1}{\eta \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1}\bigl[1-(1-s)^\nu\bigr] s^{\alpha-1} g(s,1)\,\mathrm{d}s\right)^{-1}\Biggr\}\end{aligned} \tag{8}$$

with $\eta \geqslant 1$.

(ii) *There exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \preccurlyeq u_0 \prec v_0$ *and*

$$u_0(t) \leqslant \int_0^1 G(t,s)\bigl[f\bigl(s,u_0(s),v_0(s)\bigr) + g\bigl(s,u_0(s)\bigr)\bigr]\,\mathrm{d}s, \quad t \in [0,1],$$

$$v_0(t) \geqslant \int_0^1 G(t,s)\bigl[f\bigl(s,v_0(s),u_0(s)\bigr) + g\bigl(s,v_0(s)\bigr)\bigr]\,\mathrm{d}s, \quad t \in [0,1],$$

where $h(t) = t^{\alpha-1}$, $t \in [0,1]$.

(iii) *For any $u_0, v_0 \in P_h$, constructing successively the sequences*

$$u_{n+1}(t) = \int_0^1 G(t,s)\big[f\big(s, u_n(s), v_n(s)\big) + g\big(s, u_n(s)\big)\big]\,\mathrm{d}s, \quad n = 0, 1, \dots,$$

$$v_{n+1}(t) = \int_0^1 G(t,s)\big[f\big(s, v_n(s), u_n(s)\big) + g\big(s, v_n(s)\big)\big]\,\mathrm{d}s, \quad n = 0, 1, \dots,$$

we have $\|u_n - u^*\| \to 0$ and $\|v_n - u^*\| \to 0$ as $n \to \infty$.

Proof. Firstly, we will prove operator $A\colon P_h \times P_h \to P_h$ is a mixed monotone operator and operator $B\colon P_h \to P_h$ is an increasing operator.

By (L3), for all $\lambda \in (0,1)$, $t \in (0,1)$, $u \in [0,+\infty)$, $v \in (0,+\infty)$, there exists $\beta \in (0,1)$, and one has

$$f(t,u,v) = f\big(t, \lambda\lambda^{-1}u, \lambda^{-1}\lambda v\big) \geqslant \lambda^{\beta} f\big(t, \lambda^{-1}u, \lambda v\big),$$
$$g(t,u) = g\big(t, \lambda\lambda^{-1}u\big) \geqslant \lambda g\big(t, \lambda^{-1}u\big),$$

from which we have

$$f\big(t, \lambda^{-1}u, \lambda v\big) \leqslant \frac{1}{\lambda^{\beta}} f(t,u,v) \quad \forall t \in (0,1),\ u \in [0,+\infty),\ v \in (0,+\infty), \tag{9}$$

$$g(t, \lambda^{-1}u) \leqslant \frac{1}{\lambda} g(t,u) \quad \forall t \in (0,1),\ u \in [0,+\infty). \tag{10}$$

Substituting $u = v = 1$ in (6) and (9), we derive

$$\begin{aligned} f\big(t, \lambda, \lambda^{-1}\big) &\geqslant \lambda^{\beta} f(t,1,1) \quad \forall t \in (0,1),\ \lambda \in (0,1), \\ f\big(t, \lambda^{-1}, \lambda\big) &\leqslant \frac{1}{\lambda^{\beta}} f(t,1,1) \quad \forall t \in (0,1),\ \lambda \in (0,1). \end{aligned} \tag{11}$$

Taking $u = 1$ in (7) and (10) respectively, we obtain

$$g(t,\lambda) \geqslant \lambda g(t,1), \quad g\big(t, \lambda^{-1}\big) \leqslant \frac{1}{\lambda} g(t,1) \quad \forall t \in (0,1),\ \lambda \in (0,1). \tag{12}$$

Next, we consider the function defined by $h(t) = t^{\alpha-1}$ for all $t \in [0,1]$. For any $x, y \in P_h$, we can choose a constant $\mu = \eta \geqslant 1$ such that $(1/\eta)h(t) \leqslant x(t), y(t) \leqslant \eta h(t)$ for all $t \in (0,1)$. From (L2), (6), (9), and (11), for $t \in (0,1)$, we obtain

$$\begin{aligned} f\big(t, x(t), y(t)\big) &\leqslant f\big(t, \eta h(t), \eta^{-1}h(t)\big) \leqslant f\big(t, \eta\big(h(t)\big)^{-1}, \eta^{-1}h(t)\big) \\ &\leqslant \frac{1}{(h(t))^{\beta}} f\big(t, \eta, \eta^{-1}\big) \leqslant \frac{\eta^{\beta}}{(h(t))^{\beta}} f(t,1,1) \\ &= \frac{\eta^{\beta}}{t^{\beta(\alpha-1)}} f(t,1,1). \end{aligned} \tag{13}$$

$$\begin{aligned} f\big(t, x(t), y(t)\big) &\geqslant f\big(t, \eta^{-1}h(t), \eta h(t)\big) \geqslant f\big(t, \eta^{-1}h(t), \eta\big(h(t)\big)^{-1}\big) \\ &\geqslant \big(h(t)\big)^{\beta} f\big(t, \eta^{-1}, \eta\big) \geqslant \frac{(h(t))^{\beta}}{\eta^{\beta}} f(t,1,1) \\ &= \frac{t^{\beta(\alpha-1)}}{\eta^{\beta}} f(t,1,1). \end{aligned} \tag{14}$$

Also, for any $t \in (0,1)$, it follows from (L2), (7), (10), and (12) that

$$\begin{aligned} g\big(t, x(t)\big) &\leqslant g\big(t, \eta h(t)\big) \leqslant g\big(t, \eta\big(h(t)\big)^{-1}\big) \leqslant \frac{1}{h(t)} g(t,\eta) \\ &\leqslant \frac{\eta}{h(t)} g(t,1) = \frac{\eta}{t^{\alpha-1}} g(t,1), \end{aligned} \tag{15}$$

$$\begin{aligned} g\big(t, x(t)\big) &\geqslant g\big(t, \eta^{-1}h(t)\big) \geqslant h(t) g\big(t, \eta^{-1}\big) \geqslant \frac{h(t)}{\eta} g(t,1) \\ &= \frac{t^{\alpha-1}}{\eta} g(t,1). \end{aligned} \tag{16}$$

Making use of Lemma 8, (L4), (13), and (15), we obtain

$$\begin{aligned} A(x,y)(t) &= \int_0^1 G(t,s) f\big(s, x(s), y(s)\big)\,\mathrm{d}s \\ &\leqslant \frac{t^{\alpha-1}\eta^{\beta}}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s < +\infty, \\ (Bx)(t) &= \int_0^1 G(t,s) g\big(s, x(s)\big)\,\mathrm{d}s \\ &\leqslant \frac{t^{\alpha-1}\eta}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\alpha-1}} g(s,1)\,\mathrm{d}s < +\infty. \end{aligned}$$

Then Lemma 8 implies that $A\colon P_h \times P_h \to P$, $B\colon P_h \to P$ are well defined.

In order to prove $A\colon P_h \times P_h \to P_h$, $B\colon P_h \to P_h$, we take $t \in [0,1]$, $u, v \in P_h$. Then it follows from Lemma 8 and (13) that

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s) f\big(s, u(s), v(s)\big)\,\mathrm{d}s \leqslant \eta^{\beta} \int_0^1 G(t,s) \frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s \\ &\leqslant \frac{t^{\alpha-1}\eta^{\beta}}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s. \end{aligned}$$

Considering Lemma 8 and (14), we deduce

$$
\begin{aligned}
A(u,v)(t) &\geqslant \frac{1}{\eta^\beta}\int_0^1 G(t,s)s^{\beta(\alpha-1)}f(s,1,1)\,\mathrm{d}s \\
&\geqslant \frac{t^{\alpha-1}}{\eta^\beta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\bigl[1-(1-s)^\nu\bigr]s^{\beta(\alpha-1)}f(s,1,1)\,\mathrm{d}s.
\end{aligned}
$$

Similarly, for all $t\in[0,1]$, $u\in P_h$, it follows from (15), (16) and Lemma 8 that

$$
\begin{aligned}
(Bu)(t) &= \int_0^1 G(t,s)g\bigl(s,u(s)\bigr)\,\mathrm{d}s \leqslant \eta\int_0^1 G(t,s)\frac{1}{s^{\alpha-1}}g(s,1)\,\mathrm{d}s \\
&\leqslant \frac{t^{\alpha-1}\eta}{d\,\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\alpha-1}}g(s,1)\,\mathrm{d}s, \\
(Bu)(t) &\geqslant \frac{1}{\eta}\int_0^1 G(t,s)s^{\alpha-1}g(s,1)\,\mathrm{d}s \\
&\geqslant \frac{t^{\alpha-1}}{\eta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\bigl[1-(1-s)^\nu\bigr]s^{\alpha-1}g(s,1)\,\mathrm{d}s.
\end{aligned}
$$

Let $\mu\geqslant 1$ be a constant such that (8), then we can easily obtain

$$
\frac{1}{\mu}h(t)=\frac{1}{\mu}t^{\alpha-1}\leqslant A(u,v)(t),\quad Bu(t)\leqslant \mu t^{\alpha-1}=\mu h(t),\quad t\in[0,1],
$$

which means that $A(u,v)\in P_h$, $Bu\in P_h$, and we prove that $A\colon P_h\times P_h\to P_h$, $B\colon P_h\to P_h$.

Next, we shall show that $A\colon P_h\times P_h\to P_h$ is a mixed monotone operator, $B\colon P_h\to P_h$ is an increasing operator. In fact, for any $u_i,v_i\in P_h$, $i=1,2$, with $u_1\succcurlyeq u_2$, $v_1\preccurlyeq v_2$, we know that $u_1(t)\geqslant u_2(t)$, $v_1(t)\leqslant v_2(t)$, $t\in[0,1]$, and by (L2) and Lemma 8,

$$
\begin{aligned}
A(u_1,v_1)(t) &= \int_0^1 G(t,s)f\bigl(s,u_1(s),v_1(s)\bigr)\,\mathrm{d}s \geqslant \int_0^1 G(t,s)f\bigl(s,u_2(s),v_2(s)\bigr)\,\mathrm{d}s \\
&= A(u_2,v_2)(t),
\end{aligned}
$$

which means that $A(u_1,v_1)\succcurlyeq A(u_2,v_2)$, that is, $A\colon P_h\times P_h\to P_h$ is a mixed monotone operator. Also, for any $u,v\in P_h$ with $u\preccurlyeq v$, we have $u(t)\leqslant v(t)$, $t\in[0,1]$. It follows from (L2) and Lemma 8 that

$$Bu(t) = \int_0^1 G(t,s)g\big(s,u(s)\big)\,\mathrm{d}s \leqslant \int_0^1 G(t,s)g\big(s,v(s)\big)\,\mathrm{d}s = Bv(t),$$

which implies that $Bu \preccurlyeq Bv$. Hence, $B\colon P_h \to P_h$ is an increasing operator.

Secondly, we check the operator A satisfies (2) and operator B is a subhomogeneous operator. In fact, for any $t \in [0,1]$, $\lambda \in (0,1)$, and $u, v \in P_h$, by (L3) and Lemma 8, we deduce

$$\begin{aligned} A(\lambda u, \lambda^{-1}v)(t) &= \int_0^1 G(t,s)f\big(s, \lambda u(s), \lambda^{-1}v(s)\big)\,\mathrm{d}s \\ &\geqslant \lambda^{\beta} \int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,\mathrm{d}s = \lambda^{\beta}A(u,v)(t), \end{aligned}$$

that is,

$$A\big(\lambda u, \lambda^{-1}v\big) \succcurlyeq \lambda^{\beta}A(u,v), \quad \lambda \in (0,1),\ u,v \in P_h,$$

and, for any $t \in [0,1]$, $\lambda \in (0,1)$, $u \in P_h$, we have

$$B(\lambda u)(t) = \int_0^1 G(t,s)g\big(s,\lambda u(s)\big)\,\mathrm{d}s \geqslant \lambda \int_0^1 G(t,s)g\big(t,u(s)\big)\,\mathrm{d}s = \lambda Bu(t),$$

thus we get

$$B(\lambda u) \succcurlyeq \lambda Bu, \quad \lambda \in (0,1),\ u \in P_h,$$

which implies that B is a subhomogeneous operator.

Thirdly, it follows from Lemma 8 and (L5) that for any $u, v \in P_h$ and $t \in [0,1]$,

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,\mathrm{d}s \geqslant \delta_0 \int_0^1 G(t,s)g\big(s,u(s)\big)\,\mathrm{d}s \\ &= \delta_0 Bu(t). \end{aligned}$$

So, we get $A(u,v) \geqslant \delta_0 Bu$. This proves assumption (3).

Finally, an application of Lemma 3 implies that the boundary value problem (1) has a unique positive solution u^* in P_h, which means there exists $\mu \geqslant 1$ (see (8)) such that $t^{\alpha-1}/\mu \leqslant u^*(t) \leqslant \mu t^{\alpha-1}$. So, the property of unique positive solution is more clear. Moreover, we obtain other desired results (ii)–(iii). The proof is completed. □

Corollary 1. *Let* $g(t,u) \equiv 0$. *Assume that*:

(M1) $f\colon (0,1) \times [0,+\infty) \times (0,+\infty) \to [0,+\infty)$ *is continuous and may be singular at* $t = 0$ *or* 1 *and* $v = 0$.

(M2) *$f(t,u,v)$ is increasing in $u \in [0,+\infty)$ for fixed $t \in (0,1)$ and $v \in (0,+\infty)$, decreasing in $v \in (0,+\infty)$ for fixed $t \in (0,1)$ and $u \in [0,+\infty)$.*

(M3) *There exists a constant $\beta \in (0,1)$ such that, for all $\lambda \in (0,1)$, $t \in (0,1)$, $u \in [0,+\infty)$, $v \in (0,+\infty)$,*

$$f\bigl(t, \lambda u, \lambda^{-1} v\bigr) \geqslant \lambda^{\beta} f(t,u,v).$$

(M4) *$f(t,1,1) \not\equiv 0$ and*

$$\int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s < +\infty.$$

Then the fractional boundary value problem

$$\begin{aligned}
&-D_{0+}^{\alpha}(t) = f\bigl(t, u(t), u(t)\bigr), \quad 0<t<1,\ n-1<\alpha \leqslant n,\\
&u^{i}(0) = 0, \quad i = 0,\dots,n-2,\\
&D_{0+}^{\nu} u(1) = b D_{0+}^{\nu} u(\xi), \quad n-2<\nu \leqslant n-1,
\end{aligned}$$

has a unique positive solution u^* in P_h in which μ is a positive constant defined by

$$\begin{aligned}
\mu > \max\Biggl\{1, &\left(\frac{1}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s\right)^{1/(1-\beta)},\\
&\left(\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1}\bigl[1-(1-s)^{\nu}\bigr] s^{\beta(\alpha-1)} f(s,1,1)\,\mathrm{d}s\right)^{1/(\beta-1)}\Biggr\}.
\end{aligned}$$

Besides, for any $x_0, y_0 \in P_h$, $h(t) = t^{\alpha-1}$, constructing successively the sequences

$$\begin{aligned}
u_{n+1}(t) &= \int_0^1 G(t,s) f\bigl(s, u_n(s), v_n(s)\bigr)\,\mathrm{d}s, \quad n = 0,1,\dots,\\
v_{n+1}(t) &= \int_0^1 G(t,s) f\bigl(s, v_n(s), u_n(s)\bigr)\,\mathrm{d}s, \quad n = 0,1,\dots,
\end{aligned}$$

we have $\|u_n - u^*\| \to 0$ and $\|v_n - u^*\| \to 0$ as $n \to \infty$.

Proof. By the proof of Theorem 1, we can easily prove the above conclusions with an application of the Lemma 5. □

Remark 1. If we set the constant $b = 0$ in boundary value problem (1), the obtained equations have been studied by Jleli and Samet, but they only consider the existence and uniqueness of positive solutions for nonsingular problem. In this sense, our result generalizes and supplements that of [10].

Next, let us mention the second main result, which show the other sufficient conditions guaranteeing the existence and uniqueness of positive solution to problem (1).

Theorem 2. *Suppose that* (L1)–(L2) *hold and*:

(L6) *For all* $t \in (0,1)$*,* $\lambda \in (0,1)$*,* $u \in [0,+\infty)$*,* $v \in (0,+\infty)$ *such that*

$$f\bigl(t, \lambda u, \lambda^{-1} v\bigr) \geqslant \lambda f(t,u,v), \tag{17}$$

there exists a constant $\beta \in (0,1)$ *such that, for* $\lambda \in (0,1)$*,* $t \in (0,1)$*,* $u \in [0,+\infty)$*,*

$$g(t, \lambda u) \geqslant \lambda^{\beta} g(t,u). \tag{18}$$

(L7) *There exists a constant* $\delta_0' > 0$ *such that, for* $t \in (0,1)$*,* $u \in [0,+\infty)$*,* $v \in (0,+\infty)$*,*

$$f(t,u,v) \leqslant \delta_0' g(t,u).$$

(L8) $f(t,1,1) \not\equiv 0$*,* $g(t,1) \not\equiv 0$*, and*

$$\int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\alpha-1}} f(s,1,1)\,\mathrm{d}s < +\infty,$$

$$\int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\beta(\alpha-1)}} g(s,1)\,\mathrm{d}s < +\infty.$$

Then the three-point boundary value problem (1) *has a unique positive solution* u^* *in* P_h *in which* $h(t) = t^{\alpha-1}$*, and conclusions* (ii)–(iii) *in Theorem* 1 *also hold.*

Proof. By a routine argument similar to the proof in Theorem 1, it follows from Lemma 8, (L2), (L6), and (L7) that A is a mixed monotone operator and satisfies (4), B is an increasing β-concave operator, and operators A, B satisfy (5) in Lemma 4.

In the sequel, we will prove $A\colon P_h \times P_h \to P_h$, $B\colon P_h \to P_h$. By (17) and (18), for all $\lambda \in (0,1)$, there exists $\beta \in (0,1)$ such that

$$f\bigl(t, \lambda^{-1} u, \lambda v\bigr) \leqslant \frac{1}{\lambda} f(t,u,v) \quad \forall t \in (0,1),\ u \in [0,+\infty),\ v \in (0,+\infty), \tag{19}$$

$$g\bigl(t, \lambda^{-1} u\bigr) \leqslant \frac{1}{\lambda^{\beta}} g(t,u) \quad \forall t \in (0,1),\ u \in [0,+\infty). \tag{20}$$

If $u = v = 1$, we rewrite inequalities (17)–(20) as follows:

$$\begin{aligned} f\bigl(t, \lambda, \lambda^{-1}\bigr) &\geqslant \lambda f(t,1,1) \quad \forall t \in (0,1),\ \lambda \in (0,1), \\ f\bigl(t, \lambda^{-1}, \lambda\bigr) &\leqslant \frac{1}{\lambda} f(t,1,1) \quad \forall t \in (0,1),\ \lambda \in (0,1), \end{aligned} \tag{21}$$

and

$$g(t,\lambda) \geqslant \lambda^{\beta} g(t,1), \quad g\bigl(t, \lambda^{-1}\bigr) \leqslant \frac{1}{\lambda^{\beta}} g(t,1) \quad \forall t \in (0,1),\ \lambda \in (0,1). \tag{22}$$

For any $x, y \in P_h$, here we define $h(t) = t^{\alpha-1}$, we can choose a constant $\mu = \kappa \geqslant 1$ be such that $h(t)/\kappa \leqslant x(t), y(t) \leqslant \kappa h(t)$ for all $t \in (0,1)$. By (L2) and (17), (19), (21), we see that

$$\begin{aligned} f\big(t,x(t),y(t)\big) &\leqslant f\big(t,\kappa h(t),\kappa^{-1}h(t)\big) \leqslant f\big(t,\kappa\big(h(t)\big)^{-1},\kappa^{-1}h(t)\big) \\ &\leqslant \frac{1}{h(t)} f\big(t,\kappa,\kappa^{-1}\big) \leqslant \frac{\kappa}{h(t)} f(t,1,1) \\ &= \frac{\kappa}{t^{\alpha-1}} f(t,1,1), \quad t \in (0,1). \end{aligned} \tag{23}$$

$$\begin{aligned} f\big(t,x(t),y(t)\big) &\geqslant f\big(t,\kappa^{-1}h(t),\kappa h(t)\big) \geqslant f\big(t,\kappa^{-1}h(t),\kappa\big(h(t)\big)^{-1}\big) \\ &\geqslant h(t) f\big(t,\kappa^{-1},\kappa\big) \geqslant \frac{h(t)}{\kappa} f(t,1,1) \\ &= \frac{t^{\alpha-1}}{\kappa} f(t,1,1), \quad t \in (0,1). \end{aligned} \tag{24}$$

Similarly, by (L2), (18), (20), (22) we deduce

$$g\big(t,x(t)\big) \leqslant \frac{\kappa^{\beta}}{t^{\beta(\alpha-1)}} g(t,1), \quad t \in (0,1), \tag{25}$$

$$g\big(t,x(t)\big) \geqslant \frac{t^{\beta(\alpha-1)}}{\kappa^{\beta}} g(t,1), \quad t \in (0,1). \tag{26}$$

From Lemma 8, (23), (25), and (L8) we have that $A\colon P_h \times P_h \to P$, $B\colon P_h \to P$ are well defined. For all $t \in [0,1]$, $u, v \in P_h$, from Lemma 8 and (23)–(26) we have

$$A(u,v)(t) \leqslant \frac{t^{\alpha-1}\kappa}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\alpha-1}} f(s,1,1)\,\mathrm{d}s,$$

$$A(u,v)(t) \geqslant \frac{t^{\alpha-1}}{\kappa\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1}\big[1-(1-s)^{\nu}\big] s^{\alpha-1} f(s,1,1)\,\mathrm{d}s,$$

and

$$Bu(t) \leqslant \frac{t^{\alpha-1}\kappa^{\beta}}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\beta(\alpha-1)}} g(s,1)\,\mathrm{d}s,$$

$$Bu(t) \geqslant \frac{t^{\alpha-1}}{\kappa^{\beta}\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1}\big[1-(1-s)^{\nu}\big] s^{\beta(\alpha-1)} g(s,1)\,\mathrm{d}s.$$

Let $\mu \geqslant 1$ be a positive constant defined by

$$\mu > \max\Biggl\{\kappa, \frac{\kappa}{d\,\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1} \frac{1}{s^{\alpha-1}} f(s,1,1)\,\mathrm{d}s, \\ \Biggl(\frac{1}{\kappa\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\nu-1}\big[1-(1-s)^{\nu}\big] s^{\alpha-1} f(s,1,1)\,\mathrm{d}s\Biggr)^{-1},$$

$$\frac{\kappa^\beta}{d\,\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\beta(\alpha-1)}}g(s,1)\,\mathrm{d}s,$$
$$\left(\frac{1}{\kappa^\beta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\bigl[1-(1-s)^\nu\bigr]s^{\beta(\alpha-1)}g(s,1)\,\mathrm{d}s\right)^{-1}\Biggr\}. \tag{27}$$

Then we get

$$\frac{1}{\mu}h(t)=\frac{1}{\mu}t^{\alpha-1}\leqslant A(u,v)(t),\quad Bu(t)\leqslant \mu t^{\alpha-1}=\mu h(t),\quad t\in[0,1],$$

which means $A(u,v)\in P_h$, $Bu\in P_h$. Consequently, $A\colon P_h\times P_h\to P_h$, $B\colon P_h\to P_h$.

Owing to Lemma 4, we can obtain that the boundary value problem (1) has a unique positive solution u^* in P_h, where $h(t)=t^{\alpha-1}$, $\mu\geqslant 1$, satisfies (27). Moreover, we also deduce other desired conclusions (ii)–(iii) in Theorem 1. The proof is completed. □

By means of Lemma 6 and similar to the proof of Theorem 2, we can easily prove the following conclusion.

Corollary 2. *By choosing $f(t,u,u)\equiv 0$, we may actually assume that g satisfies the following conditions*:

(N1) $g\colon (0,1)\times[0,+\infty)\to[0,+\infty)$ *is continuous and may be singular at* $t=0$ *or* 1.

(N2) $g(t,u)$ *is increasing in* $u\in[0,+\infty)$ *for fixed* $t\in(0,1)$.

(N3) *There exists a constant* $\beta\in(0,1)$ *such that*

$$g(t,\lambda u)\geqslant \lambda^\beta g(t,u)\quad \lambda\in(0,1),\ t\in(0,1),\ u\in[0,+\infty).$$

(N4) $g(t,1)\not\equiv 0$ *and*

$$\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\beta(\alpha-1)}}g(s,1)\,\mathrm{d}s<+\infty.$$

Then the fractional boundary value problem

$$\begin{aligned}
&-D_{0+}^{\alpha}(t)=g\bigl(t,u(t)\bigr),\quad 0<t<1,\ n-1<\alpha\leqslant n,\\
&u^i(0)=0,\quad i=0,\dots,n-2,\\
&D_{0+}^{\nu}u(1)=bD_{0+}^{\nu}u(\xi),\quad n-2<\nu\leqslant n-1,
\end{aligned}$$

has a unique positive solution u^* in P_h in which $h(t)=t^{\alpha-1}$, and μ is a positive constant defined by

$$\mu>\max\Biggl\{1,\left(\frac{1}{d\,\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\beta(\alpha-1)}}g(s,1)\,\mathrm{d}s\right)^{1/(1-\beta)},$$
$$\left(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\bigl[1-(1-s)^\nu\bigr]s^{\beta(\alpha-1)}g(s,1)\,\mathrm{d}s\right)^{1/(\beta-1)}\Biggr\}.$$

Remark 2. If $n = 2$, $f(t,u,u) \equiv 0$ in problem (1), which we may assume, then this construction is previously studied in [12]. Moreover, the literature [12] only concerned with the existence and multiplicity of positive solutions. So, our unique existence result generalizes and improves the results in [12].

Remark 3. The equations studied in this paper are the generalization of the equations in [13]. It is easily seen that the boundary value problem (1) reduces to the one in [13] if we set $n = 4$, $\nu = 2$, and $f(t,u,u) \equiv 0$. Besides, we provide some alternative approaches to study the existence and uniqueness of positive solution for singular fractional differential equations, which generalizes and improves the known results including nonsingular and singular problems.

4 Examples

To illustrate the main results, two examples are given as follows.

Example 1. Consider the following boundary value problem:

$$\begin{aligned} &D_{0^+}^{5/2}u(t) = t^{3/4}\sqrt{u} + \frac{t^{1/4}}{\sqrt{u(t)}} + a(t)t^{3/2}\frac{u(t)}{1+u(t)} + ct^{3/2} + mt^{3/4}\bigl(1 - t^{3/4}\bigr), \\ &u(0) = 0, \quad u'(0) = 0, \qquad D_{0^+}^{3/2}u(1) = \frac{1}{2}D_{0^+}^{3/2}u\left(\frac{1}{4}\right), \end{aligned} \tag{28}$$

where $c, m > 0$ are constants, $a\colon (0,1) \to [0,+\infty)$ is a continuous function with $a \not\equiv 0$.

Obviously, problem (28) fits the framework of problem (1). We take $c > m > 0$, $b = 1/2$, $\xi = 1/4$, $\alpha = 5/2$, and $\nu = 3/2$ such that $\alpha - \nu - 1 = 0 \geqslant 0$, and $b\xi^{\alpha-\nu-1} = 1/2 \in [0,1)$. Besides, set $\beta = 1/2$, constant $n \geqslant \max\{a(t)\colon t \in (0,1)\}$.

$$f(t,u,v) = t^{3/4}\left(\sqrt{u} + \frac{1}{\sqrt{tv}} + m\right), \qquad g(t,u) = t^{3/2}\left(\frac{u}{1+u}a(t) + c - m\right).$$

Next, let us check that all the required conditions of Theorem 1 are satisfied. Clearly, the functions $f\colon (0,1) \times [0,+\infty) \times (0,+\infty) \to [0,+\infty)$ and $g\colon (0,1) \times [0,+\infty) \to [0,+\infty)$ are continuous, and we can easily observe that $f(t,u,v)$ is increasing in $u \in [0,+\infty)$ for fixed $t \in (0,1)$ and $v \in (0,+\infty)$, decreasing in $v \in (0,+\infty)$ for fixed $t \in (0,1)$ and $u \in [0,+\infty)$, and $g(t,u)$ is increasing in $u \in [0,+\infty)$ for fixed $t \in (0,1)$. For all $\lambda \in (0,1)$, $t \in (0,1)$, $u \in [0,+\infty)$, $v \in (0,+\infty)$, we have

$$\begin{aligned} f\bigl(t,\lambda u,\lambda^{-1}v\bigr) &= t^{3/4}\left(\sqrt{\lambda u} + \frac{1}{\sqrt{t\lambda^{-1}v}} + m\right) = \lambda^{1/2}t^{3/4}\left(\sqrt{u} + \frac{1}{\sqrt{tv}} + \lambda^{-1/2}m\right) \\ &\geqslant \lambda^{1/2}t^{3/4}\left(\sqrt{u} + \frac{1}{\sqrt{tv}} + m\right) = \lambda^{\beta}f(t,u,v), \\ g(t,\lambda u) &= t^{3/2}\left(\frac{\lambda u}{1+\lambda u}a(t) + c - m\right) \geqslant t^{3/2}\left(\frac{\lambda u}{1+\lambda u}a(t) + \lambda(c-m)\right) \\ &\geqslant \lambda t^{3/2}\left(\frac{u}{1+u}a(t) + c - m\right) = \lambda g(t,u). \end{aligned}$$

Besides,

$$f(t,1,1) = t^{3/4}\left(1 + \frac{1}{\sqrt{t}} + m\right) \not\equiv 0, \qquad g(t,1) = t^{3/2}\left(\frac{a(t)}{2} + c - m\right) \not\equiv 0,$$

and

$$\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s$$
$$= \int_0^1 (1-s)^0 \frac{1}{s^{3/4}} s^{3/4}\left(1 + \frac{1}{\sqrt{s}} + m\right)\mathrm{d}s = \int_0^1 \left(1 + \frac{1}{\sqrt{s}} + m\right)\mathrm{d}s < +\infty,$$
$$\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\alpha-1}} g(s,1)\,\mathrm{d}s$$
$$= \int_0^1 (1-s)^0 \frac{1}{s^{3/2}} s^{3/2}\left(\frac{1}{2}a(s) + c - m\right)\mathrm{d}s \leqslant \int_0^1 \left(\frac{1}{2}n + c - m\right) < +\infty.$$

Moreover, for $t \in (0,1)$, $u \in [0,+\infty)$, $v \in (0,+\infty)$, if we take $\delta_0 \in (0, m/(n+c-m)]$, we can obtain

$$f(t,u,v) = t^{3/4}\left(u^{1/2} + \frac{1}{\sqrt{tv}} + m\right) \geqslant t^{3/2} m = t^{3/2}\frac{m}{n+c-m}(n+c-m)$$
$$\geqslant \delta_0 t^{3/2}\left(\frac{u}{1+u}a(t) + c - m\right) = \delta_0 g(t,u).$$

Hence, we prove that all the typotheses of Theorem 1 are satisfied. Then we deduce that problem (28) have only one positive solution $u^* \in P_h$, where $h(t) = t^{3/2}$.

Remark 4. If we set the function $a(t) = t^2$, constants $c = 2$, $m = 1$, $n = 1$, $\eta = 1$ in Example 1, we can compute that $d = 1 - b\xi^{\alpha-\nu-1} = 1/2$ and

$$\frac{\eta^\beta}{d\,\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\beta(\alpha-1)}} f(s,1,1)\,\mathrm{d}s = \frac{2}{\Gamma(\frac{5}{2})}\int_0^1 \left(2 + s^{-1/2}\right)\mathrm{d}s = 6.02,$$
$$\frac{\eta}{d\,\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\alpha-1}} g(s,1)\,\mathrm{d}s = \frac{2}{\Gamma(\frac{5}{2})}\int_0^1 \left(\frac{1}{2}s^2 + 1\right)\mathrm{d}s = 1.76,$$
$$\left(\frac{1}{\eta^\beta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\left[1 - (1-s)^\nu\right] s^{\beta(\alpha-1)} f(s,1,1)\,\mathrm{d}s\right)^{-1}$$
$$= \left(\frac{1}{\Gamma(\frac{5}{2})}\int_0^1 \left[1 - (1-s)^{3/2}\right] s^{3/2}\left(2 + s^{-1/2}\right)\mathrm{d}s\right)^{-1} = 1.28,$$

$$\left(\frac{1}{\eta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\big[1-(1-s)^{\nu}\big]s^{\alpha-1}g(s,1)\,\mathrm{d}s\right)^{-1}$$
$$=\left(\frac{1}{\Gamma(\frac{5}{2})}\int_0^1 \big[1-(1-s)^{3/2}\big]s^3\left(\frac{1}{2}s^2+1\right)\mathrm{d}s\right)^{-1}=4.43.$$

It follows from (8) that

$$\mu>\max\{1,6.02,1.28,1.76,4.43\}.$$

Here we choose $\mu=7$. Then we can obtain the unique positive solution $x^*\in(t^{3/2}/7, 7t^{3/2})$, which makes the property of unique positive solution more clear.

Example 2. Consider the three-point boundary value problem

$$\begin{aligned}&D_{0^+}^{10/3}u(t)=\big(t^{17/6}+t^{1/12}\big)u^{1/4}(t)+t^{10/3}\arctan\frac{1}{u(t)}+\frac{\pi}{2}t^{7/12}\\&u(0)=u'(0)=u''(0)=0,\qquad D_{0^+}^{7/3}u(1)=\frac{3}{4}D_{0^+}^{7/3}u\left(\frac{1}{2}\right),\end{aligned}\tag{29}$$

If we set $\alpha=10/3$, $\nu=7/3$, $b=3/4$, $\xi=1/2$, which satisfy $\alpha-\nu-1\geqslant 0$ and $b\xi^{\alpha-\nu-1}=3/4\in[0,1)$. Then the above problem can be regarded as a boundary value problem of form (1) with

$$f(t,u,v)=t^{10/3}\left(\frac{u^{1/4}}{\sqrt{t}}+\arctan\frac{1}{v}\right),\qquad g(t,u)=t^{7/12}\left(\frac{u^{1/4}}{\sqrt{t}}+\frac{\pi}{2}\right).$$

Next, we verify that conditions (L1)–(L2) and (L6)–(L8) are satisfied. At first, it is easy to check that $f(t,u,v)$ and $g(t,u)$ satisfy conditions (L1)–(L2). Moreover, for $\lambda\in(0,1)$, $t\in(0,1)$, $u\in[0,+\infty)$, $v\in(0,+\infty)$, we have

$$\begin{aligned}f\big(t,\lambda u,\lambda^{-1}v\big)&=t^{10/3}\left(\frac{(\lambda u)^{1/4}}{\sqrt{t}}+\arctan\frac{1}{\lambda^{-1}v}\right)\geqslant t^{10/3}\left(\frac{\lambda^{1/4}u^{1/4}}{\sqrt{t}}+\lambda\arctan\frac{1}{v}\right)\\&\geqslant\lambda t^{10/3}\left(\frac{u^{1/4}}{\sqrt{t}}+\arctan\frac{1}{v}\right)=\lambda f(t,u,v),\\g(t,\lambda u)&=t^{7/12}\left(\frac{(\lambda u)^{1/4}}{\sqrt{t}}+\frac{\pi}{2}\right)\geqslant\lambda^{1/4}t^{7/12}\left(\frac{u^{1/4}}{\sqrt{t}}+\frac{\pi}{2}\right)=\lambda^{\beta}g(t,u),\end{aligned}$$

where $\beta=1/4$. Therefore, (L6) is proved. At last, take $\delta_0'\in[1,+\infty)$. For $t\in(0,1)$, $u\in[0,+\infty)$, $v\in(0,+\infty)$, we get

$$\begin{aligned}f(t,u,v)&=t^{10/3}\left(\frac{u^{1/4}}{\sqrt{t}}+\arctan\frac{1}{v}\right)\leqslant t^{10/3}\left(\frac{u^{1/4}}{\sqrt{t}}+\frac{\pi}{2}\right)\leqslant t^{7/12}\left(\frac{u^{1/4}}{\sqrt{t}}+\frac{\pi}{2}\right)\\&\leqslant\delta_0'g(t,u).\end{aligned}$$

Besides,

$$f(t,1,1) = t^{10/3}\left(\frac{1}{\sqrt{t}} + \frac{\pi}{4}\right) \not\equiv 0, \qquad g(t,1) = t^{7/12}\left(\frac{1}{\sqrt{t}} + \frac{\pi}{2}\right) \not\equiv 0,$$

and

$$\begin{aligned}
&\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\alpha-1}} f(s,1,1)\,\mathrm{d}s \\
&\quad = \int_0^1 (1-s)^0 \frac{1}{s^{7/3}} s^{10/3}\left(\frac{1}{\sqrt{s}} + \frac{\pi}{4}\right)\mathrm{d}s = \int_0^1 \left(\sqrt{s} + \frac{\pi}{4}s\right)\mathrm{d}s < +\infty, \\
&\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\beta(\alpha-1)}} g(s,1)\,\mathrm{d}s \\
&\quad = \int_0^1 (1-s)^0 \frac{1}{s^{7/12}} s^{7/12}\left(\frac{1}{\sqrt{s}} + \frac{\pi}{2}\right)\mathrm{d}s = \int_0^1 \left(\frac{1}{\sqrt{s}} + \frac{\pi}{2}\right)\mathrm{d}s < +\infty.
\end{aligned}$$

So, conditions (L7) and (L8) hold. Hence, we prove that all the conditions of Theorem 2 are satisfied. By application of Theorem 2, we obtain that boundary value problem (29) has only one positive solution $u^* \in P_h$, where $h(t) = t^{7/3}$.

Remark 5. If set $\kappa = 1$, we compute $d = 1 - b\xi^{\alpha-\nu-1} = 1/4$ and

$$\begin{aligned}
&\frac{\kappa}{d\,\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\alpha-1}} f(s,1,1)\,\mathrm{d}s = \frac{4}{\Gamma(\frac{10}{3})}\int_0^1 \left(\sqrt{s} + \frac{\pi}{4}s\right)\mathrm{d}s = 1.53, \\
&\frac{\kappa^\beta}{d\,\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\frac{1}{s^{\beta(\alpha-1)}} g(s,1)\,\mathrm{d}s = \frac{4}{\Gamma(\frac{10}{3})}\int_0^1 \left(\frac{1}{\sqrt{s}} + \frac{\pi}{2}\right)\mathrm{d}s = 5.14,
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1}{\kappa\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\left[1-(1-s)^\nu\right] s^{\alpha-1} f(s,1,1)\,\mathrm{d}s\right)^{-1} \\
&\quad = \left(\frac{1}{\Gamma(\frac{10}{3})}\int_0^1 \left[1-(1-s)^{7/3}\right] s^{17/3}\left(\frac{1}{\sqrt{s}} + \frac{\pi}{4}\right)\mathrm{d}s\right)^{-1} = 10.14, \\
&\left(\frac{1}{\kappa^\beta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\nu-1}\left[1-(1-s)^\nu\right] s^{\beta(\alpha-1)} g(s,1)\,\mathrm{d}s\right)^{-1} \\
&\quad = \left(\frac{1}{\Gamma(\frac{10}{3})}\int_0^1 \left[1-(1-s)^{7/3}\right] s^{7/6}\left(\frac{1}{\sqrt{s}} + \frac{\pi}{2}\right)\mathrm{d}s\right)^{-1} = 2.46.
\end{aligned}$$

It follows from (27) that

$$\mu > \max\{1, 1.53, 10.14, 5.14, 2.46\}.$$

If we choose $\mu = 11$, then we get the unique positive solution $x^* \in (t^{7/3}/11, 11t^{7/3})$. In this case, the property of unique positive solution is clearer.

5 Conclusions

In this paper, we research a class of higher-order nonlinear singular fractional differential equations with a nonlocal-type boundary condition. By applying the properties of Green function and some fixed point theorems for sum-type operator on cone, some existence and uniqueness results are obtained successfully for the case where the nonlinearity is allowed to be singular with respect to not only the time variable but also the space variable. Moreover, our results also show that the unique positive solution can be approximated by constructing two iterative sequences for any initial point in P_h. Finally, two interesting examples are presented to demonstrate the main results. Our study improves and generalizes some previous works about positive solutions for fractional differential equations such as [10, 12, 13].

Acknowledgment. The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript.

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