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Hilfer fractional evolution hemivariational inequalities with nonlocal initial conditions and optimal controls*

Yatian Pei, Yong-Kui Chang[1]

School of Mathematics and Statistics, Xidian University,
Xi'an 710071, Shaanxi, China
ytpei@stu.xidian.edu.cn; lzchangyk@163.com

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Abstract. In this paper, we mainly consider a control system governed by a Hilfer fractional evolution hemivariational inequality with a nonlocal initial condition. We first establish sufficient conditions for the existence of mild solutions to the addressed control system via properties of generalized Clarke subdifferential and a fixed point theorem for condensing multivalued maps. Then we present the existence of optimal state-control pairs of the limited Lagrange optimal systems governed by a Hilfer fractional evolution hemivariational inequality with a nonlocal initial condition. The optimal control results are derived without uniqueness of solutions for the control system.

Keywords: Hilfer fractional evolution equation, optimal state-control pairs, hemivariational inequalities, nonlocal initial condition.

1 Introduction

As a generalization of the ordinary differentiation and integration to arbitrary noninteger order, fractional calculus has been recognized as one of the most powerful tools to describe long-memory processes in the last decades. Many phenomena from viscoelasticity, electrochemistry, nonlinear oscillation in mechanics et al. can be modelled by ordinary and partial differential equations involving fractional derivatives; see, for instance, [1, 2, 5, 7, 8, 15, 24, 28, 29] and references therein. In [13], Hilfer proposed the Hilfer fractional derivative, which covers Riemann–Liouville fractional derivative and Caputo fractional derivative as special cases and appears in theoretical simulation of dielectric relaxation in glass forming materials. In [11], Gu and Trujillo studied existence of mild solutions to an evolution equation with Hilfer fractional derivative. In [12], Harrat et al. investigated solvability and optimal controls of impulsive Hilfer fractional delay evolution inclusions with Clarke subdifferential. As indicated in [9], the nonlocal initial condition can be more natural and more precise in describing some phenomena than the

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[1]Corresponding author.

classical initial condition since some additional information is taken into account. There are some interesting results involved in nonlocal initial conditions for Hilfer fractional evolution equations. For example, in [26], Yang and Wang investigated approximate controllability of a Hilfer fractional differential inclusion with nonlocal initial conditions. In [27], Yang and Wang considered existence of mild solutions for a Hilfer fractional differential equation with nonlocal initial conditions.

The optimal control is one of the important and fundamental topics in the field of mathematical control theory, which plays a key role in control systems [17]. In recent years, solvability and optimal control governed by fractional evolution equations has attracted great interest. For instance, the existence and optimal control for semilinear Caputo fractional finite time delay evolution systems of the order $(0,1)$ was concerned in [28, Chapter 4]. Agarwal et al. investigated a survey on fuzzy fractional differential and optimal control nonlocal evolution equations in [3]. Kumar considered the existence of optimal control for the system governed by semilinear Caputo fractional evolution equation of order $(0,1)$ with fixed delay in [16]. Liu and Wang in [18] dealt with optimal controls of systems governed by semilinear Caputo fractional differential equations of order $(0,1)$ with not instantaneous impulses. Yan and Jia in [25] discussed optimal controls for Caputo fractional impulsive neutral stochastic integro-differential equations of order $(1,2)$. On the other hand, hemivariational inequality finds its important applications to models in mechanics with nonsmooth and nonconvex energy superpotentials [22]. Much attention has been paid to fractional evolution hemivariational inequalities recently. For example, Lu and Liu [19] studied the existence and controllability for a stochastic evolution hemivariational inequality in Caputo fractional derivative of order $(0,1)$. Lu, Liu et al. [20] investigated solvability and optimal controls for a semilinear fractional evolution hemivariational inequality in Caputo sense of order $(0,1)$.

Motivated by above mentioned work, the main objective of this paper is to consider the following Hilfer fractional evolution hemivariational inequality with a nonlocal initial condition:

$$\begin{aligned} &\big\langle -D_{0^+}^{\beta;\gamma}x(t) + Ax(t) + \mathfrak{B}(t)u(t),\, d\big\rangle_X + \mathcal{F}^0\big(t, x(t); d\big) \geqslant 0 \quad \forall d \in X, \\ &J_{0^+}^{(1-\beta)(1-\gamma)}\big[x(t)\big]_{t=0} + g(x) = x_0, \end{aligned} \tag{1}$$

where $D_{0^+}^{\beta;\gamma}$ denotes the Hilfer fractional derivative, $t \in I' := (0,b]$, $\beta \in [0,1]$, $\gamma \in (0,1)$, $\langle\cdot,\cdot\rangle$ denotes the inner product (induced by the duality paring) of a separable reflexible Banach space X. The notation $\mathcal{F}^0(t,\cdot;\cdot)$ represents the generalized (Clarke) directional derivative of a locally Lipschitz function $\mathcal{F}(t,\cdot) : X \to \mathbb{R}$. The state x takes values in the separable reflexible Banach space X. The control u takes its value from a separable reflexive Banach Y, and is given in a suitable admissible control set U_{ad}. The operator $\mathfrak{B} : I \to \mathcal{B}(Y,X)$, where $\mathcal{B}(Y,X)$ denotes the space of all bounded linear operators from Y into X. The operator A is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator family $\{\mathcal{S}(t)\}_{t\geqslant 0}$ on Banach space X. Let $I = [0,b]$, the function $g : C(I,X) \to X$ is continuous and compact.

In this paper, we shall establish sufficient conditions for the existence of mild solutions to system (1) and present the existence of optimal state-control pairs of the limited

Lagrange optimal systems governed by system (1). We note that the solvability for system (1) has a relation to a suitable fractional evolution inclusion with a nonlocal initial condition here. Thus the uniqueness of solutions to system (1) cannot be guaranteed by the usual condition (see condition (C3), Section 3). We shall establish optimal control results based upon the compactness result of a certain operator (see Lemma 9, Section 4).

The rest of this paper is organized as follows. Section 2 is preliminaries. Section 3 is devoted to solvability for system (1). Section 4 is involved in the existence of optimal state-control pairs of the limited Lagrange optimal control problems governed by system (1).

2 Preliminaries

In this section, we introduce some notations, definitions, and lemmas on fractional calculus, multivalued analysis and the generalized directional derivative. We can refer to [2, 6, 10, 13–15, 21, 28, 29] for detailed results and topics.

Denote by $\mathcal{B}(X)$ the space of all bounded linear operators from X into itself. Let $C(I,X)$, $C(I',X)$ denote the spaces of all continuous functions from I or I' to X, respectively. Denote $\nu=\beta+\gamma-\beta\gamma$, then $1-\nu=(1-\beta)(1-\gamma)$. Define $\mathfrak{C}(I,X) := \{x\colon\ t^{1-\nu}x(t)\in C(I,X)\}$ with the norm given by

$$\|x\|_\nu=\sup\bigl\{t^{1-\nu}\bigl\|x(t)\bigr\|,\ \nu=\beta+\gamma-\beta\gamma,\ t\in I'\bigr\}.$$

Thus, $\mathfrak{C}(I,X)$ is a Banach space. Let $L^p(I,X)$ $(1\leqslant p<+\infty)$ be the Banach space of all X-valued Bochner-integrable functions defined on I with the norm $\|x\|_{L^p} = (\int_I \|x(t)\|^p dt)^{1/p}$. For $\gamma>0$, we define

$$\mathrm{g}_\gamma(t)=\begin{cases}\frac{t^{\gamma-1}}{\Gamma(\gamma)}, & t>0,\\ 0, & t\leqslant 0,\end{cases}$$

where $\Gamma(\cdot)$ is the gamma function. We also define $\mathrm{g}_0\equiv\delta_0$, the Dirac delta.

Definition 1. Let $\gamma>0$. The γ-order Riemann–Liouville fractional integral of x is defined by $J^\gamma_{0+}x(t):=\int_0^t \mathrm{g}_\gamma(t-s)x(s)\,\mathrm{d}s$, $t\geqslant 0$. Also, we define $J^0_{0+}x(t)=x(t)$.

Lemma 1. *The Hilfer fractional derivative of order* $0\leqslant\beta\leqslant 1$, $0<\gamma<1$ *for the function* x *is defined by* $D^{\beta;\gamma}_{0+}x(t)=J^{\beta(1-\gamma)}_{0+}(\mathrm{d}/\mathrm{d}t)J^{(1-\beta)(1-\gamma)}_{0+}x(t)$.

Lemma 2. *For* $\kappa\in(0,1]$ *and* $a,b>0$, *the inequality* $|a^\kappa-b^\kappa|\leqslant|a-b|^\kappa$ *is true.*

For a Banach space Z, we denote its dual by Z^* and write the duality pairing of Z and Z^* as $\langle\cdot,\cdot\rangle$. Denote by $\mathcal{P}(Z)$ a class of nonempty subsets of Z. Denote $\mathcal{P}_{\mathrm{cl}}(Z)=\{\Omega\in\mathcal{P}(Z)\colon\ \Omega \text{ is closed}\}$, $\mathcal{P}_{\mathrm{b}}(Z)=\{\Omega\in\mathcal{P}(Z)\colon\ \Omega \text{ is bounded}\}$, and $\mathcal{P}_{\mathrm{cv}}(Z)=\{\Omega\in\mathcal{P}(Z)\colon\ \Omega \text{ is convex}\}$.

A multivalued map $G: Z\to\mathcal{P}(Z)$ has convex (closed) values if $G(z)$ is convex (closed) for all $z\in Z$. G is bounded on bounded sets if $G(\mathbb{B})=\bigcup_{z\in\mathbb{B}}G(z)$ is bounded in Z for all $\mathbb{B}\in\mathcal{P}_{\mathrm{b}}(Z)$, i.e., $\sup_{z\in\mathbb{B}}\{\sup\{\|y\|\colon\ y\in G(z)\}\}<\infty$.

The multivalued map $G : Z \to \mathcal{P}(Z)$ is called upper semicontinuous (u.s.c.) on Z if for each $z_0 \in Z$, the set $G(z_0)$ is a nonempty, closed subset of Z, and if for each open set $\mathcal{O}$ of Z containing $G(z_0)$, there exists an open neighborhood $\mathcal{V}$ of z_0 such that $G(\mathcal{V}) \subseteq \mathcal{O}$. Also, G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(Z)$. G has a fixed point if there exists $z \in Z$ such that $z \in G(Z)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $z_n \to z_*$, $h_n \to h_*$, $h_n \in G(z_n)$ imply $h_* \in G(z_*)$.

A multivalued map $\digamma : I \to \mathcal{P}(Z)$ is said to be measurable if $\digamma^{-1}(\mathbb{C}) = \{t \in I$: $\digamma(t) \cap \mathbb{C} \neq \emptyset\} \in \Sigma$ for each closed set $\mathbb{C} \subseteq Z$. If $\digamma : I \times Z \to \mathcal{P}(Z)$, then measurability of $\digamma$ means that $\digamma^{-1}(\mathbb{C}) \in \Sigma \otimes \mathscr{B}_Z$, where $\Sigma \otimes \mathscr{B}_Z$ is the σ-algebra of subsets in $I \times Z$ generated by the sets $\mathbb{A} \times \mathbb{B}$, $\mathbb{A} \in \Sigma$, $\mathbb{B} \in \mathscr{B}_Z$, and $\mathscr{B}_Z$ is the σ-algebra of the Borel sets in Z.

The generalized directional derivative (in the sense of Clarke) of a locally Lipshitz function $h : Z \to \mathbb{R}$ at x in the direction d is denoted by $h^0(x, d)$, which is given by

$$h^0(x, d) = \limsup_{y \to x,\, t \downarrow 0} \frac{h(y + td) - h(y)}{t}.$$

The Clarke subdifferential or the generalized gradient of h at x is denoted by $\partial h(x)$, which is a subset of Z^* defined by

$$\partial h(x) = \bigl\{y \in Z^*\colon h^0(x, d) \geqslant \langle y,\ d\rangle\ \forall d \in Z\bigr\}.$$

We have the following facts, which can be referred to [6] for more details.

Lemma 3. *Let $h : \mathcal{O} \to \mathbb{R}$ be a locally Lipschitz function on an open set $\mathcal{O}$ of Z. Then the following results hold*:

(i) *For each $\mathrm{d} \in Z$, one has $h^0(x; d) = \max\{\langle y, d\rangle,\ y \in \partial h(x)\}$.*

(ii) *For each $x \in \mathcal{O}$, the generalized gradient $\partial h(x)$ is a nonempty, convex, weak*-compact subset of Z^*, and $\|y\|_{Z^*} \leqslant L$ for each $y \in \partial h(x)$ (where $L > 0$ is the Lipschitz constant of h near y).*

(iii) *The graph of the generalized gradient ∂h is close in $Z \times Z^*_{\mathrm{w}^*}$ topology, i.e., if $\{x_n\} \subset \mathcal{O}$ and $\{y_n\} \subset Z^*$ are sequences such that $y_n \in \partial h(x_n)$ and $x_n \to x$ in Z, $y_n \to y$ weak* in Z^*, then $y \in \partial h(x)$ (where $Z^*_{\mathrm{w}^*}$ denotes the Banach space Z^* equipped with the w^*-topology).*

(iv) *The multifunction $\mathcal{O} \ni x \to \partial h(x) \subseteq Z^*$ is u.s.c. from $\mathcal{O}$ into $Z^*_{\mathrm{w}^*}$.*

We list the following results, which can be found in [10].

Lemma 4. *The closure and weak closure of a convex subset of a normed space are the same.*

Lemma 5. *Let $\mathfrak{D}$ be a nonempty bounded and convex subset of a Banach space Z. Suppose that $\Upsilon : \mathfrak{D} \to \mathcal{P}(\mathfrak{D})$ is an u.s.c., condensing multivalued map. If for each $x \in \mathfrak{D}$, $\Upsilon(x)$ is a closed convex set in $\mathfrak{D}$. Then Υ has a fixed point.*

In what follows, we introduce the admissible control set as [28, p. 141]. Let Y be another separable reflexive Banach space from which the control u takes values. Let $1 < p < +\infty$ and $L^p(I, Y)$ denote the usual Banach space of all Y-valued Bochner integrable functions having pth power summable norms. We assume that the multivalued map $\mathcal{U} : I \to \mathcal{P}_{\mathrm{cl,\,cv}}(Y)$ is graph measurable, $\mathcal{U}(\cdot) \subset \Omega$, where Ω is a bounded set of Y. The admissible control set is defined as

$$U_{\mathrm{ad}} = S^p_{\mathcal{U}} = \bigl\{u \in L^p(I, \Omega)\colon\ u(t) \in \mathcal{U}(t),\ \text{a.e. } t \in I\bigr\}, \quad \frac{1}{\gamma} < p < +\infty.$$

Then $U_{\mathrm{ad}} \neq \emptyset$, which can be found in [14].

3 Existence results

In order to investigate system (1), we can consider the following fractional evolution inclusion:

$$\begin{aligned} &D^{\beta;\gamma}_{0^+} x(t) \in Ax(t) + \mathfrak{B}(t)u(t) + \partial\mathcal{F}\bigl(t, x(t)\bigr), \quad t \in I', \\ &J^{(1-\nu)}_{0^+}\bigl[x(t)\bigr]_{t=0} + g(x) = x_0. \end{aligned} \tag{2}$$

We see that each solution of system (2) is also a solution of system (1). In fact, if $x(t) \in \mathfrak{C}(I, X)$ is a solution of system (1), then there exists a function $f(t) \in \partial\mathscr{F}(t, x(t))$, a.e. $t \in I$, and satisfies the following equation:

$$\begin{aligned} &D^{\beta;\gamma}_{0^+} x(t) = Ax(t) + \mathfrak{B}(t)u(t) + f(t), \quad t \in I', \\ &J^{(1-\nu)}_{0^+}\bigl[x(t)\bigr]_{t=0} + g(x) = x_0. \end{aligned}$$

In view of above equation, we obtain

$$\begin{aligned} &\bigl\langle -D^{\beta;\gamma}_{0^+} x(t) + Ax(t) + \mathfrak{B}(t)u(t),\, d\bigr\rangle_X + \bigl\langle f(t), d\bigr\rangle_X = 0, \quad \text{a.e. } t \in I',\ \forall d \in X, \\ &J^{(1-\nu)}_{0^+}\bigl[x(t)\bigr]_{t=0} + g(x) = x_0. \end{aligned}$$

Owing to the facts that $f(t) \in \partial\mathcal{F}(t, x(t))$ and $\langle f(t), d\rangle_X \leqslant \mathcal{F}^0(t, x(t); d)$, we have

$$\begin{aligned} &\bigl\langle -D^{\beta;\gamma}_{0^+} x(t) + Ax(t) + \mathfrak{B}(t)u(t),\, d\bigr\rangle_X + \mathcal{F}^0\bigl(t, x(t); d\bigr) \geqslant 0, \quad t \in I',\ \forall d \in X, \\ &J^{(1-\nu)}_{0^+}\bigl[x(t)\bigr]_{t=0} + g(x) = x_0. \end{aligned}$$

It is shown that we can investigate system (1) by the corresponding evolution inclusion system (2).

In order to define the mild solution to system (2), we now introduce the following Wright function:

$$M_\gamma(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-\mu n)}, \quad 0 < \mu < 1,\ \theta \in \mathbb{C},$$

which satisfies the equality

$$\int_0^\infty \theta^\tau M_\gamma(\theta)\,\mathrm{d}\theta = \frac{\Gamma(1+\tau)}{\Gamma(1+\gamma\tau)}, \quad \theta \geqslant 0.$$

Definition 2. (See [11].) A function $x \in \mathfrak{C}(I, X)$ is said to be a mild solution to system (2) if there exists a function $f \in L^p(I, X)$ such that $f(t) \in \partial\mathcal{F}(t, x(t))$, a.e. $t \in I$, and the following equation holds:

$$x(t) = S_{\beta,\gamma}(t)\big[x_0 - g(x)\big] + \int_0^t T_\gamma(t-s)\big[f(s) + \mathfrak{B}(s)u(s)\big]\,\mathrm{d}s, \quad t \in I',$$

where $T_\gamma(t) = t^{\gamma-1}\mathscr{P}_\gamma(t)$, $\mathscr{P}_\gamma(t) = \int_0^\infty \gamma\theta M_\gamma(\theta)\mathcal{S}(t^\gamma\theta)\,\mathrm{d}\theta$, $S_{\beta,\gamma}(t) = J_{0+}^{\beta(1-\gamma)}T_\gamma(t)$.

We now list the following conditions:

(C1) The function $t \mapsto \mathcal{S}(t)$ is continuous in $\mathcal{B}(X)$ for all $t > 0$, and there exists a constant $M > 1$ such that $\|\mathcal{S}(t)\| \leqslant M$.
(C2) The operator $\mathcal{S}(t)$ is compact for $t > 0$.
(C3) The function $\mathcal{F} : I \times X \to \mathbb{R}$ satisfies the following conditions:
  (a) For all $x \in X$, $\mathcal{F}(\cdot, x)$ is measurable;
  (b) For a.e. $t \in I$, $\mathcal{F}(t, \cdot)$ is locally Lipschitz continuous;
  (c) For a.e. $t \in I$ and $x \in X$, there exists a function $\psi(\cdot) \in L^p(I, \mathbb{R}^+)$ $(p > 1/\gamma)$ and a constant $\varrho > 0$ such that for a.e. $t \in I$ and all $x \in X$,

$$\big\|\partial\mathcal{F}(t, x)\big\| := \sup\big\{\big\|f(t)\big\|\colon\ f(t) \in \partial\mathcal{F}(t, x)\big\} \leqslant \psi(t) + \varrho\|x\|.$$

(C4) $\mathfrak{B} : I \to \mathcal{B}(Y, X)$ is essentially bounded, i.e., $\mathfrak{B} \in L^\infty(I, \mathcal{B}(Y, X))$.
(C5) There exists a constant L_g such that for every $x_1, x_2 \in \mathfrak{C}$, $\|g(x_1) - g(x_2)\| \leqslant L_g\|x_1 - x_2\|_\nu$.

Define the operator $\mathcal{N} : L^q(I, X) \to \mathcal{P}(L^p(I, X))$ $(1/p + 1/q = 1)$ as

$$\mathcal{N}(x) = \big\{w \in L^p(I, X)\colon\ w(t) \in \partial\mathcal{F}\big(t, x(t)\big),\ \text{a.e. } t \in I,\ \forall x \in L^q(J, X)\big\}.$$

Now we have the following basic results.

Lemma 6. (*See* [11].) *Under condition* (C2)*, the following results hold true*:

(i) *The operator $\mathscr{P}_\gamma(t)$ is continuous in the uniform operator topology for $t > 0$.*
(ii) *For any fixed $t > 0$, $\{T_\gamma(t)\}_{t>0}$ and $\{S_{\beta,\gamma}(t)\}_{t>0}$ are linear operators, and for each $x \in X$,*

$$\big\|T_\gamma(t)x\big\| \leqslant \frac{Mt^{\gamma-1}}{\Gamma(\gamma)}\|x\|, \qquad \big\|S_{\beta,\gamma}(t)x\big\| \leqslant \frac{Mt^{\nu-1}}{\Gamma(\beta(1-\gamma)+\gamma)}\|x\|.$$

(iii) $\{T_\gamma(t)\}_{t>0}$ *and* $\{S_{\beta,\gamma}(t)\}_{t>0}$ *are strongly continuous, i.e., for any* $x \in X$ *and* $0 < t_1, t_2 \leqslant b$, *as* $t_1 \to t_2$, *we have*

$$\big\|T_\gamma(t_1)x - T_\gamma(t_2)x\big\| \to 0 \quad \text{and} \quad \big\|S_{\beta,\gamma}(t_1)x - S_{\beta,\gamma}(t_2)x\big\| \to 0.$$

Lemma 7. (*See* [20].) *If condition* (C3) *holds, then for* $x \in L^q(I, X)$, *the set* $\mathcal{N}(x)$ *has nonempty, convex, and weakly compact values.*

Lemma 8. (*See* [20].) *Assume that condition* (C3) *holds. Let the operator* $\mathcal{N}$ *satisfy*: $z_n \to z$ *in* $L^q(I, X)$, $w_n \in \mathcal{N}(z_n)$, *and* $w_n \rightharpoonup w$ *in* $L^p(I, X)$, *then we have* $w \in \mathcal{N}(z)$.

Remark 1. (See [28, p. 141]) According to condition (C4) and the definition of the admissible set U_{ad}, it is concluded that $\mathfrak{B}u \in L^p(I, X)$ with $1/\gamma < p < \infty$ for all $u \in U_{\mathrm{ad}}$.

In what follows, we define $B_r = \{x \in \mathcal{C}(I, X)\colon \|x\|_\nu \leqslant r\}$ for each $r > 0$.

Theorem 1. *Assume that conditions* (C1)–(C5) *are satisfied. Then system* (2) *admits at least one mild solution in a suitable ball* B_r *on* I, *provided that*

$$\left[\frac{Mb^{1+\beta(\gamma-1)}\varrho}{\Gamma(1+\gamma)} + \frac{ML_g}{\Gamma(\beta(1-\gamma)+\gamma)}\right] < 1. \tag{3}$$

Proof. Now, we define the multivalued map $\Phi : \mathcal{C}(I, X) \to \mathcal{P}(\mathcal{C}(I, X))$ as

$$\Phi(x) = \Bigg\{\phi \in \mathcal{C}(I, X)\colon \phi(t) = S_{\beta,\gamma}(t)\big[x_0 - g(x)\big] + \int_0^t T_\gamma(t-s)f(s)\,\mathrm{d}s + \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s,\ f \in \mathcal{N}(x),\ t \in I'\Bigg\},$$

where $u(t) \in U_{\mathrm{ad}}$. Clearly, the fixed points of Φ are mild solutions to system (2). We show that Φ admits a fixed point. The proof will be given in several steps.

Step 1. We show that there exists $r > 0$ such that $\Phi(B_r) \subseteq B_r$.

For each $x \in \mathcal{C}(I, X)$ and $\phi \in \Phi(x)$, there exists $f \in \mathcal{N}(x)$ such that for $t \in I'$,

$$\phi(t) = S_{\beta,\gamma}(t)\big[x_0 - g(x)\big] + \int_0^t T_\gamma(t-s)f(s)\,\mathrm{d}s + \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s.$$

From condition (C3), Lemma 6, and Hölder inequality we obtain

$$\begin{aligned} t^{1-\nu}\Bigg\|\int_0^t T_\gamma(t-s)f(s)\,\mathrm{d}s\Bigg\| &\leqslant b^{1-\nu}\frac{M}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\big[\psi(s)+\varrho\big\|x(s)\big\|\big]\,\mathrm{d}s \\ &\leqslant b^{1-1/p-\beta(1-\gamma)}\frac{M}{\Gamma(\gamma)}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\|\psi\|_{L^p}+\frac{Mb^{1+\beta(\gamma-1)}\varrho}{\Gamma(1+\gamma)}r. \end{aligned} \tag{4}$$

Similarly, according to condition (C4), we also have

$$\begin{aligned} t^{1-\nu}\Bigg\|\int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s\Bigg\| &\leqslant b^{1-\nu}\frac{M}{\Gamma(\gamma)}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p} b^{\gamma-1/p}\|\mathfrak{B}u\|_{L^p} \\ &\leqslant b^{1-1/p-\beta(1-\gamma)}\frac{M}{\Gamma(\gamma)}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\|\mathfrak{B}u\|_{L^p}. \end{aligned} \tag{5}$$

Meanwhile, taking into account condition (C5), we get

$$\begin{aligned} t^{1-\nu}\big\|S_{\beta,\gamma}(t)\big[x_0-g(x)\big]\big\| &\leqslant t^{1-\nu}\big\|S_{\beta,\gamma}(t)\big\{x_0-[g(x)-g(0)]-g(0)\big\}\big\| \\ &\leqslant \frac{M}{\Gamma(\beta(1-\gamma)+\gamma)}\big[\|x_0\|+\big\|g(0)\big\|+L_g r\big]. \end{aligned} \tag{6}$$

Combined with relations (4)–(6), we have

$$\begin{aligned} t^{1-\nu}\big\|\phi(t)\big\| \leqslant{}& b^{1-1/p-\beta(1-\gamma)}\frac{M}{\Gamma(\gamma)}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\big[\|\psi\|_{L^p}+\|\mathfrak{B}u\|_{L^p}\big] \\ &+\frac{Mb^{1+\beta(\gamma-1)}\varrho}{\Gamma(1+\gamma)}r+\frac{M}{\Gamma(\beta(1-\gamma)+\gamma)}\big[\|x_0\|+\big\|g(0)\big\|+L_g r\big]. \end{aligned}$$

Owing to relation (3), we can choose a constant $r>0$ such that

$$\begin{aligned} \big\|\Phi(x)\big\|_\nu \leqslant{}& b^{1-1/p-\beta(1-\gamma)}\frac{M}{\Gamma(\gamma)}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\big[\|\psi\|_{L^p}+\|\mathfrak{B}u\|_{L^p}\big] \\ &+\frac{M}{\Gamma(\beta(1-\gamma)+\gamma)}\big[\|x_0\|+\big\|g(0)\big\|\big] \\ &+\left[\frac{M}{\Gamma(\beta(1-\gamma)+\gamma)}L_g+\frac{Mb^{1+\beta(\gamma-1)}\varrho}{\Gamma(1+\gamma)}\right]r\leqslant r. \end{aligned}$$

Hence, we obtain that $\Phi(B_r)\subseteq B_r$.

Step 2. For each $x \in \mathcal{C}(I, X)$, Φ is convex.

In fact, for any $\phi_1, \phi_2 \in \Phi(x)$, there exist $f_1, f_2 \in \mathcal{N}(x)$ such that for $t \in I'$,

$$\phi_i(t) = S_{\beta,\gamma}(t)\big[x_0 - g(x)\big] + \int_0^t T_\gamma(t-s) f_i(s)\,\mathrm{d}s + \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s, \quad i = 1, 2.$$

Let $\vartheta \in [0, 1]$, then for each $t \in I'$, we have

$$\begin{aligned}&\big[\vartheta\phi_1 + (1-\vartheta)\phi_2\big](t)\\ &\quad= S_{\beta,\gamma}(t)\big[x_0 - g(x)\big] + \int_0^t T_\gamma(t-s)\big[\vartheta f_1 + (1-\vartheta) f_2\big](s)\,\mathrm{d}s\\ &\qquad+ \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s.\end{aligned}$$

Thanks to Lemma 7, $\vartheta f_1 + (1-\vartheta) f_2 \in \mathcal{N}(x)$ for $\vartheta \in [0, 1]$, and then $\vartheta\phi_1(t) + (1-\vartheta) \times \phi_2(t) \in \Phi(x)$, i.e., Φ is convex for each $x \in \mathcal{C}(I, X)$.

Step 3. Φ is closed for each $x \in B_r$.

Let $\{\phi_n\}_{n\geqslant 1} \in \Phi(x)$ such that $\phi_n \to \phi$ in $\mathcal{C}(I, X)$. Then there exists $f_n \in \mathcal{N}(x)$ such that for each $t \in I'$,

$$\phi_n(t) = S_{\beta,\gamma}(t)\big[x_0 - g(x)\big] + \int_0^t T_\gamma(t-s) f_n(s)\,\mathrm{d}s + \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s.$$

From (C3) and Step 1 we know that $\{f_n\}_{n\geqslant 1} \subseteq L^p(I, X)$ is bounded. In view of Lemma 7, $\mathcal{N}(x)$ is weakly compact, and we may assume, passing to a subsequence if necessary, that $f_n \to \tilde{f}$, weakly in $L^p(I, X)$. Then for each $t \in I'$,

$$\begin{aligned}\phi_n(t) &\to \phi(t)\\ &= S_{\beta,\gamma}(t)\big[x_0 - g(x)\big] + \int_0^t T_\gamma(t-s) f(s)\,\mathrm{d}s\\ &\quad+ \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s \quad \text{as } n \to \infty.\end{aligned}$$

Thus, $\phi \in \Phi(x)$.

Step 4. Φ is u.s.c. and condensing.

Now we define $\Phi := \Phi_1 + \Phi_2$ as

$$(\Phi_1 x)(t) = S_{\beta,\gamma}(t)\big[x_0 - g(x)\big];$$

$$\Phi_2(x) = \Biggl\{\phi_2 \in \mathcal{C}(I, X):\ \phi_2(t) = \int_0^t T_\gamma(t-s) f(s)\,\mathrm{d}s + \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s,\ t \in I'\Biggr\}.$$

We first show that Φ_1 is a contraction. For arbitrary $x_1, x_2 \in B_r$ and each $t \in I'$, we have from Lemma 6 and condition (C5)

$$\begin{aligned} t^{1-\nu}&\big\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\big\| \\ &= t^{1-\nu}\big\|S_{\beta,\gamma}(t)\big[x_0 - g(x_1)\big] - S_{\beta,\gamma}(t)\big[x_0 - g(x_2)\big]\big\| \\ &\leqslant \frac{M}{\Gamma(\beta(1-\gamma)+\gamma)}\big\|g(x_1) - g(x_2)\big\| \leqslant \frac{ML_g}{\Gamma(\beta(1-\gamma)+\gamma)}\|x_1 - x_2\|_\nu. \end{aligned}$$

Thus

$$\|\Phi_1(x_1) - \Phi_1(x_2)\|_\nu \leqslant \frac{ML_g}{\Gamma(\beta(1-\gamma)+\gamma)}\|x_1 - x_2\|_\nu.$$

In view of relation (3), we conclude that Φ_1 is a contraction.

Next, we show that Φ_2 is u.s.c.

(i) $\Phi_2(B_r)$ is obviously bounded.

(ii) $\Phi_2(B_r)$ is equicontinuous.

In fact, for any $x \in B_r$, $\phi_2 \in \Phi_2(x)$, there exists a function $f \in \mathcal{N}(x)$ such that

$$\phi_2(t) = \int_0^t T_\gamma(t-s) f(s)\,\mathrm{d}s + \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s, \quad t \in I'.$$

Denote by $\mathcal{Y} = \{y \in C(I, X):\ y(t) = t^{1-\nu}\Phi_2(x)(t),\ y(0) = y(0^+),\ x \in B_r\}$. For $t_1 = 0$, $0 < t_2 \leqslant b$, we can easily obtain $\|y(t_2) - y(0)\| \to 0$ as $t_2 \to 0$. For $0 < t_1 < t_2 \leqslant b$ and an arbitrarily small number $\varepsilon > 0$, we have

$$\begin{aligned} \big\|y(t_2) - y(t_1)\big\| \leqslant{}& t_2^{1-\nu}\Biggl\|\int_{t_1}^{t_2} T_\gamma(t_2-s) f(s)\,\mathrm{d}s\Biggr\| \\ &+ \big(t_2^{1-\nu} - t_1^{1-\nu}\big)\Biggl\|\int_0^{t_1} T_\gamma(t_2-s) f(s)\,\mathrm{d}s\Biggr\| \\ &+ t_1^{1-\nu}\Biggl\|\int_0^{t_1-\varepsilon} \big[T_\gamma(t_2-s) - T_\gamma(t_1-s)\big] f(s)\,\mathrm{d}s\Biggr\| \end{aligned}$$

$$
\begin{aligned}
&+ t_1^{1-\nu}\left\| \int_{t_1-\varepsilon}^{t_1} \big[T_\gamma(t_2-s) - T_\gamma(t_1-s)\big] f(s)\,\mathrm{d}s \right\| \\
&+ t_2^{1-\nu}\left\| \int_{t_1}^{t_2} T_\gamma(t_2-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s \right\| \\
&+ \big(t_2^{1-\nu} - t_1^{1-\nu}\big)\left\| \int_{0}^{t_1} T_\gamma(t_2-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s \right\| \\
&+ t_1^{1-\nu}\left\| \int_{0}^{t_1-\varepsilon} \big[T_\gamma(t_2-s) - T_\gamma(t_1-s)\big] \mathfrak{B}(s)u(s)\,\mathrm{d}s \right\| \\
&+ t_1^{1-\nu}\left\| \int_{t_1-\varepsilon}^{t_1} \big[T_\gamma(t_2-s) - T_\gamma(t_1-s)\big] \mathfrak{B}(s)u(s)\,\mathrm{d}s \right\| \\
:=\ & \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3 + \mathfrak{I}_4 + \mathfrak{I}_5 + \mathfrak{I}_6 + \mathfrak{I}_7 + \mathfrak{I}_8.
\end{aligned}
$$

For terms $\mathfrak{I}_1$, $\mathfrak{I}_5$, we have as $t_2 \to t_1$,

$$
\begin{aligned}
\mathfrak{I}_1 &\leqslant b^{1-\nu}\frac{M}{\Gamma(\gamma)}\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\big(\psi(s)+r\varrho\big)\,\mathrm{d}s \\
&\leqslant b^{1-\nu}\left[\frac{M}{\Gamma(\gamma)}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\|\psi\|_{L^p}(t_2-t_1)^{\gamma-1/p} + \frac{Mr\varrho}{\Gamma(1+\gamma)}(t_2-t_1)^{\gamma}\right] \to 0, \\
\mathfrak{I}_5 &\leqslant b^{1-\nu}\frac{M}{\Gamma(\gamma)}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\|\mathfrak{B}u\|_{L^p}(t_2-t_1)^{\gamma-1/p} \to 0.
\end{aligned}
$$

As for terms $\mathfrak{I}_2$, $\mathfrak{I}_6$, based upon Lemmas 2 and 6, Hölder inequality, and conditions (C3)–(C4), we obtain when $t_2 \to t_1$,

$$
\begin{aligned}
\mathfrak{I}_2 &\leqslant \big(t_2^{1-\nu} - t_1^{1-\nu}\big)\frac{M}{\Gamma(\gamma)}\int_0^{t_1}(t_2-s)^{\gamma-1}\big[\psi(s)+r\varrho\big]\,\mathrm{d}s \\
&\leqslant \big(t_2^{1-\nu} - t_1^{1-\nu}\big)\frac{M}{\Gamma(\gamma)}\left\{\|\psi\|_{L^p}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\big[(t_2)^{(p\gamma-1)/(p-1)} \right. \\
&\quad \left. - (t_2-t_1)^{(p\gamma-1)/(p-1)}\big]^{1-1/p}\right\} \\
&\quad + \big(t_2^{1-\nu} - t_1^{1-\nu}\big)\frac{M}{\Gamma(\gamma)}r\varrho b^{1-1/p}\big[(t_2)^{1-p(1-\gamma)} - (t_2-t_1)^{1-p(1-\gamma)}\big]^{1/p} \\
&\leqslant (t_2-t_1)^{1-\nu}\frac{M}{\Gamma(\gamma)}\|\psi\|_{L^p}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p} b^{\gamma-1/p} + (t_2-t_1)^{1-\nu}\frac{M}{\Gamma(\gamma)}r\varrho b^{\gamma} \to 0,
\end{aligned}
$$

$$\begin{aligned}\mathfrak{I}_6 &\leqslant \left(t_2^{1-\nu} - t_1^{1-\nu}\right)\frac{M}{\Gamma(\gamma)}\int_0^{t_1}(t_2-s)^{\gamma-1}\left\|\mathfrak{B}(s)u(s)\right\|\mathrm{d}s\\ &\leqslant \left(t_2^{1-\nu} - t_1^{1-\nu}\right)\frac{M}{\Gamma(\gamma)}\|\mathfrak{B}u\|_{L^p}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}b^{\gamma-1/p}\\ &\leqslant (t_2-t_1)^{1-\nu}\frac{M}{\Gamma(\gamma)}\|\mathfrak{B}u\|_{L^p}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}b^{\gamma-1/p}\to 0.\end{aligned}$$

For terms $\mathfrak{I}_3$, $\mathfrak{I}_7$, we have

$$\mathfrak{I}_3 \leqslant b^{1-\nu}\frac{M}{\Gamma(\gamma)}\int_0^{t_1-\varepsilon}\left[(t_2-s)^{\gamma-1}+(t_1-s)^{\gamma-1}\right]\left(\psi(s)+r\varrho\right)\mathrm{d}s,$$

$$\mathfrak{I}_7 \leqslant b^{1-\nu}\frac{M}{\Gamma(\gamma)}\int_0^{t_1-\varepsilon}\left[(t_2-s)^{\gamma-1}+(t_1-s)^{\gamma-1}\right]\left\|\mathfrak{B}(s)u(s)\right\|\mathrm{d}s.$$

Owing to

$$\begin{aligned}&s\mapsto\left[(t_2-s)^{\gamma-1}+(t_1-s)^{\gamma-1}\right](\psi(s)+r\varrho)\in L^1\left([0,\,t_1-\varepsilon],\mathbb{R}_+\right),\\ &s\mapsto\left[(t_2-s)^{\gamma-1}+(t_1-s)^{\gamma-1}\right]\|\mathfrak{B}(s)u(s)\|\in L^1\left([0,\,t_1-\varepsilon],\mathbb{R}_+\right),\end{aligned}$$

we conclude that $\mathfrak{I}_3$, $\mathfrak{I}_7 \to 0$ as $t_2 \to t_1$ by the Lebesgue dominated convergence theorem.

For terms $\mathfrak{I}_4$, $\mathfrak{I}_8$, taking into account Lemmas 2 and 6, Hölder inequality, and conditions (C3)–(C4), we have as $\varepsilon \to 0$,

$$\begin{aligned}\mathfrak{I}_4 &\leqslant b^{1-\nu}\left[2r\varrho\varepsilon+\frac{M}{\Gamma(\gamma)}\int_{t_1-\varepsilon}^{t_1}(t_2-s)^{\gamma-1}\psi(s)\,\mathrm{d}s+\frac{M}{\Gamma(\gamma)}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\gamma-1}\psi(s)\,\mathrm{d}s\right]\\ &\leqslant 2b^{1-\nu}\left[r\varrho\varepsilon+\frac{M}{\Gamma(\gamma)}\|\psi\|_{L^p}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\varepsilon^{\gamma-1/p}\right]\to 0,\end{aligned}$$

$$\mathfrak{I}_8 \leqslant 2b^{1-\nu}\frac{M}{\Gamma(\gamma)}\|\mathfrak{B}u\|_{L^p}\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\varepsilon^{\gamma-1/p}\to 0.$$

The right-hand side of above inequalities tends to zero independently of $x \in B_r$, and thus $\Phi_2(B_r)$ is equicontinuous.

(iii) The set $V(t) = \{\phi_2(t)\colon \phi_2(t) \in \Phi_2(B_r)\}$ is relatively compact in X.

For $t = 0$, the conclusion obviously holds. Let $0 < t \leqslant b$ be fixed. Taking into account that $T_\gamma(t) = t^{\gamma-1}\mathscr{P}_\gamma(t)$, $\mathscr{P}_\gamma(t) = \int_0^\infty \gamma\theta M_\gamma(\theta)\mathcal{S}(t^\gamma\theta)\,\mathrm{d}\theta$, for any $x \in B_r$, $\phi_2 \in \Phi_2(x)$, there exists $f \in \mathcal{N}(x)$ such that

$$\phi_2(t) = \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)f(s)\,\mathrm{d}s + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s.$$

For each $\epsilon \in (0,t)$, $t \in (0,b]$, $x \in B_r$, and any $\delta > 0$, we define an operator $\Phi_2^{\epsilon,\delta}$ on B_r by $\Phi_2^{\epsilon,\delta}(x)$ the set of $\phi_2^{\epsilon,\delta}$ such that

$$\phi_2^{\epsilon,\delta}(t) = \gamma\mathcal{S}\big(\epsilon^\gamma\delta\big) \times \int_0^{t-\epsilon}\int_\delta^\infty \theta M_\gamma(\theta)(t-s)^{\gamma-1}\mathcal{S}\big((t-s)^\gamma\theta - \epsilon^\gamma\delta\big)\big[f(s)+\mathfrak{B}(s)u(s)\big]\,\mathrm{d}\theta\,\mathrm{d}s.$$

From the compactness of $\mathcal{S}(\epsilon^\gamma\delta)$ ($\epsilon^\gamma\delta > 0$) we deduce that the set $V^{\epsilon,\delta}(t) = \{\phi_2^{\epsilon,\delta}(t)\colon \phi_2^{\epsilon,\delta}(t) \in \Phi_2^{\epsilon,\delta}(B_r)\}$ is relatively compact in X for any $\epsilon \in (0,t)$ and any $\delta > 0$. Furthermore, we have

$$\begin{aligned}
&t^{1-\nu}\big\|\phi_2(t) - \phi_2^{\epsilon,\delta}(t)\big\| \\
&= t^{1-\nu}\Bigg\|\gamma\int_0^t\int_0^\infty \theta M_\gamma(\theta)(t-s)^{\gamma-1}\mathcal{S}\big((t-s)^\gamma\theta\big)\big[f(s)+\mathfrak{B}(s)u(s)\big]\,\mathrm{d}\theta\,\mathrm{d}s \\
&\quad - \int_0^{t-\epsilon}\int_\delta^\infty \theta M_\gamma(\theta)(t-s)^{\gamma-1}\mathcal{S}\big((t-s)^\gamma\theta\big)\big[f(s)+\mathfrak{B}(s)u(s)\big]\,\mathrm{d}\theta\,\mathrm{d}s\gamma\Bigg\| \\
&\leqslant \gamma b^{1-\nu}M\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\|\psi\|_{L^p}\left[b^{\gamma-1/p}\int_0^\delta \theta M_\gamma(\theta)\,\mathrm{d}\theta + \frac{1}{\Gamma(1+\gamma)}\epsilon^{\gamma-1/p}\right] \\
&\quad + b^{1-\nu}M\varrho r\left[\frac{1}{\Gamma(1+\gamma)}\epsilon^\gamma + b^\gamma\int_0^\delta \theta M_\gamma(\theta)\,\mathrm{d}\theta\right] \\
&\quad + \gamma b^{1-\nu}M\left(\frac{p-1}{p\gamma-1}\right)^{1-1/p}\|\mathfrak{B}u\|_{L^p}\left[b^{\gamma-1/p}\int_0^\delta \theta M_\gamma(\theta)\,\mathrm{d}\theta + \frac{b^{1/p}}{\Gamma(1+\gamma)}\epsilon^{\gamma-1/p}\right].
\end{aligned}$$

Since $\int_0^\infty \theta M_\gamma(\theta)\,\mathrm{d}\theta = 1/(\Gamma(1+\gamma))$, the last inequality tends to zero as $\epsilon \to 0$ and $\delta \to 0$, i.e., there are relatively compact sets arbitrarily close to the set $V(t)$ ($t > 0$). Hence, $V(t)$ is relatively compact in X for all $t \in (0,b]$. As a consequence of above steps (i)–(iii) and the Arzela–Ascoli theorem, we can deduce that Φ_2 is completely continuous.

(iv) Φ_2 has a closed graph.

Let $x_n \to \tilde{x}$, $\phi_2^{(n)} \in \Phi_2(x_n)$, and $\phi_2^{(n)} \to \tilde{\phi}_2$. We need to show that $\tilde{\phi}_2 \in \Phi_2(\tilde{x})$. The fact $\phi_2^{(n)} \in \Phi_2(x_n)$ implies that there exists $f_n \in \mathcal{N}(x_n)$ satisfying

$$\phi_2^{(n)}(t) = \int_0^t T_\gamma(t-s)f_n(s)\,\mathrm{d}s + \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s. \tag{7}$$

From condition (C3) and Step 1 we know that $\{f_n\}_{n\geqslant 1} \subseteq L^p(I,X)$ is bounded. Thus, we may assume, passing to a subsequence if necessary, that

$$f_n \to \tilde{f} \quad \text{weakly in } L^p(I,X). \tag{8}$$

Since $\phi_2^{(n)}(t) \to \tilde{\phi}_2$, it follows from (7)–(8) and Lemma 8 that, as $n \to \infty$,

$$\phi_n(t) \to \tilde{\phi}_2(t) = \int_0^t T_\gamma(t-s)\tilde{f}(s)\,\mathrm{d}s + \int_0^t T_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s. \tag{9}$$

Since $x_n \to \tilde{x}$ and $f_n \in \mathcal{N}(x_n)$, Lemma 8 and (8)–(9) infer that $\tilde{f} \in \mathcal{N}(\tilde{x})$, and thus $\tilde{\phi}_2 \in \Phi_2(\tilde{x})$, i.e., Φ_2 has a closed graph.

As a consequence, Φ_2 is an u.s.c. multivalued map. On the other hand, Φ_1 is a contraction, hence $\Phi = \Phi_1 + \Phi_2$ is u.s.c and condensing. By Lemma 5, there exists a fixed point $x(\cdot)$ for Φ on B_r. Thus, system (2) admits a mild solution in a suitable ball B_r. The proof is completed. □

4 Optimal controls

In this section, we investigate the existence of optimal state-control pairs of the limited Lagrange optimal control problems governed by system (1). Taking into account that condition (C3) cannot guarantee the uniqueness of solutions to system (2), we need the following auxiliary results.

Lemma 9. *Assume that conditions* (C1), (C4), (3) *hold. Then the operator* $\Psi : L^p(I,Y) \to \mathcal{C}(I,X)$, $p > 1/\gamma$, *defined by*

$$(\Psi u)(\cdot) := \int_0^{\cdot} (\cdot - s)^{\gamma-1}\mathscr{P}_\gamma(\cdot - s)\mathfrak{B}(s)u(s)\,\mathrm{d}s \quad \forall u(\cdot) \in U_{\mathrm{ad}} \subset L^p(I,Y),$$

is compact.

Proof. Let $\{u_k\}_{k\geqslant 1}$ be a bounded sequence in $L^p(I,Y)$ ($p > 1/\gamma$). Then condition (C4) leads to the boundedness of $\{\mathfrak{B}u_k\}_{k\geqslant 1} \subseteq L^p(I,Y)$. Thus, by a similarly conducted as Step 4(i)–(iii) in the proof of Theorem 1, we can obtain the compactness of the operator Ψ. □

Lemma 10. *Assume conditions* (C1), (3) *and the following condition* (C5′) *are satisfied*:

(C5′) *The function* $g : \mathfrak{C}(I,X) \to X$ *is compact, and there exists a constant* L_g *such that for every* $x_1, x_2 \in \mathfrak{C}$, $\|g(x_1) - g(x_2)\| \leqslant L_g\|x_1 - x_2\|_\nu$.

Then the operator $(\Phi_1 x)(t) = S_{\beta,\gamma}(t)[x_0 - g(x)]$ *is completely continuous.*

Proof. Note that for each $x \in B_r$,

$$\begin{aligned}
&S_{\beta,\gamma}(t)\big[x_0 - g(x)\big] \\
&\quad = J_{0^+}^{\beta(1-\gamma)} T_\gamma(t)\big[x_0 - g(x)\big] \\
&\quad = \frac{1}{\Gamma(\beta(1-\gamma))} \int_0^t (t-s)^{\beta(1-\gamma)-1} T_\gamma(s)\big[x_0 - g(x)\big]\,\mathrm{d}s \\
&\quad = \frac{\gamma}{\Gamma(\beta(1-\gamma))} \int_0^t (t-s)^{\beta(1-\gamma)-1} s^{\gamma-1} \int_0^\infty \theta M_\gamma(\theta) \mathcal{S}(s^\gamma\theta)\big[x_0 - g(x)\big]\,\mathrm{d}\theta\,\mathrm{d}s,
\end{aligned}$$

and define

$$\begin{aligned}
\big(\varPhi_1^{\epsilon,\delta} x\big)(t) = {}& \frac{\gamma}{\Gamma(\beta(1-\gamma))} \\
&\times \int_0^{t-\epsilon} \int_\delta^\infty \theta M_\gamma(\theta)(t-s)^{\beta(1-\gamma)-1} s^{\gamma-1} \mathcal{S}(s^\gamma\theta)\big[x_0 - g(x)\big]\,\mathrm{d}\theta\,\mathrm{d}s.
\end{aligned}$$

Analogously to Steps 4(i)–(iii) in the proof of Theorem 1, we can show that $(\varPhi_1 x)(t)$ is completely continuous. □

For any $u \in U_{\mathrm{ad}}$, let $\mathcal{S}(u)$ denote all mild solutions to systems (1) in B_r defined in Theorem 1. Denote $x^u \in B_r$ by the mild solution of system (1) corresponding to the control $u \in U_{\mathrm{ad}}$, we consider the following limited Lagrange problem:

Problem. Find $x^0 \in B_r \subseteq C(I, X)$ and $u^0 \in U_{\mathrm{ad}}$ such that for all $u \in U_{\mathrm{ad}}$, $\mathcal{J}(x^0, u^0) \leqslant \mathcal{J}(x^u, u)$, where

$$\mathcal{J}(x^u, u) = \int_0^b \mathfrak{L}\big(t, x^u(t), u(t)\big)\,\mathrm{d}t, \tag{10}$$

and $x^0 \in B_r$ denotes the mild solution to system (1) related to the control $u^0 \in U_{\mathrm{ad}}$.

We remark that under the conditions of Theorem 1, a pair $(x(\cdot), u(\cdot))$ is feasible if it verifies system (1) for $x(\cdot) \in B_r$, and if $(x^u(\cdot), u(\cdot))$ is feasible, then $x^u \in \mathcal{S}(u) \subseteq B_r$.

In order to deal with the existence of optimal state-control pairs for problem (10), we further impose the following condition:

(C6) The function $\mathfrak{L} : I \times X \times Y \to \mathbb{R} \bigcup \{\infty\}$ satisfies:

(a) The function $\mathfrak{L} : I \times X \times Y \to \mathbb{R} \bigcup \{\infty\}$ is Borel measurable;
(b) $\mathfrak{L}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times Y$ for a.e. $t \in I$;
(c) $\mathfrak{L}(t, x, \cdot)$ is convex on Y for each $x \in X$ and a.e. $t \in I$;

(d) There exist constants $c \geqslant 0$, $d > 0$, ζ is nonnegative and $\zeta \in L^1(I, \mathbb{R})$ such that
$$\mathfrak{L}(t, x, u) \geqslant \zeta(t) + c\|x\| + d\|u\|_Y^p, \quad 1 < p < \infty.$$

Theorem 2. *Assume that conditions* (C1)–(C4), (C5′)–(C6)*, and* (3) *are satisfied. Then problem* (10) *governed by system* (1) *admits at least one optimal state-control pair.*

Proof. For any given $u \in U_{\mathrm{ad}}$, we define
$$\mathcal{J}(u) = \inf_{x^u \in \mathcal{S}(u)} \mathcal{J}\big(x^u, u\big).$$

If the set $\mathcal{S}(u)$ admits only finitely many elements, there exists some $\tilde{x}^u \in \mathcal{S}(u)$ such that $\mathcal{J}(\tilde{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u) = \mathcal{J}(u)$. It is trivial if the set $\mathcal{S}(u)$ admits infinitely many elements and $\inf_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u) = +\infty$. Now, we assume that $\mathcal{J}(u) = \inf_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u) < +\infty$. By condition (C6) we have $\mathcal{J}(u) > -\infty$. For the sake of convenience, we divide the proof into the following several steps.

Step 1. Based upon the definition of infimum, there exists a sequence $\{x_n^u\} \subseteq \mathcal{S}(u)$ satisfying $J(x_n^u, u) \to J(u)$ as $n \to \infty$. Taking into account that $\{x_n^u, u\}$ is a sequence of feasible pairs, we have
$$\begin{aligned} x_n^u(t) &= S_{\beta,\gamma}(t)\big[x_0 - g(x_n^u)\big] + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s) f_n^u(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s, \quad f_n^u \in \mathcal{N}(x_n^u),\ t \in I'. \end{aligned} \tag{11}$$

Step 2. It is shown that there exists some $\tilde{x}^u \in \mathcal{S}(u)$ such that $\mathcal{J}(\tilde{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} \mathcal{J}(x^u, u) = \mathcal{J}(u)$.

To achieve this aim, we first prove that for each $u \in U_{\mathrm{ad}}$, $\{x_n^u\}$ is relatively compact in $C(I, X)$. From Step 1 we have
$$\begin{aligned} x_n^u(t) &= S_{\beta,\gamma}(t)\big[x_0 - g(x_n^u)\big] + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s) f_n^u(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s \\ &:= \mathscr{I}_1 x_n^u + \mathscr{I}_2 x_n^u + \mathscr{I}_3 x_n^u. \end{aligned}$$

In view of Lemma 10 and Steps 4(i)–(iii) in the proof of Theorem 1, we can conclude that $\{\mathscr{I}_1 x_n^u\}$, $\{\mathscr{I}_2 x_n^u\}$, $\{\mathscr{I}_3 x_n^u\}$ are all relatively compact subsets of $\mathcal{C}(I, X)$. In consequence, the set $\{x_n^u\}$ is relatively compact in $\mathcal{C}(I, X)$ for $u \in U_{\mathrm{ad}}$. Without loss of generality, we may assume that $x_n^u \to \tilde{x}^u$ in $\mathcal{C}I, X)$ for $u \in U_{\mathrm{ad}}$ as $n \to \infty$.

Moreover, by conditions (C3), (C5′), we have $f_n^u(t) \to \tilde{f}(t)$, a.e. $t \in I$, and $\|f_n^u(t)\| \leqslant \psi(t)+r\varrho$, $g(x_n^u) \to g(\tilde{x}^u)$. Let $n \to \infty$ on both sides of (11), by the Lebesgue dominated convergence theorem, we obtain that

$$\tilde{x}^u(t) = S_{\beta,\gamma}(t)\big[x_0 - g(\tilde{x}^u)\big] + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\tilde{f}(s)\,\mathrm{d}s \\ + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\mathfrak{B}(s)u(s)\,\mathrm{d}s, \quad t \in I',$$

which implies that $\tilde{x}^u \in \mathcal{S}(u)$. Thus, through the definition of a feasible pair, condition (C6) and Balder theorem [4], we have

$$\mathcal{J}(u) = \lim_{n\to\infty} \int_0^b \mathfrak{L}\big(t, x_n^u(t), u(t)\big)\,\mathrm{d}t \geqslant \int_0^b \mathfrak{L}\big(t, \tilde{x}^u(t), u(t)\big)\,\mathrm{d}t \\ = \mathcal{J}\big(\tilde{x}^u, u\big) \geqslant \mathcal{J}(u),$$

i.e., $\mathcal{J}(\tilde{x}^u, u) = \mathcal{J}(u)$. This implies that $\mathcal{J}(u)$ admits its minimum at $\tilde{x}^u \in \mathcal{C}(I, X)$ for each $u \in U_{\mathrm{ad}}$.

Step 3. It is shown that there exists $u^0 \in U_{\mathrm{ad}}$ such that $\mathcal{J}(u^0) \leqslant \mathcal{J}(u)$ for all $u \in U_{\mathrm{ad}}$.

If $\inf_{u\in U_{\mathrm{ad}}} \mathcal{J}(u) = +\infty$, it is trivial. Assume that $\inf_{u\in U_{\mathrm{ad}}} \mathcal{J}(u) < +\infty$. By condition (C6) again, we can prove that $\inf_{u\in U_{\mathrm{ad}}} \mathcal{J}(u) > -\infty$, and similarly to Step 1, there exists a sequence $\{u_n\} \subseteq U_{\mathrm{ad}}$ such that $\mathcal{J}(u_n) \to \inf_{u\in U_{\mathrm{ad}}} \mathcal{J}(u)$ as $n \to \infty$. Since $\{u_n\} \subseteq U_{\mathrm{ad}}$, $\{u_n\}$ is bounded in $L^p(I, Y)$ and $L^p(I, Y)$ is a reflexive Banach space for $1/\gamma < p < +\infty$, there exists a subsequence still denoted by $\{u_n\}$ weakly converges to some $u^0 \in L^p(I, Y)$ as $n \to \infty$. Note that U_{ad} is closed and convex, by Lemma 4 it follows that $u^0 \in U_{\mathrm{ad}}$.

Let $\tilde{x}^{u_n}$ be the mild solution to system (1) related to u_n, where $\mathcal{J}(u_n)$ attains its minimum. Then $(\tilde{x}^{u_n}, u_n)$ is a feasible pair and verifies the following integral equation

$$\tilde{x}^{u_n}(t) = S_{\beta,\gamma}(t)\big[x_0 - g(\tilde{x}^{u_n})\big] + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\tilde{f}_n(s)\,\mathrm{d}s \\ + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\mathfrak{B}(s)u_n(s)\,\mathrm{d}s, \quad \tilde{f}_n \in \mathcal{N}\big(\tilde{x}^{u_n}\big),\ t \in I'. \tag{12}$$

Let us define

$$\Lambda_1\tilde{x}^{u_n}(t) := S_{\beta,\gamma}(t)\big[x_0 - g(\tilde{x}^{u_n})\big],$$

$$\Lambda_2\tilde{x}^{u_n}(t) := \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\tilde{f}_n(s)\,\mathrm{d}s, \quad \tilde{f}_n \in \mathcal{N}\big(\tilde{x}^{u_n}\big),$$

$$\Lambda_3 u_n(t) := \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\mathfrak{B}(s)u_n(s)\,\mathrm{d}s.$$

Then

$$\tilde{x}^{u_n}(t) = \Lambda_1\tilde{x}^{u_n}(t) + \Lambda_2\tilde{x}^{u_n}(t) + \Lambda_3 u_n(t), \quad t \in I'.$$

From Lemma 10, and similarly to Steps 5(i)–(iii) in the proof of Theorem 1, we can conclude that $\{\Lambda_1\tilde{x}^{u_n}\}$, $\{\Lambda_2\tilde{x}^{u_n}\}$ are all relatively compact subsets of $\mathcal{C}(I,X)$. Additionally, by the fact $\{u_n\}$ weakly converges to some $u^0 \in L^p(I,Y)$ and Lemma 9, Λ_3 is compact and $\Lambda_3 u_n \to \Lambda_3 u^0$ as $n \to \infty$. Thus, the set $\{\tilde{x}^{u_n}\} \subset \mathcal{C}(I,X)$ is relatively compact, and there exists a subsequence still denoted by $\{\tilde{x}^{u_n}\}$, $\tilde{x}^{u^0} \in \mathcal{C}(I,X)$ such that $\tilde{x}^{u_n} \to \tilde{x}^{u^0}$ in $\mathcal{C}(I,X)$ as $n \to \infty$. Furthermore, by conditions (C3), (C5′), we have $\tilde{f}_n(t) \to \tilde{f}_*(t)$, a.e. $t \in I$, and $\|\tilde{f}_n(t)\| \leqslant \psi(t) + r\varrho$, $g(\tilde{x}^{u_n}) \to g(\tilde{x}^{u^0})$. Let $n \to \infty$ in both sides of (12), by the Lebesgue dominated convergence theorem, we have

$$\tilde{x}^{u^0}(t) = S_{\beta,\gamma}(t)\big[x_0 - g(\tilde{x}^{u^0})\big] + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\tilde{f}_*(s)(s)\,\mathrm{d}s + \int_0^t (t-s)^{\gamma-1}\mathscr{P}_\gamma(t-s)\mathfrak{B}(s)u^0(s)\,\mathrm{d}s, \quad t \in I',$$

which implies that $(\tilde{x}^{u^0}, u^0)$ is a feasible pair.

Thus, by condition (C6) and Balder theorem again [4], we obtain

$$\inf_{u\in U_{\mathrm{ad}}} \mathcal{J}(u) = \lim_{n\to\infty} \int_0^b \mathfrak{L}\big(t, \tilde{x}^{u_n}(t), u_n(t)\big)\,\mathrm{d}t \geqslant \int_0^b \mathfrak{L}\big(t, \tilde{x}^{u^0}(t), u^0(t)\big)\,\mathrm{d}t = \mathcal{J}(\tilde{x}^{u^0}, u^0) \geqslant \inf_{u\in U_{\mathrm{ad}}} \mathcal{J}(u).$$

Therefore,

$$\mathcal{J}(\tilde{x}^{u^0}, u^0) = J(u^0) = \inf_{x^{u^0}\in\mathcal{S}(u^0)} \mathcal{J}\big(x^{u^0}, u^0\big).$$

Furthermore,

$$\mathcal{J}(u^0) = \inf_{u\in U_{\mathrm{ad}}} \mathcal{J}(u),$$

i.e., $\mathcal{J}$ admits its minimum at $u^0 \in U_{\mathrm{ad}}$. This finishes the proof. □

Example 1. Finally, we end this paper with a simple example. For numerical stimulation of Hilfer fractional derivatives and approximate solutions of some fractional differential systems, we can refer to references [3, 12]. Consider the following inclusion problem:

$$\begin{aligned} &D_{0^+}^{\beta,4/7}x(t,\xi) \in \frac{\partial^2 x}{\partial \xi^2}(t,\xi) + \partial F\big(t, x(t,\xi)\big) + u(t,\xi), \quad t \in (0,1],\ \xi \in [0,\pi], \\ &x(t,0) = x(t,\pi) = 0, \quad t \in [0,1], \\ &J_{0^+}^{(3/7)(1-\beta)}\big(x(0,\xi)\big) + \sum_{i=0}^{m} \int_0^{\pi} k(t,s)x(t_i,s)\,\mathrm{d}s = x_0(\xi), \quad \xi \in [0,\pi], \end{aligned} \tag{13}$$

where $D_{0^+}^{\beta,4/7}$ is the Hilfer fractional derivative of order $4/7$ and type $\beta \in [0,1]$, $J_{0^+}^{(3/7)(1-\beta)}$ is the Riemann–Liouville integral of order $(3/7)(1-\beta)$. $k(t,s) \in L^2([0,\pi]\times[0,\pi])$, m is a positive integer and $0 < t_0 < t_1 < \cdots < t_m \leqslant 1$. Take $X = Y = L^2[0,\pi]$. Let $x(\cdot)(\xi) = x(\cdot,\xi)$, $\mathfrak{B}(\cdot)u(\cdot)(\xi) = u(\cdot,\xi)$, and

$$\mathcal{J}(x,u) = \int_0^{\pi}\int_0^{1} \big|x(t,\xi)\big|^2\,\mathrm{d}t\,\mathrm{d}\xi + \int_0^{\pi}\int_0^{1} \big|u(t,\xi)\big|^2\,\mathrm{d}t\,\mathrm{d}\xi.$$

Here ∂F denotes generalized gradient of a locally Lipschitz function F. A simple example of F satisfying condition (A2) is $F(t,\eta) = F(\eta) = \min\{f_1(\eta), f_2(\eta)\}$, where $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are convex quadratic functions (see [20, 21]).

Let operator $A : D(A) \subset X \to X$ be defined by $Av = v''$ with the domain $D(A) := \{v \in X\colon v \in H^2([0,\pi]),\ v(0) = v(\pi) = 0\}$. Then A generates a strongly continuous semigroup $\{\mathcal{S}(t)\}_{t\geqslant 0}$, which is compact for $t > 0$, analytic and self-adjoint. It is known that A has discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenvectors are given by $e_n(s) := \sqrt{2/\pi}\sin(ns)$. Moreover, $\{e_n\colon n \in \mathbb{N}\}$ is an orthonormal basis for X, and thus A can be written as $Az = \sum_{n=1}^{\infty} n^2\langle z, e_n\rangle e_n$, $z \in D(A)$. Particularly, $\|\mathcal{S}(t)\| \leqslant \mathrm{e}^{-t}$ (see [23] for details). Let $g(x)(y) = \sum_{i=0}^{m}\int_0^{\pi} k(y,z)x(t_i)(z)\,\mathrm{d}z = \sum_{i=0}^{m}\int_0^{\pi} k(y,z)x(t_i,z)\,\mathrm{d}z$, thus g satisfies condition (C5′)(see [26]). Note that problem (13) can be rewritten in the abstract form (2). According to Theorems 1–2, Eq. (13) has a mild solution for ϱ, L_g properly small, and its corresponding limited Lagrange problem admits at least one optimal feasible pair.

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