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Randomly stopped minima and maxima with exponential-type distributions

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Abstract. Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued and possibly nonidentically distributed random variables. Suppose that η is a nonnegative, nondegenerate at 0 and integer-valued random variable, which is independent of $\{\xi_1, \xi_2, \dots\}$. In this paper, we consider conditions for $\{\xi_1, \xi_2, \dots\}$ and η under which the distributions of the randomly stopped maxima and minima, as well as randomly stopped maxima of sums and randomly stopped minima of sums, belong to the class of exponential distributions.

Keywords: class of exponential distributions, counting random variable, randomly stopped maxima, randomly stopped minima, maximum of sums, minimum of sums, closure property.

1 Introduction

Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \dots\}$, and let η be a counting random variable (c.r.v.), i.e. a nonnegative, nondegenerate at 0 and integer-valued r.v. In addition, we suppose that the r.v. η and the sequence $\{\xi_1, \xi_2, \dots\}$ are independent.

- *Let* $S_0 := 0$, $S_n := \xi_1 + \xi_2 + \dots + \xi_n$ *for* $n \in \mathbb{N}$, *and let*

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the randomly stopped sum of the r.v.s $\xi_1, \xi_2, \dots$ *.*

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- *Next, let* $\xi^{(0)} := 0$, $\xi^{(n)} := \max\{0, \xi_1, \ldots, \xi_n\}$ *for* $n \in \mathbb{N}$, *and let*

$$\xi^{(\eta)} := \begin{cases} 0 & \text{if } \eta = 0, \\ \max\{0, \xi_1, \xi_2, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

be the randomly stopped maxima of the r.v.s $\xi_1, \xi_2, \ldots$.

- *Similarly, let* $\xi_{(0)} := 0$, $\xi_{(n)} := \min\{\xi_1, \xi_1, \ldots, \xi_n\}$ *for* $n \in \mathbb{N}$, *and let*

$$\xi_{(\eta)} := \begin{cases} 0 & \text{if } \eta = 0, \\ \min\{\xi_1, \xi_2, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

be the randomly stopped minima of the r.v.s $\xi_1, \xi_2, \ldots$.

- *Next, let* $S^{(n)} := \max\{S_0, S_1, \ldots, S_n\}$ *for* $n \geqslant 0$, *and let*

$$S^{(\eta)} := \max\{S_0, S_1, \ldots, S_\eta\}$$

be the randomly stopped maxima of the sums $S_0, S_1, S_2, \ldots$.

- *Similarly, let* $S_{(0)} = 0$, $S_{(n)} := \min\{S_1, S_2, \ldots, S_n\}$ *for* $n \geqslant 1$, *and let*

$$S_{(\eta)} := \begin{cases} 0 & \text{if } \eta = 0, \\ \min\{S_1, S_2, \ldots, S_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

be the randomly stopped minima of the sums $S_0, S_1, S_2, \ldots$.

We denote by $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, F_{S_η}, $F_{S_{(\eta)}}$ and $F_{S^{(\eta)}}$ the d.f.s of $\xi_{(\eta)}$, $\xi^{(\eta)}$, S_η, $S_{(\eta)}$ and $S^{(\eta)}$, respectively. Moreover, we denote by $\overline{F}$ the tail of any d.f. F, i.e. $\overline{F}(x) = 1 - F(x)$ for all $x \in \mathbb{R}$. It is obvious that the following equalities hold for all $x > 0$:

$$\overline{F}_{\xi_{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}\big(\xi_{(n)} > x\big), \qquad \overline{F}_{\xi^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}\big(\xi^{(n)} > x\big),$$

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}\big(S_n > x\big), \qquad \overline{F}_{S_{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}\big(S_{(n)} > x\big),$$

$$\overline{F}_{S^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}\big(S^{(n)} > x\big).$$

In this paper, we consider a sequence of possibly nonidentically distributed r.v.s $\{\xi_1, \xi_2, \ldots\}$. We suppose that the majority of these r.v.s belong to the class of exponential distributions and find conditions under which the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, F_{S_η}, $F_{S_{(\eta)}}$ and $F_{S^{(\eta)}}$ belong to the same class. If d.f.s F_{ξ_1}, F_{ξ_2}, … are different, then the various properties of these d.f.s and the c.r.v. η imply exponentiality of the described randomly stopped structures. Before discussing the properties of the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, F_{S_η}, $F_{S_{(\eta)}}$ and $F_{S^{(\eta)}}$, we recall the notion of exponentiality.

- *For any $\gamma \geqslant 0$, we denote by $\mathcal{L}(\gamma)$ the class of exponential-type d.f.s. It is said that $F \in \mathcal{L}(\gamma)$ if*
$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\gamma y} \quad \textit{for any } y > 0.$$
- *For $\gamma = 0$, the class $\mathcal{L}(0)$ is called the class of long-tailed distributions and denoted by $\mathcal{L}$. Consequently, $F \in \mathcal{L}$ if and only if*
$$\limsup_{x\to\infty} \frac{\overline{F}(x-1)}{\overline{F}(x)} \leqslant 1.$$

By Proposition 2.6 from [2], an absolutely continuous d.f. F belongs to the class $\mathcal{L}(\gamma)$, $\gamma \geqslant 0$, if and only if

$$\overline{F}(x) = \exp\left\{ -\int_{-\infty}^{x} \bigl(\alpha(u) + \beta(u)\bigr)\,\mathrm{d}u \right\}, \quad x \in \mathbb{R},$$

for some measurable functions α and β with $\alpha(x) + \beta(x) \geqslant 0$ for all $x \in \mathbb{R}$ such that

$$\lim_{x\to\infty} \alpha(x) = \gamma, \quad \lim_{x\to\infty} \int_{-\infty}^{x} \alpha(u)\,\mathrm{d}u = \infty \quad \text{and} \quad \lim_{x\to\infty} \int_{-\infty}^{x} \beta(u)\,\mathrm{d}u \quad \text{exists.}$$

Another representation formula, which does not require the absolute continuity of F, can be found in [27]. It states that $F \in \mathcal{L}(\gamma)$ with $\gamma \geqslant 0$ if and only if

$$\overline{F}(x) = \bar{\beta}(x) \exp\left\{ -\int_{0}^{x} \bar{\alpha}(u)\,\mathrm{d}u \right\}, \quad x \geqslant 0, \tag{1}$$

for some positive measurable functions $\bar{\alpha}$ and $\bar{\beta}$ such that

$$\lim_{x\to\infty} \bar{\alpha}(x) = \gamma \quad \text{and} \quad \lim_{x\to\infty} \bar{\beta}(x) = \bar{\beta}_0 > 0.$$

We note here that any gamma distribution belongs to the class $\mathcal{L}(\gamma)$ with some $\gamma > 0$. In particular, any exponential or Erlang distribution belongs to this class.

In addition, a d.f. from the class $\mathcal{L}(\gamma)$ can be stepped, i.e. there is such a function that describes a distribution of a discrete r.v. To be more precise, if $\gamma > 0$, we define $\overline{F}(x) = \exp(-\gamma \log\lfloor \mathrm{e}^x \rfloor)$ for all $x \geqslant 0$, where $\lfloor a \rfloor$ denotes the integer part of a real number a. It is obvious that this function F is a d.f. of a nonnegative discrete r.v. Moreover, $F \in \mathcal{L}(\gamma)$ because for any $y > 0$, we have

$$\begin{aligned}\frac{\overline{F}(x+y)}{\overline{F}(x)} &= \exp\bigl(-\gamma\bigl(\log\lfloor \mathrm{e}^{x+y} \rfloor - \log\lfloor \mathrm{e}^{x} \rfloor\bigr)\bigr) \\ &= \exp\left(-\gamma \log \frac{\mathrm{e}^{x+y} - \{\mathrm{e}^{x+y}\}}{\mathrm{e}^{x} - \{\mathrm{e}^{x}\}} \right) \underset{x\to\infty}{\longrightarrow} \mathrm{e}^{-\gamma y},\end{aligned}$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a real number a.

The class of long-tailed distributions $\mathcal{L}$ (but not the term itself) was introduced by Chistyakov [7] in the context of branching processes, whereas for $\gamma > 0$, the class $\mathcal{L}(\gamma)$ was introduced by Chover et al. [8, 9]. In this paper, we suppose that the d.f.s of the r.v.s under consideration belong to either of the following four classes:

$$\mathcal{L}, \quad \mathcal{L}(\gamma) \text{ with some } \gamma \geqslant 0, \quad \mathcal{L}^{\infty} = \bigcup_{\gamma \geqslant 0} \mathcal{L}(\gamma) \quad \text{or} \quad \mathcal{L}^{\infty}_{+} = \bigcup_{\gamma > 0} \mathcal{L}(\gamma).$$

A number of interesting and important properties of distributions from these classes can be found in the book by Foss et al. [19] and in the papers by Albin and Sundén [2], Beck et al. [4], Cheng et al. [6], Chover et al. [8, 9], Cline [10, 11], Cui et al. [12], Embrechts and Goldie [18], Klüppelberg [20], Omey et al. [23], Pakes [24, 25], Shimura and Watanabe [26], Watanabe [28], Watanabe and Yamamuro [29] and Xu et al. [30–32], Yang et al. [33], among others.

In [10, 11], Cline claimed that the d.f. F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if the r.v.s $\xi_1, \xi_2, \ldots$ are independent and identically distributed (i.i.d.) with d.f. $F_\xi \in \mathcal{L}(\gamma)$ and η is an arbitrary c.r.v. Albin [1] constructed a counterexample and showed that Cline's result is false in general. In his paper [1], Albin stated that the d.f. F_{S_η} remains in the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if the r.v.s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed with d.f. $F_\xi \in \mathcal{L}(\gamma)$ and $\mathbf{E}\,\mathrm{e}^{\delta\eta} < \infty$ for all $\delta > 0$. Watanabe and Yamamuro showed that this assertion is incorrect in the case of $\gamma > 0$ (see [29, Remark 6.1]). For $\gamma \geqslant 0$, Watanabe and Yamamuro proved the following theorem (see proof of Proposition 6.1 in [29]).

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. r.v.s distributed on $\mathbb{R}$ with d.f. F_ξ. If $F_\xi \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, then F_{S_η} belongs to the class $\mathcal{L}(\gamma)$ for any c.r.v. η distributed according to the Poisson law.*

A similar assertion but for the case $\gamma = 0$ was obtained by Leipus and Šiaulys (see [21, Thm. 6]). In such a case, the restriction on the c.r.v. η is substantially weaker. We present their result below.

Theorem 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. r.v.s distributed on $\mathbb{R}$ with d.f. F_ξ. If $F_\xi \in \mathcal{L}$, then F_{S_η} belongs to the class $\mathcal{L}$ for any c.r.v. η satisfying the condition $\mathbf{P}(\eta > \delta x) = o(\sqrt{x}\,\overline{F}_\xi(x))$ for each $\delta \in (0, 1)$.*

In the original paper, the assertion of the theorem above is formulated for nonnegative r.v.s only. But it is easy to check that the proof is identical for a more general situation. Also we should note that d.f. F_{S_η} can be exponential without requirement of exponentiality for F_ξ. The following assertion was proved by Xu et al. (see [31, Cor. 2.1(3)]).

Theorem 3. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. r.v.s distributed on $\mathbb{R}$ with d.f. F_ξ. D.f. F_{S_η} belongs to the class $\mathcal{L}$ if $F_{S_n} \in \mathcal{L}$ and $\mathbf{P}(\eta \geqslant n) > 0$ for some $n \in \mathbb{N}$ and, in addition, $\mathbf{P}(\eta > \delta x) = o(\sqrt{x}\,\overline{F}_{S_n}(x))$ for each $\delta \in (0, 1)$.*

If the sequence of r.v.s $\{\xi_1, \xi_2, \ldots\}$ consists of independent but not necessarily identically distributed r.v.s, then the following generalization of Theorem 2 was obtained by Danilenko et al. (see [13, Thm. 4]).

Theorem 4. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent r.v.s distributed on $\mathbb{R}$ such that*

$$\sup_{k\geqslant 1}\left|\frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y}\right| \underset{x\to\infty}{\longrightarrow} 0 \tag{2}$$

for some $\gamma \geqslant 0$ and each $y \geqslant 0$. In addition, let η be a c.r.v. independent of $\{\xi_1, \xi_2, \dots\}$ and such that

$$\frac{\mathbf{P}(\eta = k+1)}{\mathbf{P}(\eta = k)} \underset{k\to\infty}{\longrightarrow} 0. \tag{3}$$

Then $F_{S_\eta} \in \mathcal{L}(\gamma)$.

Motivated by the presented assertions and the results of papers [3, 5, 13–17, 22], we continue to consider conditions under which the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, $F_{S_{(\eta)}}$ or $F_{S^{(\eta)}}$ belong to either of the following classes: $\mathcal{L}$, $\mathcal{L}(\gamma)$ with some $\gamma \geqslant 0$, $\mathcal{L}^\infty$ or $\mathcal{L}_+^\infty$. As we mentioned above, we deal with the case where the sequence $\{\xi_1, \xi_2, \dots\}$ consists of independent but possibly nonidentically distributed r.v.s.

The rest of the paper is organized as follows. In Section 2, we present our main results. Section 3 consists of two auxiliary lemmas. The proofs of the main results are given in Section 4. Finally, in Section 5, we present two examples to expose the usefulness of our results.

2 Main results

In this section, we present the main results of this paper.

In all the assertions below, we suppose that the sequence $\{\xi_1, \xi_2, \dots\}$ consists of independent r.v.s and the c.r.v. η is independent of this sequence. By default, we suppose also that r.v.s $\xi_1, \xi_2, \dots$ are distributed on $\mathbb{R}$, i.e. they can take positive and negative values.

The first theorem describes properties of the randomly stopped minima.

Theorem 5. *For the randomly stopped minima, the following assertions hold*:

(i) *If $F_{\xi_k} \in \mathcal{L}_+^\infty$ for each $k \in \mathbb{N}$, then $F_{\xi_{(\eta)}} \in \mathcal{L}_+^\infty$ for any c.r.v. η.*
(ii) *$F_{\xi_k} \in \mathcal{L}^\infty$ $\big(F_{\xi_k} \in \mathcal{L}\big)$ for all $k \in \mathbb{N}$ if and only if $F_{\xi_{(\eta)}} \in \mathcal{L}^\infty$ $(F_{\xi_{(\eta)}} \in \mathcal{L})$ for any c.r.v. η.*

The second theorem describes properties of the randomly stopped maxima. We note that it gives only sufficient conditions.

Theorem 6. *Let $\mathcal{B}$ denote one of d.f.s classes*: *$\mathcal{L}(\gamma)$ with some $\gamma \geqslant 0$, $\mathcal{L}_+^\infty$, $\mathcal{L}^\infty$. If $F_{\xi_k} \in \mathcal{B}$ for each $k \leqslant \varkappa$ and some $\varkappa \in \operatorname{supp}\eta := \{n\colon \mathbf{P}(\eta = n) > 0\}$, and there is a positive sequence $\varphi(n)$, $n \in \mathbb{N}$, such that*

$$\limsup_{x\to\infty}\sup_{n\geqslant\varkappa}\left|\frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x)}\sum_{k=1}^{n}\overline{F}_{\xi_k}(x) - 1\right| = 0 \quad \textit{and} \quad \mathbf{E}\big(\varphi(\eta)\mathbf{1}_{\{\eta\geqslant 1\}}\big) < \infty, \tag{4}$$

then $F_{\xi^{(\eta)}} \in \mathcal{B}$.

The following assertion gives sufficient conditions under which the randomly stopped minima of sums preserves exponentiality.

Theorem 7. *Suppose that $\mathcal{B}$ denotes the same object as in Theorem* 6*. For the randomly stopped minima of sums, the following two assertions hold*:

(i) *If $F_{\xi_1} \in \mathcal{B}$ and $\mathbf{P}(\xi_k \geqslant 0) = 1$ for each $k > \varkappa$ and some $\varkappa \geqslant 1$, then $F_{S_{(\eta)}} \in \mathcal{B}$ for any c.r.v. η.*

(ii) *If $F_{\xi_1} \in \mathcal{B}$, then $F_{S_{(\eta)}} \in \mathcal{B}$ for any c.r.v. η with property* (3)*.*

The last assertion describes sufficient conditions under which the d.f. of the randomly stopped maxima of sums remains in the class of exponential distributions. We note that Theorem 8 below is related to the results of the paper by Danilenko et al. [13], where the authors consider conditions under which the randomly stopped sums preserve exponentiality.

Theorem 8. *Suppose that $\mathcal{B}$ denotes the same object as in Theorem* 6*. Then the following assertions hold*:

(i) *If $F_{\xi_k} \in \mathcal{B}$ for each $k \leqslant \varkappa$ and some $\varkappa \geqslant 1$, and $\mathbf{P}(\xi_k \leqslant 0) = 1$ for all $k > \varkappa$, then $F_{S^{(\eta)}} \in \mathcal{B}$ for any c.r.v. η.*

(ii) *If $\{\xi_1, \xi_2, \ldots\}$ is a sequence of nonnegative r.v.s such that condition* (2) *holds for some $\gamma \geqslant 0$ and each $y \geqslant 0$, then $F_{S^{(\eta)}} \in \mathcal{L}(\gamma)$ for any c.r.v. satisfying condition* (3)*.*

3 Auxiliary lemmas

In this section, we give two auxiliary assertions, which are used in the proofs of our main results. The first lemma is an extension of Lemma 3.1 from [17].

Lemma 1. *Let X and Y be two independent r.v.s with d.f.s F and G, respectively, and let H be the d.f. of $\max(X, Y)$. Then*

$$\frac{\overline{H}(x-t)}{\overline{H}(x)} \leqslant \max\left\{\frac{\overline{F}(x-t)}{\overline{F}(x)}, \frac{\overline{G}(x-t)}{\overline{G}(x)}\right\}$$

if $\overline{F}(x) > 0$, $\overline{G}(x) > 0$ and $t \geqslant 0$.

In addition,

$$\frac{\overline{H}(x-t)}{\overline{H}(x)} \geqslant \min\left\{\frac{G(x-t)}{G(x)}\frac{\overline{F}(x-t)}{\overline{F}(x)}, \frac{\overline{G}(x-t)}{\overline{G}(x)}\right\}$$

if $\overline{F}(x) > 0$, $\overline{G}(x) \in (0,1)$ and $t \geqslant 0$.

Proof. The derivation of the upper bound can be found in the proof of Lemma 3.1 in [17]. For the lower bound, we observe that

$$\overline{H}(x) = G(x)\overline{F}(x) + \overline{G}(x).$$

If $\overline{F}(x) > 0$, $\overline{G}(x) \in (0,1)$ and $t > 0$, then

$$\frac{\overline{H}(x-t)}{\overline{H}(x)} = \frac{G(x-t)\overline{F}(x-t) + \overline{G}(x-t)}{G(x)\overline{F}(x) + \overline{G}(x)} \geqslant \min\left\{\frac{G(x-t)}{G(x)}\frac{\overline{F}(x-t)}{\overline{F}(x)}, \frac{\overline{G}(x-t)}{\overline{G}(x)}\right\}$$

by the inequality

$$\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_r}{b_r}\right\} \leqslant \frac{a_1 + a_2 + \dots + a_r}{b_1 + b_2 + \dots + b_r} \leqslant \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_r}{b_r}\right\}, \tag{5}$$

which holds for all $a_i \geqslant 0$ and $b_i > 0$, $i = 1, 2, \dots, r$, $r \geqslant 1$. □

The next lemma was proved by Embrechts and Goldie (see [18, Thm. 3]). Note that in its assertion "$*$" stands for the convolution of d.f.s.

Lemma 2. *Let F and G be two d.f.s. If $F \in \mathcal{L}(\gamma)$ with $\gamma \geqslant 0$, then $F * G \in \mathcal{L}(\gamma)$ if either $G \in \mathcal{L}(\gamma)$ or $\overline{G}(x) = o(\overline{F}(x))$.*

4 Proofs of main results

In this section, we give detailed proofs of all our main results.

Proof of Theorem 5. Let η be an arbitrary c.r.v., and set

$$\varkappa := \min\bigl\{n \geqslant 1\colon \mathbf{P}(\eta = n) > 0\bigr\}.$$

Then for any $x > 0$, we have

$$\begin{aligned}
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbf{P}(\eta = n) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbf{P}(\eta = \varkappa) + \sum_{n=\varkappa+1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbf{P}(\eta = n) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbf{P}(\eta = \varkappa)\left(1 + \sum_{n=\varkappa+1}^{\infty}\left(\prod_{k=\varkappa+1}^{n} \overline{F}_{\xi_k}(x)\right)\frac{\mathbf{P}(\eta = n)}{\mathbf{P}(\eta = \varkappa)}\right) \\
&\leqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbf{P}(\eta = \varkappa)\left(1 + \overline{F}_{\xi_{\varkappa+1}}(x)\frac{\mathbf{P}(\eta \geqslant \varkappa + 1)}{\mathbf{P}(\eta = \varkappa)}\right)
\end{aligned}$$

and

$$\overline{F}_{\xi_{(\eta)}}(x) \geqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbf{P}(\eta = \varkappa).$$

Therefore, we obtain

$$\overline{F}_{\xi_{(\eta)}}(x) \underset{x\to\infty}{\sim} \mathbf{P}(\eta = \varkappa)\,\overline{F}_{\xi_{(\varkappa)}}(x). \tag{6}$$

Part (i) If $F_{\xi_k} \in \mathcal{L}_+^\infty$ for all $k \in \mathbb{N}$, then $F_{\xi_1} \in \mathcal{L}(\gamma_1),\ F_{\xi_2} \in \mathcal{L}(\gamma_2),\ \dots$ for some positive parameters $\gamma_1, \gamma_2, \dots$. Hence,

$$F_{\xi_{(\varkappa)}} \in \mathcal{L}(\gamma_1 + \gamma_2 + \dots + \gamma_\varkappa) \tag{7}$$

for the finite nonrandom $\varkappa$ because

$$\lim_{x\to\infty} \frac{\overline{F}_{\xi_{(\varkappa)}}(x+y)}{\overline{F}_{\xi_{(\varkappa)}}(x)} = \lim_{x\to\infty} \frac{\prod_{k=1}^{\varkappa} \overline{F}_{\xi_k}(x+y)}{\prod_{k=1}^{\varkappa} \overline{F}_{\xi_k}(x)} = \prod_{k=1}^{\varkappa} \mathrm{e}^{-y\gamma_k}$$

for each $y > 0$. Thus, it follows from (6) and (7) that

$$F_{\xi_{(\eta)}} \in \mathcal{L}(\gamma_1 + \dots + \gamma_\varkappa) \subset \mathcal{L}_+^\infty$$

for any c.r.v. η, which proves part (i) of the theorem.

Part (ii) Let us consider the class $\mathcal{L}^\infty$. If $F_{\xi_k} \in \mathcal{L}^\infty$ for each $k \in \mathbb{N}$, then, applying arguments similar to those in the proof of part (i), we deduce that $F_{\xi_{(\eta)}} \in \mathcal{L}^\infty$ for an arbitrary c.r.v. η.

If $F_{\xi_{(\eta)}} \in \mathcal{L}^\infty$ for an arbitrary c.r.v. η, then from (6) it follows that $F_{\xi_{(m)}} \in \mathcal{L}^\infty$ for any $m \in \mathbb{N}$. This implies that $F_{\xi_k} \in \mathcal{L}^\infty$ for each $k \in \mathbb{N}$ because for all $x > 0$ and $y > 0$, we have

$$\frac{\overline{F}_{\xi_1}(x+y)}{\overline{F}_{\xi_1}(x)} = \frac{\overline{F}_{\xi_{(1)}}(x+y)}{\overline{F}_{\xi_{(1)}}(x)}$$

and

$$\frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} = \frac{\overline{F}_{\xi_{(k)}}(x+y)}{\overline{F}_{\xi_{(k)}}(x)} \bigg/ \frac{\overline{F}_{\xi_{(k-1)}}(x+y)}{\overline{F}_{\xi_{(k-1)}}(x)}, \quad k \in \{2, 3, \dots\},$$

which completes the proof of part (ii).

If $F_{\xi_k} \in \mathcal{L}$ for each index $k \in \mathbb{N}$, then proof of the assertion is analogous to the presented proof for the class $\mathcal{L}^\infty$. □

Proof of Theorem 6. We consider the proof separately for different classes.

• Let us consider the case $\mathcal{B} = \mathcal{L}(\gamma)$ with some $\gamma \geqslant 0$. First, we suppose that $\mathbf{P}(\eta < \varkappa) > 0$. Let $y > 0$ be a real number. By (5), for any $x > y$ and $K > \varkappa$, we have

$$\frac{\overline{F}_{\xi^{(\eta)}}(x-y)}{\overline{F}_{\xi^{(\eta)}}(x)} \leqslant \max\{\mathcal{J}_1, \mathcal{J}_2\}, \tag{8}$$

where

$$\mathcal{J}_1 = \frac{\sum_{n<\varkappa} \overline{F}_{\xi^{(n)}}(x-y)\mathbf{P}(\eta = n)}{\sum_{n<\varkappa} \overline{F}_{\xi^{(n)}}(x)\mathbf{P}(\eta = n)} \quad \text{and} \quad \mathcal{J}_2 = \frac{\sum_{n\geqslant\varkappa} \overline{F}_{\xi^{(n)}}(x-y)\mathbf{P}(\eta = n)}{\sum_{n=\varkappa}^{K} \overline{F}_{\xi^{(n)}}(x)\mathbf{P}(\eta = n)}.$$

Applying the upper bound from Lemma 1, we obtain

$$\frac{\overline{F}_{\xi^{(n)}}(x-y)}{\overline{F}_{\xi^{(n)}}(x)} \leqslant \max_{1\leqslant k\leqslant n} \frac{\overline{F}_{\xi_k}(x-y)}{\overline{F}_{\xi_k}(x)}.$$

This, together with (5), yields

$$\mathcal{J}_1 \leqslant \max_{\substack{n<\varkappa \\ n\in\operatorname{supp}\eta}} \left\{ \max_{1\leqslant k\leqslant n} \frac{\overline{F}_{\xi_k}(x-y)}{\overline{F}_{\xi_k}(x)} \right\}. \tag{9}$$

For x large enough, application of the Bonferroni inequality gives that

$$\begin{aligned}
\mathcal{J}_2 &\leqslant \frac{\sum_{n\geqslant\varkappa}\sum_{k=1}^n \overline{F}_{\xi_k}(x-y)\mathbf{P}(\eta=n)}{\sum_{n=\varkappa}^K(\sum_{k=1}^n \overline{F}_{\xi_k}(x) - \sum_{1\leqslant k_1<k_2\leqslant n} \overline{F}_{\xi_{k_1}}(x)\overline{F}_{\xi_{k_2}}(x))\mathbf{P}(\eta=n)} \\
&\leqslant \frac{\overline{F}_{\xi_\varkappa}(x-y)\sup_{n\geqslant\varkappa}\{\frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x-y)}\sum_{k=1}^n \overline{F}_{\xi_k}(x-y)\}}{\overline{F}_{\xi_\varkappa}(x)(1-\sum_{k=1}^K \overline{F}_{\xi_k}(x))\inf_{n\geqslant\varkappa}\{\frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x)}\sum_{k=1}^n \overline{F}_{\xi_k}(x)\}} \\
&\quad\times \frac{\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\eta\geqslant\varkappa\}})}{\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\varkappa\leqslant\eta\leqslant K\}})}.
\end{aligned} \tag{10}$$

Combining inequalities (8), (9) and (10) and taking into account conditions (4), we conclude that

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi^{(\eta)}}(x-y)}{\overline{F}_{\xi^{(\eta)}}(x)} \leqslant \max\left\{ \mathrm{e}^{\gamma y},\, \mathrm{e}^{\gamma y} \frac{\mathbf{E}\big(\varphi(\eta)\mathbf{1}_{\{\eta\geqslant\varkappa\}}\big)}{\mathbf{E}\big(\varphi(\eta)\mathbf{1}_{\{\varkappa\leqslant\eta\leqslant K\}}\big)} \right\}$$

for all $K > \varkappa$, which implies

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi^{(\eta)}}(x-y)}{\overline{F}_{\xi^{(\eta)}}(x)} \leqslant \mathrm{e}^{\gamma y} \tag{11}$$

for any $y > 0$.

Applying arguments similar to those for deriving (8), we obtain

$$\frac{\overline{F}_{\xi^{(\eta)}}(x-y)}{\overline{F}_{\xi^{(\eta)}}(x)} \geqslant \min\{\mathcal{J}_1, \mathcal{J}_3\}, \tag{12}$$

where

$$\mathcal{J}_3 = \frac{\sum_{n=\varkappa}^K \overline{F}_{\xi^{(n)}}(x-y)\mathbf{P}(\eta=n)}{\sum_{n\geqslant\varkappa} \overline{F}_{\xi^{(n)}}(x)\mathbf{P}(\eta=n)}.$$

Similarly to (10), we get

$$\begin{aligned}
\mathcal{J}_3 &\geqslant \frac{\overline{F}_{\xi_\varkappa}(x-y)(1-\sum_{k=1}^K \overline{F}_{\xi_k}(x-y))\inf_{n\geqslant\varkappa}\{\frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x-y)}\sum_{k=1}^n \overline{F}_{\xi_k}(x-y)\}}{\overline{F}_{\xi_\varkappa}(x)\sup_{n\geqslant\varkappa}\{\frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x)}\sum_{k=1}^n \overline{F}_{\xi_k}(x)\}} \\
&\quad\times \frac{\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\varkappa\leqslant\eta\leqslant K\}})}{\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\eta\geqslant\varkappa\}})}.
\end{aligned} \tag{13}$$

Applying the lower bound from Lemma 1, we obtain

$$\frac{\overline{F}_{\xi^{(n)}}(x-y)}{\overline{F}_{\xi^{(n)}}(x)} \geqslant \min_{1\leqslant k\leqslant n}\left\{\frac{\overline{F}_{\xi_k}(x-y)}{\overline{F}_{\xi_k}(x)}\prod_{l=1}^{k-1}\frac{F_{\xi_l}(x-y)}{F_{\xi_l}(x)}\right\}.$$

Therefore, by (5), we have

$$\mathcal{J}_1 \geqslant \min_{\substack{n<\varkappa\\ n\in\operatorname{supp}\eta}}\left\{\min_{1\leqslant k\leqslant n}\left\{\frac{\overline{F}_{\xi_k}(x-y)}{\overline{F}_{\xi_k}(x)}\prod_{l=1}^{k-1}\frac{F_{\xi_l}(x-y)}{F_{\xi_l}(x)}\right\}\right\}. \tag{14}$$

Combining inequalities (12), (13) and (14) and taking into account conditions (4), we deduce that

$$\liminf_{x\to\infty}\frac{\overline{F}_{\xi^{(\eta)}}(x-y)}{\overline{F}_{\xi^{(\eta)}}(x)} \geqslant \mathrm{e}^{\gamma y}.$$

The last inequality together with (11) implies that $F_{\xi^{(\eta)}} \in \mathcal{L}(\gamma)$, which is the desired conclusion for the case $\mathbf{P}(\eta < \varkappa) > 0$. If $\mathbf{P}(\eta < \varkappa) = 0$, then the proof is similar with the only difference that

$$\overline{F}_{\xi^{(\eta)}}(x) = \sum_{n\geqslant\varkappa}\overline{F}_{\xi^{(n)}}(x)\mathbf{P}(\eta = n)$$

for all $x > 0$. This completes the proof provided that $\mathcal{B} = \mathcal{L}(\gamma)$ with some $\gamma \geqslant 0$.

• Let now $\mathcal{B} = \mathcal{L}_+^\infty$. First, we suppose that $\mathbf{P}(\eta < \varkappa) > 0$ and $\varkappa \geqslant 2$. In this case, $F_{\xi_k} \in \mathcal{L}(\gamma_k)$ with some parameters $\gamma_k > 0$ for all $k \in \{1, 2, \dots, \varkappa\}$. It is easy to check that

$$\lim_{x\to\infty}\frac{\overline{F}_{\xi^{(2)}}(x-y)}{\overline{F}_{\xi^{(2)}}(x)} = \lim_{x\to\infty}\frac{\overline{F}_{\xi_1}(x-y)+\overline{F}_{\xi_2}(x-y)}{\overline{F}_{\xi_1}(x)+\overline{F}_{\xi_2}(x)} \tag{15}$$

for any $y > 0$.

If $F_{\xi_1} \in \mathcal{L}(\gamma_1)$ and $F_{\xi_2} \in \mathcal{L}(\gamma_2)$ with $\gamma_1 = \gamma_2$, then applying (5), we conclude that $F_{\xi^{(2)}} \in \mathcal{L}(\gamma_1)$. If $F_{\xi_1} \in \mathcal{L}(\gamma_1)$ and $F_{\xi_2} \in \mathcal{L}(\gamma_2)$ with $\gamma_1 < \gamma_2$, then representation (1) implies that $\overline{F}_{\xi_2}(x) = o(\overline{F}_{\xi_1}(x))$, and hence, from relation (15) we deduce that $F_{\xi^{(2)}} \in \mathcal{L}(\gamma_1)$. Consequently, $F_{\xi^{(2)}} \in \mathcal{L}\big(\min\{\gamma_1, \gamma_2\}\big)$ in all possible cases.

Since $\xi^{(n)} = \max\{\xi^{(n-1)}, \xi_n\}$, $n \geqslant 2$, using induction on n, we obtain

$$F_{\xi^{(n)}} \in \mathcal{L}\big(\min\{\gamma_1, \gamma_2, \dots, \gamma_n\}\big) \tag{16}$$

for each $n \in \{1, 2, \dots, \varkappa\}$.

Applying arguments similar to those for deriving (8), (10), (12) and (13), we have

$$\frac{\overline{F}_{\xi^{(\eta)}}(x-y)}{\overline{F}_{\xi^{(\eta)}}(x)} \leqslant \frac{\mathcal{I}_1(x-y) + \overline{F}_{\xi_\varkappa}(x-y)\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\eta\geqslant\varkappa\}})\sup_{n\geqslant\varkappa}\mathcal{I}_{2,n}(x-y)}{\mathcal{I}_1(x)+\overline{F}_{\xi_\varkappa}(x)(1-\sum_{k=1}^K\overline{F}_{\xi_k}(x))\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\varkappa\leqslant\eta\leqslant K\}})\inf_{n\geqslant\varkappa}\mathcal{I}_{2,n}(x)}$$

and

$$\frac{\overline{F}_{\xi^{(\eta)}}(x-y)}{\overline{F}_{\xi^{(\eta)}}(x)} \geqslant \frac{\mathcal{I}_1(x-y)}{\mathcal{I}_1(x) + \overline{F}_{\xi_\varkappa}(x)\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\varkappa\leqslant\eta\leqslant K\}})\sup_{n\geqslant\varkappa}\mathcal{I}_{2,n}(x)}$$
$$+\frac{\overline{F}_{\xi_\varkappa}(x-y)(1-\sum_{k=1}^{K}\overline{F}_{\xi_k}(x-y))\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\eta\geqslant\varkappa\}})\inf_{n\geqslant\varkappa}\mathcal{I}_{2,n}(x-y)}{\mathcal{I}_1(x) + \overline{F}_{\xi_\varkappa}(x)\mathbf{E}(\varphi(\eta)\mathbf{1}_{\{\varkappa\leqslant\eta\leqslant K\}})\sup_{n\geqslant\varkappa}\mathcal{I}_{2,n}(x)}$$

for all $y > 0$, $K > \varkappa$ and sufficiently large x, where

$$\mathcal{I}_1(x) = \sum_{1\leqslant n<\varkappa} \overline{F}_{\xi^{(n)}}(x)\mathbf{P}(\eta = n) \quad \text{and} \quad \mathcal{I}_{2,n}(x) = \frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x)}\sum_{k=1}^{n}\overline{F}_{\xi_k}(x).$$

The last two inequalities, condition (4) and relation (16) yield

$$F_{\xi^{(\eta)}} \in \mathcal{L}(\gamma^*) \subset \mathcal{L}_+^\infty \quad \text{with } \gamma^* = \min_{\substack{1\leqslant k\leqslant\varkappa \\ k\in\operatorname{supp}\eta}}\{\gamma_k\}.$$

If either $\mathbf{P}(\eta < \varkappa) = 0$ or $\varkappa = 1$, then we have $\mathcal{I}_1(x) = 0$ for all $x > 0$. Therefore, from the last two inequalities and condition (4) we see that $F_{\xi^{(\eta)}} \in \mathcal{L}(\gamma_\varkappa) \subset \mathcal{L}_+^\infty$ in this case, which completes the proof provided that $\mathcal{B} = \mathcal{L}_+^\infty$.

• If $\mathcal{B} = \mathcal{L}^\infty$, then the proof can be obtained along the lines of the proof for the class $\mathcal{L}_+^\infty$. □

Proof of Theorem 7. We give the proof only for the case $\mathcal{B} = \mathcal{L}(\gamma)$ with some $\gamma \geqslant 0$. The other two cases can be considered similarly.

Let $F_{\xi_1} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. For all $x > 0$, it is obvious that

$$\begin{aligned}\overline{F}_{S_{(2)}}(x) &= \mathbf{P}(\xi_1 > x,\ \xi_1 + \xi_2 > x) \\ &= \int_{(-\infty,0)} \overline{F}_{\xi_1}(x-z)\,\mathrm{d}\,F_{\xi_2}(z) + \overline{F}_{\xi_1}(x)\mathbf{P}(\xi_2 \geqslant 0).\end{aligned} \tag{17}$$

Consequently, by inequality (5), for all $x > 0$ and $y > 0$, we have

$$\begin{aligned}\frac{\overline{F}_{S_{(2)}}(x+y)}{\overline{F}_{S_{(2)}}(x)} &\leqslant \max\left\{\sup_{z<0}\frac{\overline{F}_{\xi_1}(x+y-z)}{\overline{F}_{\xi_1}(x-z)}, \frac{\overline{F}_{\xi_1}(x+y)}{\overline{F}_{\xi_1}(x)}\right\} \\ &= \sup_{v\geqslant x}\frac{\overline{F}_{\xi_1}(v+y)}{\overline{F}_{\xi_1}(v)}.\end{aligned} \tag{18}$$

Similarly, for all $x > 0$ and $y > 0$, we get

$$\frac{\overline{F}_{S_{(2)}}(x+y)}{\overline{F}_{S_{(2)}}(x)} \geqslant \inf_{v\geqslant x}\frac{\overline{F}_{\xi_1}(v+y)}{\overline{F}_{\xi_1}(v)}. \tag{19}$$

Combining (18) and (19), we conclude that $F_{S_{(2)}} \in \mathcal{L}(\gamma)$.

Applying arguments similar to those for deriving (17), we obtain

$$\begin{aligned}\overline{F}_{S_{(3)}}(x) &= \iint\limits_{\mathbb{R}^2} \mathbf{P}(\xi_1 > x,\ \xi_1 > x - z_1,\ \xi_1 > x - z_1 - z_2)\,\mathrm{d}\mathbf{P}(\xi_2 \leqslant z_1,\ \xi_3 \leqslant z_2)\\ &= \overline{F}_{\xi_1}(x)\mathbf{P}(\xi_2 \geqslant 0,\ \xi_3 \geqslant -\xi_2) + \iint\limits_{\substack{z_1<0\\ z_2\geqslant 0}} \overline{F}_{\xi_1}(x - z_1)\,\mathrm{d}\mathbf{P}(\xi_2 \leqslant z_1,\ \xi_3 \leqslant z_2)\\ &\quad + \iint\limits_{\substack{z_1<0\\ z_1<-z_2}} \overline{F}_{\xi_1}(x - z_1 - z_2)\,\mathrm{d}\mathbf{P}(\xi_2 \leqslant z_1,\ \xi_3 \leqslant z_2)\end{aligned}$$

for all $x > 0$. From this and inequality (5) it follows that

$$\inf_{v\geqslant x} \frac{\overline{F}_{\xi_1}(v+y)}{\overline{F}_{\xi_1}(v)} \leqslant \frac{\overline{F}_{S_{(3)}}(x+y)}{\overline{F}_{S_{(3)}}(x)} \leqslant \sup_{v\geqslant x} \frac{\overline{F}_{\xi_1}(v+y)}{\overline{F}_{\xi_1}(v)}$$

for all $x > 0$ and $y > 0$. Letting $x \to \infty$ in the last inequality, we see that $F_{S_{(3)}} \in \mathcal{L}(\gamma)$. Continuing in the same way, we deduce that $F_{S_{(n)}} \in \mathcal{L}(\gamma)$ for each $n \in \mathbb{N}$.

Part (i) In the case under consideration, we have

$$\begin{aligned}\overline{F}_{S_{(\eta)}}(x) &= \sum_{n=1}^{\varkappa} \overline{F}_{S_{(n)}}(x)\mathbf{P}(\eta = n) + \sum_{n=\varkappa+1}^{\infty} \overline{F}_{S_{(n)}}(x)\mathbf{P}(\eta = n)\\ &= \sum_{n=1}^{\varkappa} \overline{F}_{S_{(n)}}(x)\mathbf{P}(\eta = n) + \overline{F}_{S_{(\varkappa)}}(x)\mathbf{P}(\eta > \varkappa).\end{aligned}$$

Applying (5) yields

$$\min_{\substack{1\leqslant n\leqslant\varkappa\\ n\in\operatorname{supp}\eta}} \frac{\overline{F}_{S_{(n)}}(x+y)}{\overline{F}_{S_{(n)}}(x)} \leqslant \frac{\overline{F}_{S_{(\eta)}}(x+y)}{\overline{F}_{S_{(\eta)}}(x)} \leqslant \max_{\substack{1\leqslant n\leqslant\varkappa\\ n\in\operatorname{supp}\eta}} \frac{\overline{F}_{S_{(n)}}(x+y)}{\overline{F}_{S_{(n)}}(x)}$$

for all $x > 0$ and $y > 0$. Since $F_{S_{(n)}} \in \mathcal{L}(\gamma)$ for each n, the last inequality implies that $F_{S_{(\eta)}} \in \mathcal{L}(\gamma)$ as well, which proves part (i) of the theorem.

Part (ii) It is easily seen that the condition on the c.r.v. η implies that $\mathbf{P}(\eta = k) > 0$ for all k large enough. In addition, we observe that requirement (3) implies relation

$$\frac{\mathbf{P}(\eta \geqslant k+1)}{\mathbf{P}(\eta = k)} \underset{k\to\infty}{\longrightarrow} 0, \tag{20}$$

which follows from the equality

$$\frac{\mathbf{P}(\eta \geqslant k+1)}{\mathbf{P}(\eta = k)} = \sum_{j=1}^{\infty} \frac{\mathbf{P}(\eta = k+j)}{\mathbf{P}(\eta = k)} = \sum_{j=1}^{\infty} \prod_{r=0}^{j-1} \frac{\mathbf{P}(\eta = k+r+1)}{\mathbf{P}(\eta = k+r)},$$

provided for k is sufficiently large.

We now choose any K large enough. Then on the one hand, we have

$$\begin{aligned}\overline{F}_{S_{(\eta)}}(x) &= \sum_{n=1}^{K-1} \overline{F}_{S_{(n)}}(x)\mathbf{P}(\eta = n) + \overline{F}_{S_{(K)}}(x)\mathbf{P}(\eta = K) \\ &\quad + \sum_{n=K+1}^{\infty} \overline{F}_{S_{(n)}}(x)\mathbf{P}(\eta = n) \\ &\leqslant \sum_{n=1}^{K-1} \overline{F}_{S_{(n)}}(x)\mathbf{P}(\eta = n) + \overline{F}_{S_{(K)}}(x)\mathbf{P}(\eta = K)\left(1 + \frac{\mathbf{P}(\eta \geqslant K+1)}{\mathbf{P}(\eta = K)}\right)\end{aligned}$$

for all $x > 0$ because

$$\begin{aligned}\overline{F}_{S_{(n)}}(x) &= \mathbf{P}\bigl(\min\{S_1, S_2, \ldots, S_K, \ldots, S_n\} > x\bigr) \\ &\leqslant \mathbf{P}\bigl(\min\{S_1, S_2, \ldots, S_K\} > x\bigr) = \overline{F}_{S_{(K)}}(x)\end{aligned}$$

for any $n \in \{K, K+1, \ldots\}$ in the case under consideration.

On the other hand, it is evident that

$$\overline{F}_{S_{(\eta)}}(x) \geqslant \sum_{n=1}^{K-1} \overline{F}_{S_{(n)}}(x)\mathbf{P}(\eta = n) + \overline{F}_{S_{(K)}}(x)\mathbf{P}(\eta = K).$$

Therefore, by inequality (5), for all $x > 0$ and $y > 0$, we obtain

$$\frac{\overline{F}_{S_{(\eta)}}(x+y)}{\overline{F}_{S_{(\eta)}}(x)} \leqslant \max\left\{\max_{\substack{1 \leqslant n \leqslant K-1 \\ n \in \operatorname{supp} \eta}} \frac{\overline{F}_{S_{(n)}}(x+y)}{\overline{F}_{S_{(n)}}(x)}, \frac{\overline{F}_{S_{(K)}}(x+y)}{\overline{F}_{S_{(K)}}(x)}\left(1 + \frac{\mathbf{P}(\eta \geqslant K+1)}{\mathbf{P}(\eta = K)}\right)\right\}.$$

Since $F_{S_{(n)}} \in \mathcal{L}(\gamma)$ for each n, letting $x \to \infty$ yields

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_{(\eta)}}(x+y)}{\overline{F}_{S_{(\eta)}}(x)} \leqslant \mathrm{e}^{-\gamma y}\left(1 + \frac{\mathbf{P}(\eta \geqslant K+1)}{\mathbf{P}(\eta = K)}\right)$$

for any fixed $y > 0$ and all K large enough.

Similarly, we get

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_{(\eta)}}(x+y)}{\overline{F}_{S_{(\eta)}}(x)} \geqslant \mathrm{e}^{-\gamma y}\left(1 + \frac{\mathbf{P}(\eta \geqslant K+1)}{\mathbf{P}(\eta = K)}\right)^{-1}$$

for any fixed $y > 0$ and all K large enough.

The last two inequalities together with (20) imply that $F_{S_{(\eta)}} \in \mathcal{L}(\gamma)$ for any c.r.v. η satisfying property (3), which proves part (ii) of the theorem. □

Proof of Theorem 8. Part (i) We consider the proof separately for different classes.

• Let now $\mathcal{B} = \mathcal{L}(\gamma)$ with some $\gamma \geqslant 0$. From Lemma 2 it may be concluded that the d.f.s of the r.v.s

$$S_n = \xi_1 + \xi_2 + \cdots + \xi_n \quad \text{and} \quad S_n^{(+)} := \xi_1^+ + \xi_2^+ + \cdots + \xi_n^+$$

belong to the class $\mathcal{L}(\gamma)$ for any $n \leqslant \varkappa$, where ξ_k^+ denotes the positive part of ξ_k, $1 \leqslant k \leqslant \varkappa$. Moreover, it is obvious that

$$\begin{aligned} \mathbf{P}(S_n > x) &\leqslant \mathbf{P}\bigl(\max\{\xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \xi_2 + \cdots + \xi_n\} > x\bigr) \\ &= \mathbf{P}\bigl(S^{(n)} > x\bigr) \leqslant \mathbf{P}\bigl(\xi_1^+ + \xi_2^+ + \cdots + \xi_n^+ > x\bigr) \\ &= \mathbf{P}\bigl(S_n^{(+)} > x\bigr) \end{aligned} \tag{21}$$

for any $x > 0$ and $n \leqslant \varkappa$. Consequently, $F_{S^{(n)}} \in \mathcal{L}(\gamma)$ for each $n \leqslant \varkappa$.

To get the desired assertion, it suffices to notice that

$$\overline{F}_{S^{(\eta)}}(x) = \sum_{n=1}^{\varkappa} \overline{F}_{S^{(n)}}(x)\mathbf{P}(\eta = n) + \overline{F}_{S^{(\varkappa)}}(x)\mathbf{P}(\eta > \varkappa) \tag{22}$$

and apply arguments similar to those in the proof of part (i) of Theorem 7.

• We now deal with the case $\mathcal{B} = \mathcal{L}_+^\infty$ and suppose that

$$F_{\xi_1} \in \mathcal{L}(\gamma_1)\,, F_{\xi_2} \in \mathcal{L}(\gamma_2),\ \ldots,\ F_{\xi_\varkappa} \in \mathcal{L}(\gamma_\varkappa)$$

for some collection of positive parameters $\{\gamma_1, \gamma_2, \ldots, \gamma_\varkappa\}$.

From Lemma 2 and inequality (21) it follows that

$$F_{S^{(n)}} \in \mathcal{L}\bigl(\min\{\gamma_1, \gamma_2, \ldots, \gamma_n\}\bigr)$$

for any $n \leqslant \varkappa$.

Hence, from equality (22) we have

$$\overline{F}_{S^{(\eta)}}(x) \sim \sum_{k=1}^{\varkappa-1} \overline{F}_{S^{(n)}}(x)\mathbf{P}(\eta = n) + \overline{F}_{S^{(\varkappa)}}(x)\mathbf{P}(\eta \geqslant \varkappa), \tag{23}$$

which is equivalent to

$$\overline{F}_{S^{(\eta)}}(x) \sim \sum_{k=1}^{\widehat{\varkappa}} q_k \overline{F}_{S^{(n_k)}}(x) := \widehat{F}(x) \tag{24}$$

for some collection of positive coefficients $\{q_1, q_2, \ldots, q_{\widehat{\varkappa}}\}$, where $1 \leqslant \widehat{\varkappa} \leqslant \varkappa$, $\{n_1, n_2, \ldots, n_{\widehat{\varkappa}}\} \subset \{1, 2, \ldots, \varkappa\}$ and $F_{S^{(n_k)}} \in \mathcal{L}(\min\{\gamma_1, \gamma_2, \ldots, \gamma_\varkappa\})$ for all indices n_k. Note that the sum in (24) contains only those summands of the sum (23) for which $F_{S^{(n)}} \in \mathcal{L}(\min\{\gamma_1, \gamma_2, \ldots, \gamma_\varkappa\})$, and the coefficients q_k in (24) stand for the corresponding probabilities.

Applying inequality (5), we see that a d.f. with the tail function $\widehat{F}$ belongs to the class $\mathcal{L}\bigl(\min\{\gamma_1, \gamma_2, \ldots, \gamma_\varkappa\}\bigr)$. Therefore, $F_{S^{(\eta)}} \in \mathcal{L}(\min\{\gamma_1, \gamma_2, \ldots, \gamma_\varkappa\}) \subset \mathcal{L}_+^\infty$ as well.

• The case of the class $\mathcal{L}^{\infty}$ can be considered similarly, which proves part (i) of the theorem.

Part (ii) This part of the theorem follows immediately from Theorem 4 because distributions of the r.v.s S_η and $S^{(\eta)}$ coincide provided that all the r.v.s in the sequence $\{\xi_1, \xi_2, \ldots\}$ are nonnegative. This completes the proof of the theorem. □

5 Examples

In this section, we present two examples which demonstrate the applicability of the obtained results.

Example 1. Let $\{\xi_1, \xi_2, \ldots\}$ be the seasonal sequence of independent r.v.s with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$ defined as follows:

$$F_{\xi_k}(x) = \begin{cases} (1 - \mathrm{e}^{-x})\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 1 \bmod 3, \\ (1 - \mathrm{e}^{-2x}(1+2x))\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 2 \bmod 3, \\ (1 - \mathrm{e}^{-2x}(1+2x+2x^2))\mathbf{1}_{[0,\infty)}(x) & \text{if } k \equiv 3 \bmod 3. \end{cases}$$

It is clear that $F_{\xi_k} \in \mathcal{L}(1)$ for $k \equiv 1 \bmod 3$ and $F_{\xi_k} \in \mathcal{L}(2)$ for $k \not\equiv 1 \bmod 3$. According to Theorem 5, $F_{\xi_{(\eta)}} \in \mathcal{L}_+^{\infty}$ for any c.r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$. While Theorem 7 implies that d.f. $F_{S_{(\eta)}}$ belongs to the class $\mathcal{L}(1)$ for each c.r.v η independent of $\{\xi_1, \xi_2, \ldots\}$ and satisfying condition (3).

Example 2. Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s with shifted exponential d.f.s

$$F_{\xi_k}(x) = \bigl(1 - \mathrm{e}^{-(x+k-1)}\bigr)\mathbf{1}_{[0,\infty)}(x), \quad k \in \mathbb{N},$$

and let η be the Poisson r.v. independent of $\{\xi_1, \xi_2, \ldots\}$.

In such a case, condition (2) holds, and we can say that d.f.s F_{ξ_k}, $k \in \mathbb{N}$, belong to the class $\mathcal{L}(1)$ uniformly. Since r.v. η satisfies condition (3), $F_{S_\eta} \in \mathcal{L}(1)$ due to Theorem 4, $F_{\xi_{(\eta)}} \in \mathcal{L}_+^{\infty}$ due to Theorem 5, $F_{\xi^{(\eta)}} \in \mathcal{L}(1)$ due to Theorem 6, $F_{S_{(\eta)}} \in \mathcal{L}(1)$ due to Theorem 7 and $F_{S^{(\eta)}} \in \mathcal{L}(1)$ due to Theorem 8.

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