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Global dynamics of a fourth-order parabolic equation describing crystal surface growth*

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Abstract. In this paper, we study the global dynamics for the solution semiflow of a fourth-order parabolic equation describing crystal surface growth. We show that the equation has a global attractor in $H^4_{\mathrm{per}}(\Omega)$ when the initial value belongs to $H^1_{\mathrm{per}}(\Omega)$.

Keywords: global dynamics, fourth-order parabolic equation, absorbing set.

1 Introduction

In the field of infinite-dimensional dynamical systems, one of the most important issues is to obtain the existence of global attractors for the semigroups of solutions associated with some concrete partial differential equations. There are many studies on the existence of global attractors for diffusion equations. For the classical results, we refer the reader to [2, 3, 8, 19, 20, 24] and the reference cited therein.

The model we studied here arises from the study of molecular beam epitaxy. Suppose that F denotes the incident mass flux out of the molecular beam, the height $H(x,t)$ of the surface above the substrate plane satisfies a continuity equation

$$\frac{\partial}{\partial t}H + \nabla \cdot J_{\mathrm{surface}}\{H\} = F. \tag{1}$$

In general, the systematic current J_{surface} depends on the whole surface configuration. Keeping only the most important terms in a gradient expansion, subtracting the mean height $H = Fu$ and using appropriately rescaled units of height, distance and time [18], Eq. (1) attains the following dimensionless form:

$$\frac{\partial u}{\partial t} = -\Delta^2 u - \nabla \cdot \left[f\left(\nabla u^2\right)\nabla u\right],$$

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where $\Delta^2 u$ describes relaxation through adatom diffusion driven by the surface free energy [13], $\nabla \cdot [f(\nabla u^2)\nabla u]$ models the nonequilibrium current [10], respectively. Assuming in-plane symmetry, it follows that the nonequilibrium current is (anti)parallel to the local tilt ∇u with a magnitude $f(\nabla u^2)$ depending only on the magnitude of the title. Within a Burton–Cabrera–Frank-type theory [11], for small tilts, the current is proportional to $|\nabla u|$, and the opposite limit is proportional to $|\nabla u|^{-1}$. Hence, by the interpolation formula $f(s^2) = 1/(1+s^2)$ [9, 17], we obtain the following equation:

$$u_t = -a\Delta^2 u - \mu\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right), \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{2}$$

where a and μ are positive constants, $\Omega \subset \mathbb{R}^2$ is a bounded domain, respectively.

In this paper, we study the global dynamics of solutions to Eq. (2), which describes the crystal surface growth. We suppose that $\Omega = [0, L] \times [0, L]$, where $L > 0$. Moreover, on the basis of physical considerations, the equation is supplemented by the following boundary conditions:

$$\varphi|_{x_i=0} = \varphi|_{x_i=L}, \quad i = 1, 2, \tag{3}$$

for u and the derivatives of u at least of order $\leqslant 3$, and the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{4}$$

Remark 1. Since the derivation procedure of (2) is attached to a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$, our study focus on the 2D case, which seems meaningful in physical.

During the past years, many authors have paid much attention to Eq. (2). For example, Rost and Krug [17] studied the unstable epitaxy on singular surfaces using Eq. (2) with a prescribed slope-dependent surface current. In the limit of weak desorption, Pierre-Louis et al. [15] derived Eq. (2) for a vicinal surface growing in the step flow mode. This limit turned out to be singular, and nonlinearities of arbitrary order need to be taken into account. Recently, Grasselli et al. [7] showed that Eq. (2) endowed with no-flux boundary conditions generates a dissipative dynamical system under very general assumptions on $\partial\Omega$ on a phase-space of L^2-type. They proved that the system possesses a global as well as an exponential attractor. In [27], based on the iteration technique for regularity estimates and the classical existence theorem of global attractors, Zhao and Liu proved the existence of global attractor for Eq. (2) on some affine space of H^k ($0 \leqslant k < +\infty$) when the initial value belongs to H^k space. Zhao et al. [26] also consider the existence and uniqueness of time-periodic generalized solutions for Eq. (2) in 1D case. Very recently, Zhao and Cao [25], Duan and Zhao [5] invistigated the optimal control problem for Eq. (2) in one-dimensional and two-dimensional cases, respectively. There are also some other papers related to the well-posedness of molecular beam epitaxy equations in $\mathbb{R}^N$ and $\mathbb{T}^N$; we refer the reader to [6, 12] and the reference cited therein.

In this paper, we are interested in the existence of global attractors for problem (2)–(4). The outline of this paper is as follows. We begin by giving some preparations and the main results on the existence of global attractor in Section 2. Then, in Section 3, we establish some uniform estimates. In Section 4, we prove the main result. The conclusion of this paper is postponed in the last section.

2 Preliminaries

The weak formulation of problem (2)–(4) is obtained by multiplying (2) by a test function $v \in H^2_{\mathrm{per}}(\Omega)$ and using the Green formula and the boundary condition. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, v) + a(\Delta u, \Delta v) = \mu\left(\frac{\nabla u}{1 + |\nabla u|^2}, \nabla v\right) \quad \forall v \in H^2_{\mathrm{per}}(\Omega). \tag{5}$$

It is worth pointing out that the total mass of the solution $u(x,t)$ is conserved. Indeed, when we replace v by 1 in (5), we find

$$\frac{\partial}{\partial t}\int\limits_{\Omega} u(x,t)\,\mathrm{d}x = 0, \qquad \int\limits_{\Omega} u(x,t)\,\mathrm{d}x = \int\limits_{\Omega} u_0(x)\,\mathrm{d}x \quad \forall t > 0.$$

We assume that the initial function satisfies $\int_{\Omega} u_0(x)\,\mathrm{d}x = 0$, then it follows that $\int_{\Omega} u(x,t)\,\mathrm{d}x = 0$ for $t > 0$. Set

$$\dot{H}^k_{\mathrm{per}} = \left\{u \;\middle|\; u \in H^k_{\mathrm{per}}(\Omega), \int\limits_{\Omega} u(x,t)\,\mathrm{d}x = 0\right\}, \quad k = 1, 2, \dots .$$

For convenience, in this section, using the same method as [7], we summarize the result on the existence and uniqueness of global solution for problem (2)–(4).

Lemma 1. *Let $u_0 \in \dot{H}^1_{\mathrm{per}}(\Omega)$. Then problem* (2)–(4) *possesses a unique global solution $u(x,t)$ such that*

$$u \in \mathcal{C}\big([0,\infty); \dot{H}^1_{\mathrm{per}}(\Omega)\big) \cap \mathcal{C}^1\big((0,\infty); L^2(\Omega)\big) \cap \mathcal{C}\big((0,\infty); \dot{H}^4_{\mathrm{per}}(\Omega)\big).$$

Remark 2. A mild solution to problem (2)–(4) can also be yielded with initial data $u_0 \in \dot{H}^2_{\mathrm{per}}(\Omega)$. There are also some classical results on the mild solution to higher-order parabolic equations in the subcritical case of the scale of Banach spaces embedding into $L^q(\Omega)$-spaces; we refer the reader to [4, 16] and the reference cited therein.

By virtue of Lemma 1 we define the operator semigroup

$$S(t)u_0 : \dot{H}^1_{\mathrm{per}}(\Omega) \times \mathbb{R}^+ \to \dot{H}^1_{\mathrm{per}}(\Omega),$$

which is $(\dot{H}^1_{\mathrm{per}}, \dot{H}^1_{\mathrm{per}})$-continuous. In what follows, we always assume that $\{S(t)\}_{t\geqslant 0}$ is the semigroup generated by the weak solutions of problem (2)–(4). It is sufficiently to see that the restriction of $\{S(t)\}$ on the affined space $\dot{H}^1_{\mathrm{per}}(\Omega)$ is a well-defined semigroup.

In order to prove the existence of global attractor, we give some definitions and results.

Definition 1. (See [2, 23].) Let B be a bounded subset of $H^4(\Omega)$. B is said to be a bounded (H^1, H^4)-absorbing set for $\{S(t)\}_{t\geqslant 0}$ if for every bounded subset E in H^1, there exists $T > 0$ depending on B such that

$$S(t)E \subseteq B \quad \forall t \geqslant T.$$

Definition 2. (See [2, 23].) $\{S(t)\}_{t\geqslant 0}$ is said to be (H^1, H^4)-asymptotically compact if for any bounded $\{u_{0,n}\}_{n=1}^{\infty}$ in H^1 and $t_n \to \infty$, $\{S(t_n)u_{0,n}\}_{n=1}^{\infty}$ has a convergent subsequence in $H^4(\Omega)$.

Remark 3. The assumption that the operator $S(t)$ is compact on a separable Banach space for all $t > 0$ (semigroup of compact operators) is met by large classes of dynamical system of physical interest. Triggiani [21, 22] pointed out that parabolic PDE defined on bounded spatial domains represent an important subclass of dynamical systems, whose correspondent semigroups are compact for all $t > 0$. In this paper, in order to let the proof process more complete and systematic, we also give the definition and proof of compactness for $S(t)$—Definition 2 and Lemma 9.

Definition 3. (See [2, 23].) Let $\mathcal{A}$ be a subset of $H^4(\Omega)$. $\mathcal{A}$ is said to be an (H^1, H^4)-global attractor if the following conditions are satisfied:

(i) $\mathcal{A}$ is compact in $H^4(\Omega)$;
(ii) $\mathcal{A}$ is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$;
(iii) $\mathcal{A}$ attracts every bounded subset of H^1 with respect to the norm of $H^4(\Omega)$, that is, if E is bounded in H^1, then

$$\operatorname{dist}_{H^4}\bigl(S(t)E, \mathcal{A}\bigr) \to 0 \quad \text{as } t \to \infty.$$

Proposition 1. (*See* [2,23].) *Let $\mathcal{A}$ be an (H^1, H^1)-global attractor for $\{S(t)\}_{t\geqslant 0}$. Then $\mathcal{A}$ is also an (H^1, H^4)-global attractor if and only if*

(i) $\{S(t)\}_{t\geqslant 0}$ *has a bounded (H^1, H^4)-absorbing set*;
(ii) $\{S(t)\}_{t\geqslant 0}$ *is (H^1, H^4)-asymptotically compact.*

The main result of this article is given by the following theorem, which provides the existence of global attractors of problem (2)–(4).

Theorem 1. *Suppose that $u_0 \in \dot{H}^1_{\mathrm{per}}(\Omega)$, the coefficient a is sufficiently large, then problem* (2)–(4) *has a $(\dot{H}^1_{\mathrm{per}}, \dot{H}^4_{\mathrm{per}})$-global attractor for the solution $u(x,t)$, which is invariant and compact in $\dot{H}^4_{\mathrm{per}}(\Omega)$ and attracts every bounded subset of $\dot{H}^1_{\mathrm{per}}(\Omega)$ with respect to the norm topology of $\dot{H}^4_{\mathrm{per}}(\Omega)$.*

Remark 4. In [27], by using iterative principle and the properties of sectorial operator, the authors established the existence of $H^k(\Omega)$-global attractor for problem (2)–(4), provided that $u_0 \in H^k_{\mathrm{per}}(\Omega)$ ($k \in \mathbb{R}^+$). Here, we only assume the initial data $u_0 \in H^1_{\mathrm{per}}(\Omega)$, and we prove that problem (2)–(4) has a global attractor in $H^4_{\mathrm{per}}(\Omega)$. Our assumption on the initial data seems more relax than [27].

3 Uniform estimates of solutions

In this section, we establish the uniform estimates of solutions of problem (2)–(4) as $t \to \infty$. These estimates are necessary to prove the existence of global attractors.

Lemma 2. *Suppose that* $u_0 \in L^2(\Omega)$, *then for problem* (2)–(4), *we have*

$$\|u(t)\| \leqslant M_0 \quad \textit{and} \quad \int_t^{t+1} \|\Delta u(t)\|^2 \,\mathrm{d}\tau \leqslant M_0 \quad \forall t \geqslant T_0.$$

Here, $M_0 = M_0(a)$ *is a positive constant depending on* a, $T_0 = T_0(a, R)$ *depends on* a *and* R, *where* $\|u_0\|^2 \leqslant R^2$.

Proof. Multiplying Eq. (2) by u and integrating the resulting relation over Ω, we derive that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + a\|\Delta u\|^2 + \mu \int_\Omega \frac{|\nabla u|^2}{1 + |\nabla u|^2}\,\mathrm{d}x = 0,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2a\|\Delta u\|^2 \leqslant 0. \tag{6}$$

Note that $\int_\Omega u(x,t)\,\mathrm{d}x = 0$. By virtue of Poincaré's inequality [20], we obtain

$$\|u\|^2 \leqslant C'\|\nabla u\|^2. \tag{7}$$

Moreover, we also have

$$\|\nabla u\|^2 = \int_\Omega |\nabla u|^2\,\mathrm{d}x = -\int_\Omega u\Delta u\,\mathrm{d}x \leqslant \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\Delta u\|^2. \tag{8}$$

Combining (7) and (8) together gives

$$\|u\|^2 \leqslant C_1\|\Delta u\|^2. \tag{9}$$

It then follows from (6) and (9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + C_1\|u\|^2 \leqslant 0.$$

Applying Gronwall's inequality [20], we deduce that

$$\|u\|^2 \leqslant \mathrm{e}^{-C_1 t}\|u_0\|^2 \leqslant C_2 \quad \forall t \geqslant T^*, \tag{10}$$

$T^* = (1/C_1)\ln C_2 R^2$. Integrating (6) over $(t,\, t+1)$ with $t \geqslant T^*$ yields

$$\int_t^{t+1} \|\Delta u\|^2\,\mathrm{d}\tau \leqslant C_3. \tag{11}$$

Applying a mean value theorem for integrals, we obtain the existence of a time $t_0' \in (T^*,\, T^*+1)$ such that the following estimate holds uniformly:

$$\|\Delta u(t_0')\|^2 \leqslant C_4. \qquad \square$$

Lemma 3. *Suppose that $u_0 \in \dot{H}^1_{\mathrm{per}}(\Omega)$, then for problem* (2)–(4)*, we have*

$$\big\|\nabla u(t)\big\| \leqslant M_1 \quad \textit{and} \quad \int\limits_t^{t+1} \big\|\nabla\Delta u(t)\big\|^2 \,\mathrm{d}\tau \leqslant M_1 \quad \forall t \geqslant T_1.$$

Here, $M_1 = M_1(a)$ is a positive constant depending on a, $T_1 = T_1(a, R)$ depends on a and R, where $\|u_0\|^2_{H^1_{\mathrm{per}}} \leqslant R^2$.

Proof. Multiplying Eq. (2) by $-\Delta u$ and integrating the resulting relation over Ω, by using Young's inequality [3], we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + a\|\nabla\Delta u\|^2 &\leqslant \frac{a}{2}\|\nabla\Delta u\|^2 + \frac{\mu^2}{a}\int\limits_\Omega \frac{|\nabla u|^2}{(1+|\nabla u|^2)^2}\,\mathrm{d}x \\ &\leqslant \frac{a}{2}\|\nabla\Delta u\|^2 + \frac{\mu^2}{4a}|\Omega|,\end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + a\|\nabla\Delta u\|^2 \leqslant C_5. \tag{12}$$

Using the Gagliardo–Nirenberg inequality [3, 14], we have

$$\|\nabla u\|^2 \leqslant \big(C_1'\|\nabla\Delta u\|^{1/3}\|u\|^{2/3} + C_2'\|u\|\big)^2 \leqslant a\|\nabla\Delta u\|^2 + C. \tag{13}$$

Owning to (12) and (13), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \|\nabla u\|^2 \leqslant C_6.$$

By using Gronwall's inequality, we derive that

$$\|\nabla u\|^2 \leqslant \mathrm{e}^{-t}\|\nabla u_0\|^2 + C_6 \leqslant 2C_6 \quad \forall t \geqslant T', \tag{14}$$

$T' = \max\{T^*, \ln(R^2/C_6)\}$. Integrating (12) over $(t,\ t+1)$ with $t \geqslant T'$ yields

$$\int\limits_t^{t+1} \|\nabla\Delta u\|^2 \,\mathrm{d}\tau \leqslant C_7.$$

Using a mean value theorem for integrals, we obtain the existence of a time $t_0 \in (T', T'+1)$ such that the following estimate holds uniformly:

$$\big\|\nabla\Delta u(t_0)\big\|^2 \leqslant C_8,$$

this complete the proof. □

Lemma 4. *Suppose that $u_0 \in \dot{H}^1_{\mathrm{per}}(\Omega)$, then for problem* (2)–(4)*, we have*

$$\|\Delta u(t)\| \leqslant M_2 \quad \textit{and} \quad \int\limits_t^{t+1} \|u_t\|^2 \, \mathrm{d}\tau \leqslant M_2 \quad \forall t \geqslant T_2.$$

Here, $M_2 = M_2(a)$ is a positive constant depending on a, $T_2 = T_2(a, R)$ depends on a and R, where $\|u_0\|^2_{H^1_{\mathrm{per}}} \leqslant R^2$.

Proof. Multiplying Eq. (2) by $\Delta^2 u$ and integrating the resulting relation over Ω, using Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + a\|\Delta^2 u\|^2 \\
&\quad = -\mu \int\limits_\Omega \nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2}\right) \Delta^2 u \, \mathrm{d}x \\
&\quad \leqslant \frac{a}{4}\|\Delta^2 u\|^2 + \frac{\mu^2}{a} \int\limits_\Omega \left[\frac{\Delta u}{1 + |\nabla u|^2} - \frac{2|\nabla u|^2 \Delta u}{(1 + |\nabla u|^2)^2}\right]^2 \mathrm{d}x \\
&\quad \leqslant \frac{a}{2}\|\Delta^2 u\|^2 + C \int\limits_\Omega \left(\frac{\Delta u}{1 + |\nabla u|^2}\right)^2 \mathrm{d}x + C \int\limits_\Omega \frac{|\nabla u|^4 |\Delta u|^2}{(1 + |\nabla u|^2)^4} \, \mathrm{d}x \\
&\quad \leqslant \frac{a}{2}\|\Delta^2 u\|^2 + C\|\Delta u\|^2 + C\|\Delta u\|^2,
\end{aligned}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + a\|\Delta^2 u\|^2 \leqslant C\|\Delta u\|^2. \tag{15}$$

Applying the Gagliardo–Nirenberg inequality, it yields

$$\|\Delta u\|^2 \leqslant \left(C_1\|\Delta^2 u\|^{1/2}\|u\|^{1/2} + C_2'\|u\|\right)^2 \leqslant \varepsilon\|\Delta^2 u\|^2 + C_\varepsilon. \tag{16}$$

Letting ε small enough, combining (15) and (16) together gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \frac{a}{2}\|\Delta^2 u\|^2 \leqslant C_9. \tag{17}$$

Owning to (17) and (16), we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \|\Delta u\|^2 \leqslant C_{10}.$$

Applying Gronwall's inequality, we get

$$\|\Delta u\|^2 \leqslant \mathrm{e}^{-(t-t_0')}\|\Delta u(t_0')\|^2 + C_{10} \leqslant C_4 \mathrm{e}^{-(t-t_0')} + C_{10} \leqslant 2C_{10} \quad \forall t \geqslant T_0', \tag{18}$$

$T_0' = \max\{T_0, t_0' + \ln(C_4/C_{10})\}$. Setting $t \geqslant T_0{}'$, taking $s \in (t, t+1)$, integrating (17) over $(s, t+1)$, it yields

$$\big\|\Delta u(t+1)\big\|^2 \leqslant C + \big\|\Delta u(s)\big\|^2.$$

Integrating the above inequality with respect to s in $(t, t+1)$, using (11), we obtain

$$\big\|\Delta u(t+1)\big\|^2 \leqslant C + \int\limits_t^{t+1} \big\|\Delta u(s)\big\|^2 \,\mathrm{d}\tau \leqslant C_{11} \quad \forall t \geqslant T_0'. \tag{19}$$

Multiplying Eq. (2) by u_t, integrating the resulting relation over Ω, we derive that

$$\|u_t\|^2 + \frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 - \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \ln\big(1 + |\nabla u|^2\big)\,\mathrm{d}\tau = 0. \tag{20}$$

Integrating (20) over $(t+1,\ t+2)$, by using (19), it yields that

$$\int\limits_{t+1}^{t+2} \|u_t\|^2 \,\mathrm{d}\tau \leqslant C_{12} \quad \forall t \geqslant T_0'.$$

Then, by using a mean value theorem for integrals, we obtain the existence of a time $t_1 \in (T_0{}' + 1,\ T_0{}' + 2)$ such that the following estimate holds uniformly:

$$\big\|u_t(t_1)\big\|^2 \leqslant C_{13},$$

this complete the proof. □

Lemma 5. *Suppose that $u_0 \in \dot{H}^1_{\mathrm{per}}(\Omega)$, then for problem* (2)–(4)*, we have*

$$\big\|\nabla\Delta u(t)\big\| \leqslant M_3 \quad \textit{and} \quad \int\limits_t^{t+1} \big\|\nabla u_t(t)\big\|^2 \,\mathrm{d}t \leqslant M_3 \quad \forall t \geqslant T_3.$$

Here, $M_3 = M_3(a)$ is a positive constant depending on a, $T_3 = T_3(a, R)$ depends on a and R, where $\|u_0\|^2_{H^1} \leqslant R^2$.

Proof. Acting the Laplace operator on (2), we obtain

$$\frac{\partial \Delta u}{\partial t} + a\Delta^3 u + \mu\Delta\bigg[\nabla\cdot\bigg(\frac{\nabla u}{1 + |\nabla u|^2}\bigg)\bigg] = 0. \tag{21}$$

Equation (21) is supplemented with the boundary by the following boundary conditions:

$$\varphi|_{x_i=0} = \varphi|_{x_i=L_i}, \quad i = 1, 2,$$

for u and the derivatives of u at least of order $\geqslant 2$ and $\leqslant 5$.

Multiplying (21) by $\Delta^2 u$ and integrating on Ω, applying the boundary conditions, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 + a\|\nabla\Delta^2 u\|^2 \\
&\quad= -\mu\int_\Omega \Delta\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\nabla\Delta^2 u\,\mathrm{d}x \\
&\quad\leqslant \frac{a}{2}\|\nabla\Delta^2 u\|^2 + \frac{\mu^2}{2a}\int_\Omega \left|\Delta\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right|^2\mathrm{d}x \\
&\quad\leqslant \frac{a}{2}\|\nabla\Delta^2 u\|^2 + C\left(\int_\Omega \frac{|\nabla\Delta u|^2}{(1+|\nabla u|^2)^2}\,\mathrm{d}x + \int_\Omega \frac{|\nabla u|^2|\Delta u|^4}{(1+|\nabla u|^2)^4}\,\mathrm{d}x\right. \\
&\qquad\left. + \int_\Omega \frac{|\nabla u|^4|\nabla\Delta u|^2}{(1+|\nabla u|^2)^4}\,\mathrm{d}x + \int_\Omega \frac{|\nabla u|^6|\Delta u|^4}{(1+|\nabla u|^2)^6}\,\mathrm{d}x\right) \\
&\quad\leqslant \frac{a}{2}\|\nabla\Delta^2 u\|^2 + \frac{C_{14}}{2}\|\nabla\Delta u\|^2 + \frac{C_{15}}{2}\|\Delta u\|_4^4.
\end{aligned} \tag{22}$$

Using the Gagliardo–Nirenberg inequality, we deduce that

$$\|\Delta u\|_4^4 \leqslant \left(C_1'\|\nabla\Delta^2 u\|^{1/6}\|\Delta u\|^{5/6} + C_2'\|\Delta u\|\right)^4 \leqslant \varepsilon\|\nabla\Delta^2 u\|^2 + C_{16}, \tag{23}$$

$$\|\nabla\Delta u\|^2 \leqslant \left(C_1'\|\nabla\Delta^2 u\|^{1/3}\|\Delta u\|^{2/3} + C_2'\|\Delta u\|\right)^2 \leqslant \varepsilon\|\nabla\Delta^2 u\|^2 + C_{17}. \tag{24}$$

Letting ε small enough in (23) and (24), combining (22)–(24) together gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 + \frac{a}{2}\|\nabla\Delta^2 u\|^2 \leqslant C_{18}. \tag{25}$$

Owning to (24) and (25), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 + \|\nabla\Delta u\|^2 \leqslant C_{19}.$$

Using Gronwall's inequality, we derive that

$$\begin{aligned}
\|\nabla\Delta u\|^2 &\leqslant \mathrm{e}^{-(t-t_0)}\|\nabla\Delta u(t_0)\|^2 + C_{19} \leqslant C_8\mathrm{e}^{-(t-t_0)} + C_{19} \\
&\leqslant 2C_{19} \quad \forall t \geqslant T_1^*,
\end{aligned} \tag{26}$$

$T_1^* = \max\{T_2,\, t_0 + \ln(C_8/C_{19})\}$. Combining (10), (14), (18) and (26) together gives

$$\|\nabla u\|_\infty \leqslant C_{20}, \quad \|\Delta u\|_q \leqslant C_{21}, \quad 1 \leqslant q < +\infty,\ t \geqslant T_1^*.$$

Setting $v = u_t$, multiplying Eq. (21) by v, integrating the resulting relation over Ω, we obtain

$$\begin{aligned}\|\nabla v\|^2 + \frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 &= -\mu\int_\Omega \Delta\left[\frac{\nabla u}{1+|\nabla u|^2}\right]\nabla v\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\|\nabla v\|^2 + \frac{\mu^2}{2}\int_\Omega \left|\Delta\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right|^2\mathrm{d}x \\ &\leqslant \frac{1}{2}\|\nabla v\|^2 + C_{22}\big(\|\nabla\Delta u\|^2 + \|\Delta u\|_4^4\big) \\ &\leqslant \frac{1}{2}\|\nabla v\|^2 + \frac{C_{23}}{2},\end{aligned}$$

which yields

$$\|\nabla v\|^2 + a\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 \leqslant C_{23}, \tag{27}$$

that is,

$$a\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 \leqslant C_{23}.$$

Letting $t \geqslant T_1^*$, taking $s \in (t, t+1)$, integrating the above inequality over $(s, t+1)$, we obtain

$$\big\|\nabla\Delta u(t+1)\big\|^2 \leqslant \frac{1}{a}\big(C_{23} + \big\|\nabla\Delta u(s)\big\|^2\big).$$

Integrating the above inequality with respect to s in $(t,\ t+1)$, we have

$$\big\|\nabla\Delta u(t+1)\big\|^2 \leqslant \frac{1}{a}\left(C_{23} + \int_t^{t+1}\big\|\nabla\Delta u(s)\big\|^2\,\mathrm{d}s\right) \leqslant C_{24} \quad \forall t \geqslant T_1^*. \tag{28}$$

Integrating (27) over $(t+1,\ t+2)$, using (28), we get

$$\int_{t+1}^{t+2}\|\nabla v\|^2\,\mathrm{d}\tau \leqslant C_{25} \quad \forall t \geqslant T_1^*.$$

Applying a mean value theorem for integrals, we obtain the existence of a time $t_2 \in (T_1^* + 1,\ T_1^* + 2)$ such that the following estimate holds uniformly:

$$\big\|\nabla v(t_2)\big\|^2 \leqslant C_{26},$$

this complete the proof. □

Lemma 6. *Suppose that $u_0 \in \dot{H}^1_{\mathrm{per}}(\Omega)$, a is sufficiently large, then for problem* (2)–(4)*, we have*

$$\|u_t\| \leqslant M_4 \quad \forall t \geqslant T_4.$$

Here, $M_4 = M_4(a)$ is a positive constant depending on a, $T_4 = T_4(a, R)$ depends on a and R, where $\|u_0\|^2_{H^1_{\mathrm{per}}} \leqslant R^2$.

Proof. Differentiating (2) with respect to the time t, setting $v = u_t$, we deduce that

$$v_t + a\Delta^2 v + \mu\left[\nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2}\right)\right]_t = 0. \tag{29}$$

Multiplying (29) by v, integrating the resulting relation over Ω, we derive that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + a\|\Delta v\|^2 &= \mu \int_\Omega \left(\frac{\nabla u}{1 + |\nabla u|^2}\right)_t \nabla v \,\mathrm{d}x \\ &= \mu \int_\Omega \frac{|\nabla v|^2}{1 + |\nabla u|^2}\,\mathrm{d}x + \mu \int_\Omega \frac{2|\nabla u|^2|\nabla v|^2}{(1 + |\nabla u|^2)^2}\,\mathrm{d}x \\ &\leqslant \frac{3\mu}{2}\|\nabla v\|^2 \leqslant \frac{a}{2}\|\Delta v\|^2 + \frac{9\mu^2}{8a}\|v\|^2,\end{aligned}$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + a\|\Delta v\|^2 \leqslant \frac{9\mu^2}{2a}\|v\|^2. \tag{30}$$

Using Poincaré's inequality two times, we have

$$\|v\|^2 \leqslant \frac{1}{C'}\|\Delta v\|^2.$$

It follows from (30) and the above inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \left(C'a - \frac{9\mu^2}{2a}\right)\|v\|^2 \leqslant C_{27},$$

where a is sufficiently large, it satisfies $C'a - 9\mu^2/(2a) > 0$. Using Gronwall's inequality, we derive that

$$\begin{aligned}\|v\|^2 &\leqslant \mathrm{e}^{-(C'a - 9\mu^2/(2a))(t-t_1)}\big\|v(t_1)\big\|^2 + \frac{2aC_{27}}{2a^2C' - 9\mu^2} \\ &\leqslant C_{13}\mathrm{e}^{-(C'a - 9\mu^2/(2a))(t-t_1)} + \frac{2aC_{27}}{2a^2C' - 9\mu^2} \\ &\leqslant \frac{4aC_{27}}{2a^2C' - 9\mu^2} \quad \forall t \geqslant t_1 + \frac{2a}{2a^2C' - 9\mu^2}\ln\frac{C_{13}(2a^2C' - 9\mu^2)}{2aC_{27}}.\end{aligned}$$

The proof is complete. □

Lemma 7. *Suppose that $u_0 \in \dot{H}^1_{\mathrm{per}}(\Omega)$, the coefficient a is sufficiently large, then for problem* (2)–(4)*, we have*

$$\big\|\nabla v_t(t)\big\| \leqslant M_5 \quad \forall t \geqslant T_5.$$

Here, $M_5 = M_5(a)$ is a positive constant depending on a, $T_5 = T_5(a, R)$ depends on a and R, where $\|u_0\|^2_{H^1_{\mathrm{per}}} \leqslant R^2$.

Proof. Multiplying (30) by Δv, integrating the resulting relation over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + a\|\nabla\Delta v\|^2 \\
&\quad = \mu\int\limits_{\Omega}\left(\frac{\nabla u}{1+|\nabla u|^2}\right)_t \nabla\Delta v\,\mathrm{d}x \leqslant \frac{a}{4}\|\nabla\Delta v\|^2 + \frac{\mu^2}{a}\int\limits_{\Omega}\left(\frac{\nabla u}{1+|\nabla u|^2}\right)_t^2 \mathrm{d}x \\
&\quad \leqslant \frac{a}{4}\|\nabla\Delta v\|^2 + C\int\limits_{\Omega}\left(\frac{\nabla v}{1+|\nabla u|^2}\right)^2 \mathrm{d}x + C\int\limits_{\Omega}\left(\frac{2|\nabla u|^2\nabla v}{(1+|\nabla u|^2)^2}\right)^2 \mathrm{d}x \\
&\quad \leqslant \frac{a}{4}\|\nabla\Delta v\|^2 + C\|\nabla v\|^2. \qquad (31)
\end{aligned}$$

Applying the Gagliardo–Nirenberg inequality, we arrive at

$$\|\nabla v\|^2 \leqslant \left(C_1'\|\nabla\Delta v\|^{1/3}\|v\|^{2/3} + C_2'\|v\|\right)^2 \leqslant \varepsilon\|\nabla\Delta v\|^2 + C_{28}. \qquad (32)$$

Letting ε small enough in (32), combining (31) and (32) together gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + a\|\nabla\Delta v\|^2 \leqslant C_{29}.$$

Owning to (32) and (30), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + \|\nabla v\|^2 \leqslant C_{30}.$$

Then, by Gronwall's inequality, we derive that

$$\begin{aligned}
\|\nabla v\|^2 &\leqslant \mathrm{e}^{-(t-t_2)}\big\|\nabla v(t_2)\big\|^2 + C_{30} \leqslant C_{26}\mathrm{e}^{-(t-t_2)} + C_{30} \\
&\leqslant 2C_{30} \quad \forall t \geqslant t_2 + \ln\frac{C_{26}}{C_{30}}.
\end{aligned}$$

The proof is complete. □

Lemma 8. *Suppose that $u_0 \in \dot H^1_{\mathrm{per}}(\Omega)$, the coefficient a is sufficiently large, then for problem* (2)–(4)*, we have*

$$\big\|\Delta^2 u(t)\big\| \leqslant M_6 \quad \forall t \geqslant T_6.$$

Here, $M_6 = M_6(a)$ is a positive constant depending on a, $T_6 = T_6(a,R)$ depends on a and R, where $\|u_0\|^2_{H^1_{\mathrm{per}}} \leqslant R^2$.

Proof. For Eq. (2), by Lemmas 2–7, we have

$$\begin{aligned}
\big\|\Delta^2 u\big\| &\leqslant \frac{1}{a}\left(\|u_t\| + \left\|\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right\|\right) \\
&\leqslant \frac{1}{a}\left(\|u_t\| + 2\left\|\frac{\Delta u}{1+|\nabla u|^2}\right\| + 4\left\|\frac{|\nabla u|^2\Delta u}{(1+|\nabla u|^2)^2}\right\|\right) \\
&\leqslant \frac{1}{a}\big(\|u_t\| + 2\|\Delta u\| + \|\Delta u\|\big) \leqslant C_{31} \quad \forall t \geqslant T.
\end{aligned}$$

By using Sobolev's embedding theorem [1], we arrive at

$$\|\Delta u\|_\infty \leqslant C_{32}.$$

Then, the proof is complete. □

4 Proof of Theorem 1

Consider problem (2)–(4), we first show that $\{S(t)\}_{t\geqslant 0}$ has a $(H^1_{\mathrm{per}}, H^1_{\mathrm{per}})$-global attractor, and then we prove that this attractor is actually an $(H^1_{\mathrm{per}}, H^4_{\mathrm{per}})$-attractor for the solution u of problem (2)–(4) by Proposition 1.

We suppose that M_1 and M_6 are the constants in Lemmas 3 and 8, respectively. Denote

$$\begin{aligned} B_1 &= \bigl\{u \in \dot H^1_{\mathrm{per}}\colon\ \|\nabla u\| \leqslant M_1\bigr\},\\ B_2 &= \bigl\{u \in \dot H^4_{\mathrm{per}}\colon\ \bigl\|\Delta^2 u\bigr\| \leqslant M_6\bigr\}. \end{aligned} \tag{33}$$

Using Lemmas 3 and 8, we know that B_1 is a bounded $(\dot H^1_{\mathrm{per}}, \dot H^1_{\mathrm{per}})$-absorbing set for $\{S(t)\}_{t\geqslant 0}$ and B_2 is a bounded $(\dot H^1_{\mathrm{per}}, \dot H^4_{\mathrm{per}})$-absorbing set for $\{S(t)\}_{t\geqslant 0}$, respectively. Applying the compactness of embedding $\dot H^4_{\mathrm{per}} \hookrightarrow \dot H^1_{\mathrm{per}}$ and Lemma 4, we find that $\{S(t)\}_{t\geqslant 0}$ is $(\dot H^1_{\mathrm{per}}, \dot H^1_{\mathrm{per}})$-asymptotically compact. Therefore, based on the standard attractors theory (see [8, 19, 20]), $\{S(t)\}_{t\geqslant 0}$ has a $(\dot H^1_{\mathrm{per}}, \dot H^1_{\mathrm{per}})$-global attractor $\mathcal{A}$. In the following, we show that $\mathcal{A}$ is actually an $(\dot H^1_{\mathrm{per}}, \dot H^4_{\mathrm{per}})$-global attractor for $\{S(t)\}_{t\geqslant 0}$. To this end, we have to prove that $\{S(t)\}_{t\geqslant 0}$ is $(H^1_{\mathrm{per}}, \dot H^4_{\mathrm{per}})$-asymptotically compact, which is given by the following lemma.

Lemma 9. *Suppose that $u_0 \in \dot H^1_{\mathrm{per}}(\Omega)$, the coefficient a is sufficiently large, then for the solution $u(x,t)$ of problem* (2)–(4)*, the dynamical system $\{S(t)\}_{t\geqslant 0}$ is $(\dot H^1_{\mathrm{per}}, \dot H^4_{\mathrm{per}})$-asymptotically compact.*

Proof. Suppose that $\{u_{0,n}\}_{n=1}^\infty$ is bounded in $\dot H^1_{\mathrm{per}}(\Omega)$ and $t_n \to \infty$. In the following, we prove that $\{S(t_n)u_{0,n}\}_{n=1}^\infty$ has a convergent subsequence in $\dot H^4_{\mathrm{per}}(\Omega)$. Denote

$$u_n(t) = S(t)u_{0,n} \quad \text{and} \quad v_n(t_n) = \frac{\mathrm{d}v_n}{\mathrm{d}t}\bigg|_{t=t_n}.$$

It follows from (2) that

$$a\Delta^2 u = -u_t - \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right).$$

Since $\{u_{0,n}\}_{n=1}^\infty$ is bounded in $\dot H^1_{\mathrm{per}}$, there exists $R>0$ such that

$$\|u_{0,n} + \nabla u_{0,n}\| \leqslant R \quad \forall n = 1,2,\dots .$$

By Lemmas 7 and 8 there exists $T>0$ such that for all $t \geqslant T$,

$$\left\|\frac{\mathrm{d}u_n}{\mathrm{d}t}\right\|_{H^1_{\mathrm{per}}} \leqslant M_5, \quad \|u_n\|_{H^4_{\mathrm{per}}} \leqslant M_6 \quad \forall n = 1,2,\dots, \tag{34}$$

where M_5 and M_6 are positive constants in Section 2, respectively. Since $t_n \to \infty$, there exists $N > 0$ such that $t_n \geqslant T$ for all $n \geqslant N$. Therefore, owning to (34), we arrive at

$$\left\|v_n(t_n)\right\|_{H^1_{\mathrm{per}}} \leqslant M_5, \quad \left\|u_n(t_n)\right\|_{H^4_{\mathrm{per}}} \leqslant M_6 \quad \forall n \geqslant N. \tag{35}$$

On the basis of the compactness of embedding $H^1 \hookrightarrow H$ and $H^4 \hookrightarrow H^2$, we find from (34) that there exist $v \in \dot{H}^1_{\mathrm{per}}(\Omega)$, $\Delta u \in \dot{H}^2_{\mathrm{per}}(\Omega)$, $\nabla u \in \dot{H}^3_{\mathrm{per}}(\Omega)$ and $u \in \dot{H}^4_{\mathrm{per}}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} v_n(t_n) &\to v && \text{strongly in } H,\\ \Delta u_n(t_n) &\to \Delta u && \text{strongly in } \dot{H}^1_{\mathrm{per}},\\ \nabla u_n(t_n) &\to \nabla u && \text{strongly in } \dot{H}^2_{\mathrm{per}},\\ u_n(t_n) &\to u && \text{strongly in } \dot{H}^3_{\mathrm{per}}. \end{aligned} \tag{36}$$

Hence, it follows from (35) and Sobolev's embedding theorem [1] that

$$\left\|u_n(t_n)\right\|_{W^{2,\infty}} \leqslant C \quad \forall n \geqslant N.$$

Owning to (34) and (36), we derive that

$$\left\|u_n(t_n) - u\right\| \to 0, \quad \left\|v_n(t_n) - v\right\|^2 \to 0, \quad \left\|\Delta u_n(t_n) - \Delta u\right\|^2 \to 0,$$

and

$$\begin{aligned} &\left\|\nabla \cdot \left(\frac{\nabla u_n(t_n)}{1 + |\nabla u_n(t_n)|^2}\right) - \nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2}\right)\right\| \\ &\quad \leqslant C\left\|\frac{\Delta u_n(t_n)}{1 + |\nabla u_n(t_n)|^2} - \frac{\Delta u}{1 + |\nabla u_n(t_n)|^2} + \frac{\Delta u}{1 + |\nabla u_n(t_n)|^2} - \frac{\Delta u}{1 + |\nabla u|^2}\right\| \\ &\qquad + C\left\|\frac{|\nabla u_n(t_n)|^2}{(1 + |\nabla u_n(t_n)|^2)^2}\left(\Delta u_n(t_n) - \Delta u\right)\right. \\ &\qquad \left. + \Delta u\left(\frac{|\nabla u_n(t_n)|^2}{(1 + |\nabla u_n(t_n)|^2)^2} - \frac{|\nabla u|^2}{(1 + |\nabla u|^2)^2}\right)\right\|. \end{aligned}$$

A simple calculation shows that

$$\begin{aligned} &\left\|\nabla \cdot \left(\frac{\nabla u_n(t_n)}{1 + |\nabla u_n(t_n)|^2}\right) - \nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2}\right)\right\| \\ &\quad \leqslant C\left(\left\|\frac{1}{1 + |\nabla u_n(t_n)|^2}\right\|_\infty \left\|\Delta u_n(t_n) - \Delta u\right\|\right. \\ &\qquad + \|\Delta u\|_\infty \left\|\frac{2\nabla\psi\Delta\psi}{(1 + |\nabla\psi|^2)^2}\right\|_\infty \left\|\nabla u_n(t_n) - \nabla u\right\| \end{aligned}$$

$$
\begin{aligned}
&+ \left\| \frac{|\nabla u_n(t_n)|^2}{(1+|\nabla u_n(t_n)|^2)^2} \right\|_\infty \big\|\Delta u_n(t_n) - \Delta u\big\| \\
&+ \|\Delta u\|_\infty \left\| \frac{2\nabla\kappa\Delta\kappa}{(1+|\nabla\kappa|^2)^2} - \frac{4\nabla\kappa^3\Delta\kappa}{(1+|\nabla\kappa|^2)^3} \right\|_\infty \big\|\nabla u_n(t_n) - \nabla u\big\| \Big) \\
&\leqslant C\big(\big\|\Delta u_n(t_n) - \Delta u\big\| + \big\|\nabla u_n(t_n) - \nabla u\big\|\big) \to 0,
\end{aligned}
$$

where $\psi = \theta_1 u_n(t_n) + (1-\theta_1)u$, $\kappa = \theta_2 u_n(t_n) + (1-\theta_2)u$, $\theta_1, \theta_2 \in (0,1)$. Therefore,

$$
a\Delta^2 u_n(t_n) \to -u_t - \mu\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right) \quad \text{strongly in } H,
$$

that is, $\{u_n(t_n)\}_{n=1}^\infty$ converges to $(1/a)\Delta^{-2}[-u_t - \mu\nabla\cdot(\nabla u/(1+|\nabla u|^2))]$ in $\dot H^4_{\mathrm{per}}(\Omega)$, this complete the proof. $\square$

Now we give the proof of the main result.

Proof of Theorem 1. Note that $\{S(t)\}_{t\geqslant 0}$ has a $(\dot H^1_{\mathrm{per}}, \dot H^1_{\mathrm{per}})$-global attractor $\mathcal{A}$ as mentioned above. By Lemma 8, the bounded set B_2 given by (33) is a bounded $(\dot H^1_{\mathrm{per}}, \dot H^4_{\mathrm{per}})$-absorbing set for $\{S(t)\}_{t\geqslant 0}$. In addition, Lemma 9 shows that $\{S(t)\}_{t\geqslant 0}$ is $(\dot H^1_{\mathrm{per}}, \dot H^4_{\mathrm{per}})$-asymptotically compact. Then, by Proposition 1, $\mathcal{A}$ is actually an $(\dot H^1_{\mathrm{per}}, \dot H^4_{\mathrm{per}})$-global attractor for $\{S(t)\}_{t\geqslant 0}$. The proof is complete. $\square$

5 Conclusion

The dynamic properties of diffusion equation and diffusion system such as the global asymptotical behaviors of solutions and global attractors are important for the study of diffusion model. In this paper, we show that problem (2)–(4), which models the crystal surface growth, has a H^4_{per}-global attractor provided that the initial data $u_0 \in H^1_{\mathrm{per}}(\Omega)$. The results on the existence of global attractor have an analytical complexity slightly about what material scientists normally encounter, then potentially making the analysis more difficult to interpret for a non-mathematician. We also believe that our approach is more satisfying than multiple numerical simulations because with computed solutions there is always the question of whether all interesting states of the system have been detected.

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