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A new eigenvalue problem for the difference operator with nonlocal conditions

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Abstract. In the paper, the spectrum structure of one-dimensional differential operator with nonlocal conditions and of the difference operator, corresponding to it, has been exhaustively investigated. It has been proved that the eigenvalue problem of difference operator is not equivalent to that of matrix eigenvalue problem $Au = \lambda u$, but it is equivalent to the generalized eigenvalue problem $Au = \lambda Bu$ with a degenerate matrix B. Also, it has been proved that there are such critical values of nonlocal condition parameters under which the spectrum of both the differential and difference operator are continuous. It has been established that the number of eigenvalues of difference problem depends on the values of these parameters. The condition has been found under which the spectrum of a difference problem is an empty set. An elementary example, illustrating theoretical expression, is presented.

Keywords: eigenvalue problem, nonlocal condition, difference operator.

1 Introduction and problem statement

During last several decades of the development of differential equations theory and numerical analysis, there is an increased interest in problems with various types of nonlocal conditions. A separate class of these problems is eigenvalue problems of differential and difference operators. Eigenvalue problems of differential operator with nonlocal conditions can be interpreted as a separate case of the non-self-adjoint operators theory [12].

In papers [4, 16, 18, 20, 21], eigenvalue problems of one- and two-dimensional differential operators with various nonlocal boundary conditions were analyzed.

Eigenvalue problems of difference operators with nonlocal conditions usually arise when solving boundary problems by the finite difference method. The spectrum properties of difference operators with various nonlocal boundary conditions were explored for investigation of the stability of difference schemes [1, 2, 4, 7, 8, 10, 11]. Another sphere of such a spectrum analysis application is convergence of iterative methods for the systems of difference equations [17, 19, 22], in particular, for nonlinear elliptic equations with integral boundary conditions [22, 23].

Many articles on the investigation of the partial differential equations with various types of nonlocal conditions were published presenting new mathematical models in heat conduction, thermoelasticity, underground water flow, biochemistry and so on. References to the original papers can be found in [2, 19, 21]. Solving these problems by the finite difference method, we meet unavoidably the problem of the structure of the spectrum of difference operator. Therefore, the eigenvalue problems could be interpreted as one of the methods of modeling.

In [5, 6], the eigenvalue problem was investigated in connection with the existence, uniqueness and multiplicity of the solution of differential problems with nonlocal conditions.

The spectrum of differential and difference operators with nonlocal conditions is much more diverse and complicated as compared to the spectrum in the case of the classical boundary conditions (Dirichlet or Neumann). Let us take such an eigenvalue problem with the Bitsadze–Samarskii nonlocal condition [20]:

$$\begin{aligned}&\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 0, \quad x \in (0,1),\\ &u(0) = 0, \quad u(1) = \gamma u(\xi), \quad \xi \in (0,1),\end{aligned}$$

where γ, ξ are given real numbers. It has been proved that, depending on the values of these parameters, in the spectrum of both differential and difference operators, there can be zero, positive, negative or complex values. Besides, though the matrix of a difference problem is non-symmetrical (except the case $\gamma = 0$), we can determine intervals of γ and ξ in which all the eigenvalues are real and positive.

Next, let us take the corresponding difference problem

$$\begin{aligned}&\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \quad i = 1, \ldots, N-1,\\ &u_0 = 0, \qquad u_N = \gamma u_s,\end{aligned}$$

where $h = 1/N$, $\xi = Sh$, which is equivalent to $(N-1)$-order matrix A eigenvalue problem $Au = \lambda u$, $u = \{u_i\}$, $i = i, \ldots, N-1$. It has been proved that, under certain values of γ and ξ, matrix A may have a parasitic eigenvalue without any correspondence as $h \to 0$. For example, if

$$\gamma = \frac{(-1)^{N-S}}{\xi},$$

there exists a matrix eigenvalue $\lambda = 4/h^2$ with the corresponding eigenvector

$$u = \{u_i\} = \bigl\{(-1)^i cih\bigr\}, \quad i = 1, \dots, N-1,$$

where $c \neq 0$ is any real number. As $h \to 0$, the limit of the eigenvector does not exist.

In paper [18], one more singularity of the spectrum of a difference operator with nonlocal conditions is indicated. Let us take a differential eigenvalue problem with integral conditions

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 0, \quad x \in (0,1),$$

$$u(0) = \gamma_1 \int_0^1 u(x)\,\mathrm{d}x, \qquad u(1) = \gamma_2 \int_0^1 u(x)\,\mathrm{d}x.$$

The following difference problem corresponds to it with the approximation error $O(h^2)$:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \quad i = 1, \dots, N-1,$$

$$u_0 = \gamma_1 h\left(\frac{u_0 + u_N}{2} + \sum_{i=1}^{N-1} u_i\right), \qquad u_N = \gamma_2 h\left(\frac{u_0 + u_N}{2} + \sum_{i=1}^{N-1} u_i\right).$$

This difference eigenvalue problem for all the values γ_1, γ_2 and h, except one case as $h = 2/(\gamma_1 + \gamma_2)$, is equivalent to the eigenvalue problem $Au = \lambda u$, where A is the $(N-1)$-order matrix. If $\gamma_1 + \gamma_2 > 2$ and $h < 2/(\gamma_1 + \gamma_2)$, then all eigenvalues of difference operator are positive except one that is negative. This negative eigenvalue tends to infinity $(-\infty)$ as $h \to 2/(\gamma_1 + \gamma_2)$. This fact is well illustrated by numerical experiment when $\gamma_1 + \gamma_2$ is quite large positive number. If $h > 2/(\gamma_1 + \gamma_2)$, then all eigenvalues are positive, and one of them tends to infinity $(+\infty)$ as $h \to 2/(\gamma_1 + \gamma_2)$. In the case $h = 2/(\gamma_1 + \gamma_2)$, difference eigenvalue problem cannot be written in the matrix form $Au = \lambda u$.

In this paper, we consider a differential eigenvalue problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 0, \quad x \in (0,1), \tag{1}$$

$$u(0) = \gamma_1 u(1), \tag{2}$$

$$u(\xi) = \gamma_2 u(1-\xi), \quad 0 < \xi < 1, \tag{3}$$

and a difference eigenvalue problem, corresponding to it. Such a difference eigenvalue problem was investigated in paper [3] in which some necessary conditions for the parameters γ_1, γ_2 and ξ were obtained in order that zero, positive, negative or complex eigenvalues might exist.

In this paper, we have investigated in detail the spectrum of differential and difference operators and drew a new qualitative conclusions. Particularly, we have proved that, depending on the parameters γ_1, γ_2, ξ and h, the spectrum structure of a difference operator can be essentially different from the spectrum structure both of differential operator and that of matrix.

2 Eigenvalue problem of a differential operator

We investigate the spectrum structure of a differential operator, defined by formulas (1)–(3). First, we analyze when there exist real eigenvalues by separate three cases: $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$.

Theorem 1. *The number* $\lambda = 0$ *is an eigenvalue of differential problem* (1)–(3) *if and only if the following condition is true*:

$$(\gamma_1\gamma_2 - 1)\xi = (\gamma_1 - \gamma_2)(1 - \xi). \tag{4}$$

Proof. As $\lambda = 0$, the general solution of equation (1) is

$$u(x) = c_1 x + c_2,$$

where c_1 and c_2 are arbitrary constants. By substituting this expression into nonlocal conditions (2) and (3), we obtain a system of equations with two unknowns c_1 and c_2

$$\begin{aligned} &-\gamma_1 c_1 + (1 - \gamma_1)c_2 = 0, \\ &\bigl(\xi - \gamma_2(1 - \xi)\bigr)c_1 + (1 - \gamma_2)c_2 = 0. \end{aligned} \tag{5}$$

This system has a nontrivial solution if and only if

$$D = \begin{vmatrix} -\gamma_1 & 1 - \gamma_1 \\ \xi - \gamma_2(1 - \xi) & 1 - \gamma_2 \end{vmatrix} = 0.$$

After elementary rearrangement, it follows (4) from this equality. □

Remark 1. In the coordinate plane (γ_1, γ_2), as ξ is a fixed number, equation (4) determines a hyperbola. Points $(-1, -1)$ and $(1, 1)$ always belong to the hyperbola independent of the ξ value (see Fig. 1).

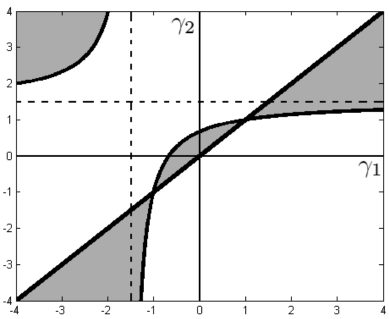

Figure 1. The graph of hyperbola (4) in the case $\xi = 0.4$. The grey areas correspond to the values of γ_1, γ_2 for which there exists one negative eigenvalue.

Lemma 1. *Any positive eigenvalue of differential operator* (1)–(3) *is defined by the formula*

$$\lambda_k = \alpha_k^2, \tag{6}$$

where α_k are roots of the equation

$$(\gamma_1\gamma_2 - 1)\sin\alpha\xi = (\gamma_1 - \gamma_2)\sin\alpha(1-\xi). \tag{7}$$

Proof. As $\lambda > 0$, the general solution of (1) is

$$u(x) = c_1\cos\alpha x + c_2\sin\alpha x, \quad \alpha = \sqrt{\lambda} > 0.$$

After substituting this solution into nonlocal conditions (2) and (3), we have a system

$$\begin{aligned} &(1-\gamma_1\cos\alpha)c_1 - (\gamma_1\sin\alpha)c_2 = 0, \\ &\bigl(\cos\alpha\xi - \gamma_2\cos\alpha(1-\xi)\bigr)c_1 + \bigl(\sin\alpha\xi - \gamma_2\sin\alpha(1-\xi)\bigr)c_2 = 0. \end{aligned} \tag{8}$$

For nontrivial solution (c_1, c_2) of this system, the necessary and sufficient condition is

$$D = \begin{vmatrix} 1-\gamma_1\cos\alpha & -\gamma_1\sin\alpha \\ \cos\alpha\xi - \gamma_2\cos\alpha(1-\xi) & \sin\xi - \gamma_2\sin\alpha(1-\xi) \end{vmatrix} = 0.$$

After elementary rearrangements, hence it follows (7). □

Theorem 2. *For all values of γ_1 and γ_2, except two cases $\gamma_1=\gamma_2=1$ and $\gamma_1=\gamma_2=-1$, and all values of ξ, there exist a countable set of positive eigenvalues of form* (6).

Proof. Let us consider three qualitatively different cases of the parameters γ_1 and γ_2.

(i) $\gamma_1 = \gamma_2 \neq \pm 1$, i.e., $\gamma_1 - \gamma_2 = 0$, $\gamma_1\gamma_2 - 1 \neq 0$. In this case, equation (7) becomes as follows:

$$\sin\alpha\xi = 0.$$

Hence, we derive

$$\alpha_k = \frac{k\pi}{\xi}, \quad k = 1, 2, \ldots .$$

(ii) $\gamma_1 = 1/\gamma_2$, $\gamma_1 \neq \pm 1$, i.e., $\gamma_1\gamma_2 - 1 = 0$, $\gamma_1 - \gamma_2 \neq 0$. From (7) we obtain

$$\begin{aligned} &\sin\alpha(1-\xi) = 0, \\ &\alpha_k = \frac{k\pi}{1-\xi}, \quad k = 1, 2, \ldots . \end{aligned}$$

(iii) $\gamma_1 - \gamma_2 \neq 0$, $\gamma_1\gamma_2 - 1 \neq 0$. Denote

$$\varphi_1(\alpha) = (\gamma_1\gamma_2 - 1)\sin\alpha\xi, \qquad \varphi_2(\alpha) = (\gamma_1 - \gamma_2)\sin\alpha(1-\xi).$$

The functions $\varphi_1(\alpha)$ and $\varphi_2(\alpha)$ are continuous periodical functions with the periods $2\pi/\xi$ and $2\pi/(1-\xi)$, respectively. Since $\xi < 1-\xi$, in a longer interval $\alpha \in (0, 2\pi/\xi)$, the graphs of both functions intersect at least one time (or several times) in case ξ, γ_1 and γ_2 are fixed. Thus, in all three cases, equation (7) has a countable set of roots α_k, i.e., there exists a countable set of positive eigenvalues $\lambda_k = \alpha_k^2$. □

Corollary 1. *If* $\gamma_1 = \gamma_2 = 1$ *or* $\gamma_1 = \gamma_2 = -1$*, equations* (7) *become an identity for all values of* λ*. In other words, any positive number* λ *is an eigenvalue. It means that the spectrum of differential problem* (1)–(3) *is continuous.*

The phenomenon of continuous spectrum also takes a place in the theory of boundary value problems for degenerate elliptic equations [14, 15].

Lemma 2. *The negative eigenvalue of differential operator* (1)–(3)*, if it exists, is defined by the formula*

$$\lambda = -\beta_0^2, \tag{9}$$

where $\beta_0 > 0$ *is the root of the equation*

$$(\gamma_1\gamma_2 - 1)\sinh\beta\xi = (\gamma_1 - \gamma_2)\sinh\beta(1-\xi). \tag{10}$$

Proof. As $\lambda < 0$, the general solution of equation (1) is

$$u(x) = c_1\cosh\beta x + c_2\sinh\beta x, \quad \beta = \sqrt{-\lambda} > 0.$$

After substituting this expression into nonlocal condition (2) and (3), we get

$$\begin{aligned} &c_1 = \gamma_1(c_1\cosh\beta + c_2\sinh\beta),\\ &c_1\cosh\beta\xi + c_2\sinh\beta\xi = \gamma_2\bigl(c_1\cosh\beta(1-\xi) + c_2\sinh\beta(1-\xi)\bigr). \end{aligned} \tag{11}$$

The necessary and sufficient condition for the existence of a nontrivial solution (c_1, c_2) of this system is as follows:

$$D = \begin{vmatrix} 1 - \gamma_1\cosh\beta & -\gamma_1\sinh\beta \\ \cosh\beta\xi - \gamma_2\cosh\beta(1-\xi) & \sinh\beta\xi - \gamma_2\sinh\beta(1-\xi) \end{vmatrix} = 0.$$

As earlier, it follows (10) from this equality. □

Next, we explore when equation (10) has at least one root.

Theorem 3. *Except two cases* $\gamma_1 = \gamma_2 = 1$ *and* $\gamma_1 = \gamma_2 = -1$*, equation* (10) *has a unique root* $\beta_0 \in (0, \infty)$ *if and only if*

$$\frac{\gamma_1\gamma_2 - 1}{\gamma_1 - \gamma_2} > \frac{1-\xi}{\xi}, \quad \gamma_1 \neq \gamma_2. \tag{12}$$

If condition (12) *is satisfied, there exists a unique negative eigenvalue of differential problem* (1)–(3)*, and it is defined by formula* (9)*.*

Proof. According to the assumption of the theorem, let us consider all real values of γ_1 and γ_2 except $\gamma_1 = \gamma_2 = \pm 1$. Thus, if $\gamma_1 = \gamma_2$, then equation (10) has only a root $\beta = 0$. Therefore, the condition $\gamma_1 - \gamma_2 \neq 0$ is the necessary condition for the existence of the root $\beta > 0$. In this case, we can rewrite equation (10) as follows:

$$\frac{\sinh\beta(1-\xi)}{\sinh\beta\xi} = \frac{\gamma_1\gamma_2 - 1}{\gamma_1 - \gamma_2}. \tag{13}$$

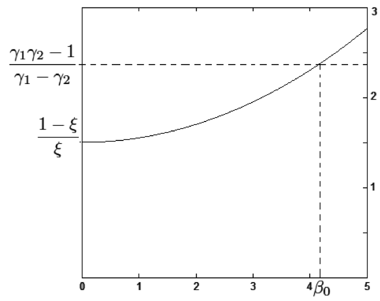

Figure 2. The graph of the function $\varphi(\beta) = \sinh\beta(1-\xi)/\sinh\beta\xi$ in the case $\xi = 0.4$, $\gamma_1 = 2.4$, $\gamma_2 = 1.4$.

The function $\varphi(\beta) = \sinh\beta(1-\xi)/\sinh\beta\xi$, as $\xi \in (0, 1/2)$, has in the interval $\beta \in (0,\infty)$ the following properties:

(i) it is continuous, and $\varphi(\beta) > 1$;

(ii) it is monotonously increasing function since

$$\varphi'(\beta) = \frac{(1-\xi)\cosh\beta(1-\xi)\sinh\beta\xi - \xi\sinh\beta(1-\xi)\cosh\beta\xi}{\sinh^2(\beta\xi)} = \frac{\cosh\beta(1-\xi)-\xi}{\sinh\beta\xi} > 0;$$

(iii) $$\varphi(0) = \lim_{\beta\to 0}\frac{\sinh\beta(1-\xi)}{\sinh\beta\xi} = \lim_{\beta\to 0}\frac{(1-\xi)\cosh\beta(1-\xi)}{\xi\cosh\beta\xi} = \frac{1-\xi}{\xi} > 1;$$

(iv) $$\lim_{\beta\to\infty}\frac{\sinh\beta(1-\xi)}{\sinh\beta\xi} = +\infty.$$

It follows from these properties that equation (13) has a unique root in interval $(0,\infty)$ if and only if condition (12) is fulfilled (see Fig. 2). $\square$

Corollary 2. *It follows from Lemma 2 that, in the case* $\gamma_1 = \gamma_2 = \pm 1$, *i.e., if* $\gamma_1\gamma_2 - 1 = 0$ *and* $\gamma_1 - \gamma_2 = 0$, *equation* (10) *turns into identity for all values of* $\beta > 0$. *Thus, if* $\gamma_1 = \gamma_2 = \pm 1$, *then any negative number* $\lambda = -\beta^2$ *is the eigenvalue of differential problem* (1)–(3).

3 Eigenvalue problem of a difference operator

Let us write a difference problem of eigenvalues that approximates differential problem (1)–(3) with the approximation error $O(h^2)$:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \quad i = 1,\dots,N-1, \tag{14}$$

$$u_0 = \gamma_1 u_N, \qquad u_S = \gamma_2 u_{N-S}. \tag{15}$$

We denote here $h = 1/N$, $\xi = Sh$, $1 - \xi = (N - S)h$; N and S are integer numbers. In other words, the grid is uniform, and ξ is a mesh point ($1 - \xi$ is also a mesh point). Note that $1 \leqslant S \leqslant N/2 - 1$ and $h \leqslant \xi \leqslant 1/2 - h$.

We call the number λ an eigenvalue of the difference problem, if with this number there exists a nontrivial solution (eigenvector) of problem (14)–(15).

Let us analyze the spectrum (the set of all eigenvalues) of difference problem (14)–(15). The proofs of lemmas and theorems presented below, according to methodology are analogous to that used in Section 2. Therefore, in our proofs, we emphasize only that what is different.

Theorem 4. *The number $\lambda = 0$ is the eigenvalue of difference problem* (14)–(15) *if and only if*

$$(\gamma_1\gamma_2 - 1)\xi = (\gamma_1 - \gamma_2)(1 - \xi). \tag{16}$$

Proof. First, let us pay attention that condition (16) is coincident with condition (4) of Theorem 1.

As $\lambda = 0$, the general solution of difference equation (14) is

$$u_i = c_1 ih + c_2, \quad i = 1, \dots, N - 1,$$

where c_1 and c_2 are arbitrary constants. By substituting this expression of u_i into nonlocal conditions (15), we obtain a system just like system (5) in the proof of Theorem 1. □

Remark 1 is right for difference operator (14)–(15) as well as for differential operator (1)–(3).

Lemma 3. *Positive eigenvalues of difference problem* (14)–(15) *satisfying the inequality*

$$0 < \lambda < \frac{4}{h^2} \tag{17}$$

are defined by the formula

$$\lambda_k = \frac{4}{h^2} \sin^2 \frac{\alpha_k h}{2}, \tag{18}$$

where α_k are roots of the equation

$$(\gamma_1\gamma_2 - 1) \sin \alpha\xi = (\gamma_1 - \gamma_2) \sin \alpha(1 - \xi) \tag{19}$$

in the interval $(0, \pi/h)$.

Proof. First of all, note that equation (19) is coincident with equation (7) in Lemma 1. However, expressions and numbers of eigenvalues are different than in Lemma 1.

Since the inequality

$$\left|1 - \frac{\lambda h^2}{2}\right| < 1$$

follows from condition (17), we can introduce into equation (14) a new unknown α instead of λ:

$$\cos \alpha h = 1 - \frac{\lambda h^2}{2}, \quad 0 < \alpha < \frac{\pi}{h}. \tag{20}$$

Hence, formula (18) follows. Now, equation (14) becomes as follows:

$$u_{i-1} - 2(\cos\alpha h)u_i + u_{i+1} = 0,$$

and its general solution is

$$u_i = c_1 \cos\alpha ih + c_2 \sin\alpha ih. \tag{21}$$

By substituting this expression of u_i into nonlocal conditions (15), we obtain a system coincident with system (8) in the demonstration of Lemma 1. So, the rest of this proof now is coincident with that one of Lemma 1. □

Theorem 5. *The number of positive eigenvalues of form* (18) *is finite with all the values of* γ_1, γ_2 *except two cases*: $\gamma_1 = \gamma_2 = 1$ *and* $\gamma_1 = \gamma_2 = -1$.

Proof. According to assumption of the theorem, all possible values of γ_1 and γ_2 can be separated into three qualitatively different cases.

(i) $\gamma_1 = \gamma_2 \neq \pm 1$; then $\gamma_1\gamma_2 - 1 \neq 0$ and $\gamma_1 - \gamma_2 = 0$. Equation (19), in this case, is

$$\sin\alpha\xi = 0,$$

and its roots are

$$\alpha_k = \frac{k\pi}{\xi}, \quad 0 < \alpha_k < \frac{\pi}{h},$$

i.e., $0 < k < S$. Thus,

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{k\pi h}{2\xi} = \frac{4}{h^2}\sin^2\frac{k\pi}{2S}, \quad k = 1, \ldots, S-1, \tag{22}$$

where $S \geqslant 2$.

(ii) $\gamma_1 = 1/\gamma_2 \neq \pm 1$, i.e., $\gamma_1\gamma_2 - 1 = 0$, $\gamma_1 - \gamma_2 \neq 0$. Then equation (19) is

$$\sin\alpha(1-\xi) = 0.$$

Analogously as in the first case, we derive

$$\alpha_k = \frac{k\pi}{1-\xi}, \quad k < N - S,$$

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\pi k}{2(N-S)}, \quad k = 1, \ldots, N-S-1. \tag{23}$$

(iii) $\gamma_1 - \gamma_2 \neq 0$ and $\gamma_1\gamma_2 - 1 \neq 0$. As mentioned in the proof of Theorem 2, in the interval $(0, 2\pi/\xi)$, equation (19) has one or several roots. Since $\alpha \in (0, \pi/h)$, the number of roots in this interval is finite (in the general case it depends on four parameters: γ_1, γ_2, ξ and h). □

We shall indicate one interesting fact.

Remark 2. If $\xi = h$, i.e., $S = 1$ and $\gamma_1 = \gamma_2 \neq \pm 1$, then it follows from formula (22) that difference problem (14)–(15) has no positive eigenvalue λ that satisfies the inequality $0 < \lambda < 4/h^2$. Thereby we admit that from the equality $\xi = h$ it not follows (1)–(3) is ill-possessed problem. The matter is that $h \to 0$, but $\xi = \mathrm{const}$. Hence, the spectrum of the difference problem (14)–(15) is empty set with only one concrete value h, i.e., $h = \xi$ (see Section 6, case 2)). With increasing or decreasing value h this phenomenon disappears.

Corollary 3. *If $\gamma_1 = \gamma_2 = 1$ or $\gamma_1 = \gamma_2 = -1$, then the spectrum of difference problem* (14)–(15) *is continuous for all the values of ξ and h, i.e., any number $\lambda \in (0, 4/h^2)$ is an eigenvalue.*

The conclusion follows directly from equation (19).

Let us find an eigenvector as $\gamma_1 = \gamma_2 = 1$, i.e., in the case of continuous spectrum. Consider any fixed number $\lambda_0 \in (0, 4/h^2)$ as an eigenvalue. In accordance with (20), we calculate $\alpha_0 \in (0, \pi/h)$ from the equality

$$\cos \alpha_0 h = 1 - \frac{\lambda_0 h^2}{2}. \tag{24}$$

The eigenvector is of form (21). Choosing $c_1 = 1$, from system (8) we calculate

$$c_2 = \frac{1 - \cos \alpha_0}{\sin \alpha_0}.$$

Thus, the eigenvector corresponding to the eigenvalue λ_0 is as follows:

$$u = \{u_i\} = \left\{ \cos \alpha_0 ih + \frac{1 - \cos \alpha_0}{\sin \alpha_0} \sin \alpha_0 ih \right\}, \quad i = 0, \ldots, N,$$

where α_0 satisfies equality (24).

Note that, differently than in the case of the differential problem (Theorem 2), we have found not all the positive eigenvalues of the difference problem, but only the eigenvalues from the interval $(0, 4/h^2)$. Under certain additional conditions, there may exist one more eigenvalue $\lambda \geqslant 4/h^2$ of the difference problem (see below Theorems 7 and 8). As far as the authors are acquainted, for the first time, the existence of such an eigenvalue in the case of nonlocal conditions was noticed most likely in paper [20].

Lemma 4. *The negative eigenvalue of difference problem* (14)–(15)*, if it exists, is defined by the formula*

$$\lambda = -\frac{4}{h^2} \sinh^2 \frac{\beta h}{2}, \quad \beta > 0, \tag{25}$$

where $\beta > 0$ is the root of the equation

$$(\gamma_1 \gamma_2 - 1) \sinh \beta ih = (\gamma_1 - \gamma_2) \sinh \beta (N - i) h. \tag{26}$$

Proof. If $\lambda < 0$, then $1 - \lambda h^2/2 > 1$, therefore we can introduce a new unknown β by the relation

$$\cosh \beta h = 1 - \frac{\lambda h^2}{2}, \quad \beta > 0.$$

Hence, formula (25) follows. Equation (14) becomes as follows:

$$u_{i-1} - 2(\cosh \beta h)u_i + u_{i+1} = 0.$$

The general solution of this equation is

$$u_i = c_1 \cosh \beta ih + c_2 \sinh \beta ih.$$

By substituting this expression into nonlocal conditions (15), we get system (11). A further proof of the lemma is coincident with that of Lemma 2. □

Like as Theorem 3, the next theorem is proved.

Theorem 6. *Except two cases $\gamma_1 = \gamma_2 = 1$ and $\gamma_1 = \gamma_2 = -1$, there exists a unique negative eigenvalue of form* (25) *of difference problem* (14)–(15) *if and only if condition* (12) *is true.*

Corollary 4. *If $\gamma_1 = \gamma_2 = \pm 1$, then any number $\lambda < 0$ is the eigenvalue of difference operator* (14)–(15)*. Indeed, in this case, equation* (26) *is an identity $0 = 0$ for all values of β.*

In Fig. 1, in the presence of the fixed value $\xi = 0.4$, in the grey areas of the coordinate plane (γ_1, γ_2), there exists a negative eigenvalue of difference problem (14)–(15).

According to Theorems 3 and 6, there exists a negative eigenvalue of both differential and difference operators under the same conditions (12).

Now we can return to conditions under which there exists the positive eigenvalue $\lambda \geqslant 4/h^2$.

Theorem 7. *The number $\lambda = 4/h^2$ is the eigenvalue of difference problem* (14)–(15) *if and only if the condition*

$$(-1)^S(\gamma_1\gamma_2 - 1)\xi = (-1)^{N-S}(\gamma_1 - \gamma_2)(1 - \xi) \tag{27}$$

is satisfied.

Proof. When $\lambda = 4/h^2$, equation (14) becomes as follows:

$$u_{i-1} + 2u_i + u_{i+1} = 0.$$

The general solution of this equation is

$$u_i = (-1)^i(c_1 ih + c_2). \tag{28}$$

After substituting the expression of this solution into conditions (15), we obtain

$$c_2 = \gamma_1(-1)^N(c_1 + c_2),$$
$$(-1)^S(c_1\xi + c_2) = \gamma_2(-1)^{N-S}\big(c_1(1 - \xi) + c_2\big).$$

Hence, the necessary and sufficient condition for the existence of nontrivial solution (c_1, c_2) is

$$D = \begin{vmatrix} -(-1)^N\gamma_1 & 1-(-1)^N\gamma_1 \\ (-1)^S\xi - (-1)^{N-S}\gamma_2(1-\xi) & (-1)^S - (-1)^{N-S}\gamma_2 \end{vmatrix} = 0.$$

After elementary rearrangement, hence we derive (27). □

Assume that N is an even number. Then condition (27) is coincident with condition (16). Thus, we obtained the following conclusion.

Corollary 5. *As N is an even number, the existence condition of the eigenvalues $\lambda = 0$ and $\lambda = 4/h^2$ is the same.*

Theorem 8. *Except two cases $\gamma_1 = \gamma_2 = 1$ and $\gamma_1 = \gamma_2 = -1$, difference problem* (14)–(15) *has one eigenvalue $\lambda > 4/h^2$ if and only if*

$$(-1)^N \frac{\gamma_1\gamma_2 - 1}{\gamma_1 - \gamma_2} > \frac{1-\xi}{\xi}. \tag{29}$$

If this condition is satisfied, then

$$\lambda = \frac{4}{h^2}\cosh^2\frac{\beta h}{2}, \tag{30}$$

where β is a unique root of the equation

$$(-1)^N(\gamma_1\gamma_2 - 1)\sinh\beta\xi = (\gamma_1 - \gamma_2)\sinh\beta(1-\xi) \tag{31}$$

in the interval $(0, \infty)$.

Proof. If $\lambda > 4/h^2$, then $1 - \lambda h^2/2 < -1$. Therefore, we can introduce in equation (14) a new unknown $\beta > 0$ by the equality

$$\cosh\beta h = \frac{\lambda h^2}{2} - 1, \quad \beta > 0.$$

Hence, expression (30) follows, and equation (14) becomes such as follows:

$$u_{i-1} + 2(\cosh\beta h)u_i + u_{i+1} = 0.$$

The general solution of this equation is

$$u_i = (-1)^i(c_1\cosh\beta ih + c_2\sinh\beta ih). \tag{32}$$

After substituting this expression into (15), we obtain a system, analogous to system (11):

$$c_1 = \gamma_1(-1)^N(c_1\cosh\beta + c_2\sinh\beta),$$
$$c_1\cosh\beta\xi + c_2\sinh\beta\xi = \gamma_2(-1)^N\bigl(c_1\cosh\beta(1-\xi) + c_2\sinh\beta(1-\xi)\bigr).$$

By equating a determinant of this system to zero, after elementary rearrangement, we obtain (31). The further proof of the theorem is analogous to that of Theorem 3. □

The inferences analogous to Corollaries 3 and 5 are true.

Corollary 6. *If $\gamma_1 = \gamma_2 = \pm 1$, then any number $\lambda > 4/h^2$ is the eigenvalue of difference problem* (14)–(15).

Corollary 7. *As N is an even number, the existence condition of the eigenvalues $\lambda < 0$ and $\lambda > 4/h^2$ is the same.*

Remark 3. The result of Theorems 7 and 8 on the existence conditions for the eigenvalues $\lambda = 4/h^2$ and $\lambda > 4/h^2$ is proper on for a difference but not a differential problem. The matter is that eigenvectors (28) and (32), corresponding to these eigenvalues, have no limit as $h \to 0$.

4 Complex eigenvalues

Differential operator of (1)–(3) is not self-adjoint. Therefore, there may exist complex eigenvalues. Such a statement is also right for difference operator of (14)–(15). We investigate when there exist complex eigenvalues of a difference operator, since for a differential operator, both the investigation methods and their results are analogous.

In this section, we denote an imaginary unit by the letter i, i.e., $\mathrm{i} = \sqrt{-1}$. Therefore, in equation (14), we shall use the index j instead of index i.

Lemma 5. *If there exist complex eigenvalues λ_k of difference problem* (14)–(15)*, they are defined by the following formula*:

$$\lambda_k = \frac{4}{h^2} \sin^2 \frac{q_k h}{2}, \tag{33}$$

where $q_k = \alpha_k \pm \mathrm{i}\beta$ are complex roots of the equation

$$(\gamma_1\gamma_2 - 1) \sin q\xi = (\gamma_1 - \gamma_2) \sin q(1 - \xi). \tag{34}$$

Proof. If λ is a complex number in equation (14), we can introduce a new complex quantity q by the formula

$$\cos qh = 1 - \frac{\lambda h^2}{2}, \tag{35}$$

where $q = \alpha \pm \mathrm{i}\beta$. Hence, it follows that

$$\lambda = \frac{4}{h^2} \sin^2 \frac{qh}{2}.$$

Note that, in the case where λ is a complex number, there must be $\alpha \neq 0$ and $\beta \neq 0$. The condition $\beta = 0$ is coincident with the condition that q is a real number, so λ is positive. If $\alpha = 0$, then

$$\lambda = \frac{4}{h^2} \sin^2 \frac{\mathrm{i}\beta h}{2} = \frac{4}{h^2} \left(\mathrm{i} \sinh \frac{\beta h}{2} \right)^2 = -\frac{4}{h^2} \sinh^2 \frac{\beta h}{2},$$

i.e., the case $\alpha = 0$, $\beta \neq 0$ corresponds to $\lambda < 0$. In case $\alpha = \beta = 0$, it follows $\lambda = 0$.

After replacement (35), equation (14) becomes as follows:

$$u_{j-1} - 2(\cos qh)u_j + u_{j+1} = 0.$$

Its general solution is

$$u_j = c_1 \cos qjh + c_2 \sin qjh, \quad q = \alpha \pm \mathrm{i}\beta,$$

where c_1 and c_2 are arbitrary complex constants.

By substituting this expression in nonlocal conditions (15) we obtain

$$c_1 = \gamma_1(c_1 \cos q + c_2 \sin q),$$
$$c_1 \cos q\xi + c_2 \sin q\xi = \gamma_2\bigl(c_1 \cos q(1-\xi) + c_2 \sin q(1-\xi)\bigr).$$

Hence just like in the proof of Lemma 3, we get that the nontrivial solution (c_1, c_2) exists if and only if condition (34) is fulfilled. In case this equation has complex roots $q_k = \alpha_k \pm \mathrm{i}\beta_k$, $\alpha_k \neq 0$, $\beta_k \neq 0$, then the corresponding eigenvalue λ_k is defined by (33). □

We can specify three elementary cases where difference problem (14)–(15) has no complex eigenvalues. We formulate these cases as the corollaries of Lemma 5.

Corollary 8. *If $\gamma_1 = \gamma_2 \neq \pm 1$, there are no complex eigenvalues.*

Indeed, in this case, $\gamma_1 - \gamma_2 = 0$, $\gamma_1\gamma_2 - 1 \neq 0$. Thus, equation (34) becomes as follows:

$$\sin q\xi = 0.$$

If ξ is a real numbers, then all the roots q_k are real.

Corollary 9. *In case $\gamma_1 = 1/\gamma_1 \neq 1$, there are no complex eigenvalues.*

In this case, $\gamma_1\gamma_2 - 1 = 0$, $\gamma_1 - \gamma_2 \neq 0$ and equation (34) is as follows:

$$\sin q(1-\xi) = 0,$$

which implies that the roots q_k are real numbers.

Corollary 10. *In case $\gamma_1\gamma_2 - 1 = \gamma_2 - \gamma_1 = a \neq 0$, there are no complex eigenvalues.*

In this case, we can rewrite equation (34) as follows:

$$2a \sin q\left(\xi + \frac{1}{2}\right) \cos\frac{q}{2} = 0,$$

the roots q_k of which are only real.

An analogous proposition is right in the case

$$\gamma_1\gamma_1 - 1 = -(\gamma_1 - \gamma_2) = a \neq 0.$$

We analyze one more complicated case where all eigenvalues of the difference operator are real.

Let us introduce a generalized parameter

$$\tilde{\gamma} = \frac{\gamma_1\gamma_2 - 1}{\gamma_1 - \gamma_2}. \tag{36}$$

This value was used in many lemmas and theorems.

Theorem 9. *If* $|\tilde{\gamma}| \leqslant 1$*, then all the eigenvalues of difference operator* (14)–(15) *are real.*

Proof. Suppose that under condition $|\tilde{\gamma}| \leqslant 1$, equation (34) has a complex root $q = \alpha \pm \mathrm{i}\beta$, $\alpha \neq 0$, $\beta \neq 0$. Since $|\tilde{\gamma}| \leqslant 1$, it means that $\gamma_1 - \gamma_2 \neq 0$. We rewrite equation (34) in the following form:

$$\sin(\alpha \pm \mathrm{i}\beta)(1 - \xi) - \tilde{\gamma}\sin(\alpha \pm \mathrm{i}\beta)\xi = 0.$$

Hence,

$$\begin{aligned} &\sin\alpha(1-\xi)\cosh\beta(1-\xi) \pm \mathrm{i}\sinh\beta(1-\xi)\cos\alpha(1-\xi) \\ &\quad - \tilde{\gamma}(\sin\alpha\xi\cosh\beta\xi \pm \mathrm{i}\sinh\beta\xi\cos\alpha\xi) = 0. \end{aligned}$$

We separate the real and imaginary parts:

$$\sin\alpha(1-\xi)\cosh\beta(1-\xi) - \tilde{\gamma}\sin\alpha\xi\cosh\beta\xi = 0, \tag{37}$$

$$\sinh\beta(1-\xi)\cos\alpha(1-\xi) - \tilde{\gamma}\sinh\beta\xi\cos\alpha\xi = 0. \tag{38}$$

We express $\sin\alpha(1-\xi)$ from equation (37) and $\cos\alpha(1-\xi)$ from equation (38) and substitute the obtained expressions into the identity

$$\sin^2\alpha(1-\xi) + \cos^2\alpha(1-\xi) = 1.$$

By substituting, we get

$$\left(\tilde{\gamma}\frac{\cosh\beta\xi}{\cosh\beta(1-\xi)}\right)^2 \sin^2\alpha\xi + \left(\tilde{\gamma}\frac{\sinh\beta\xi}{\sinh\beta(1-\xi)}\right)^2 \cos^2\alpha\xi = 1. \tag{39}$$

Since $\xi < 1 - \xi$, $\beta \neq 0$, $|\tilde{\gamma}| \leqslant 1$, we derive

$$\left(\tilde{\gamma}\frac{\cosh\beta\xi}{\cosh\beta(1-\xi)}\right)^2 < 1, \qquad \left(\tilde{\gamma}\frac{\sinh\beta\xi}{\sinh\beta(1-\xi)}\right)^2 < 1.$$

Consequently, equality (39) is impossible. We have got a contradiction from which it follows that q cannot be complex. □

Corollary 11. *In case* $\gamma_1 = \gamma_2 = \pm 1$*, any complex number* q *is the root of equation* (34)*, i.e., any complex number* λ *is the eigenvalue of difference problem* (14)–(15).

5 The difference eigenvalue problem as a generalized matrix eigenvalue problem

In paper [3], it is stated difference eigenvalue problem (14)–(15) can be written as a matrix eigenvalue problem

$$Au = \lambda u, \tag{40}$$

where A is $(N-1)$-order matrix, $u = (u_1, u_2, \dots, u_{N-1})^{\mathrm{T}}$. The expression of matrix A is written as well. Though the expression of matrix A is written correctly, problem (14)–(15) is not equivalent to problem (40).

We shall repeat these reasoning, presented in paper [3], and will correct one inaccuracy in it. At the same time, we note that the authors of paper, after writing problem (14)–(15) in incorrect form (40), nowhere use such a form. We will write a slightly different matrix from of problem (14)–(15), and we will comment it in Section 6.

Let us write equation (14) more in detail:

$$\begin{aligned}
&h^{-2}(-u_0 + 2u_1 - u_2) = \lambda u_1,\\
&h^{-2}(-u_1 + 2u_2 - u_3) = \lambda u_2,\\
&\dots\\
&h^{-2}(-u_{S-2} + 2u_{S-1} - u_S) = \lambda u_{S-1},\\
&h^{-2}(-u_{S-1} + 2u_S - u_{S+1}) = \lambda u_S,\\
&h^{-2}(-u_S + 2u_{S+1} - u_{S+2}) = \lambda u_{S+1},\\
&\dots\\
&h^{-2}(-u_{N-3} + 2u_{N-2} - u_{N-1}) = \lambda u_{N-2},\\
&h^{-2}(-u_{N-2} + 2u_{N-1} - u_N) = \lambda u_{N-1}.
\end{aligned} \tag{41}$$

We substitute the expression $u_0 = \gamma_1 u_N$ from (15) into the first equation of system (41). Analogously, we substitute the expression $u_S = \gamma_2 u_{N-S}$ from (15) into three equations of system (41) as $i = S-1, S, S+1$. After substitution, we obtain a new form of problem (14)–(15)

$$\begin{aligned}
&h^{-2}(2u_1 - u_2 - \gamma_1 u_N) = \lambda u_1,\\
&h^{-2}(-u_1 + 2u_2 - u_3) = \lambda u_2,\\
&\dots\\
&h^{-2}(-u_{S-2} + 2u_{S-1} - \gamma_2 u_{N-S}) = \lambda u_{S-1},\\
&h^{-2}(-u_{S-1} - u_{S+1} + 2\gamma_2 u_{N-S}) = \lambda u_S,\\
&h^{-2}(2u_{S+1} - u_{S+2} - \gamma_2 u_{N-S}) = \lambda u_{S+1},\\
&\dots\\
&h^{-2}(-u_{N-3} + 2u_{N-2} - u_{N-1}) = \lambda u_{N-2},\\
&h^{-2}(-u_{N-2} + 2u_{N-1} - u_N) = \lambda u_{N-1}.
\end{aligned} \tag{42}$$

System (42), together with (15), is equivalent to system (14)–(15). Now system (42) can be written in the matrix form

$$Au^{(1)} = \lambda u^{(2)},$$

where

$$u^{(1)} = (u_1, u_2, \dots, u_{S-1}, u_{S+1}, \dots, u_{N-1}, u_N)^{\mathrm{T}},$$
$$u^{(2)} = (u_1, u_2, \dots, u_{S-1}, u_S, \dots, u_{N-2}, u_{N-1})^{\mathrm{T}}.$$

The expression of the matrix A is written correct in [3]. Since $u^{(1)} \neq u^{(2)}$, the problem $Au^{(1)} = \lambda u^{(2)}$ is not an eigenvalue problem. Note that, neither by the method proposed in [3], not by any other way, difference eigenvalue problem (14)–(15) cannot be written in form (40). However, this problem (14)–(15) can be written as generalized matrix eigenvalue problem [9, 13]

$$Au = \lambda Bu, \tag{43}$$

where A and B are the N-order matrices. A specific feature of such a problem is that B is a singular matrix (B^{-1} does not exists).

Let us write system (14)–(15) in the following way:

$$u_0 = \gamma_1 u_N, \tag{44}$$
$$u_{i-1} - 2u_i + u_{i+1} + \lambda h^2 u_i = 0, \quad i = 1, \dots, N-1, \tag{45}$$
$$u_S = \gamma_2 u_{N-S}. \tag{46}$$

In this system, we take two steps of equivalent rearrangement.

The first step: write the expression u_0 from (44) into equation (45) in which $i = 1$. So we obtain

$$-2u_1 + u_2 + \gamma_1 u_N + \lambda h^2 u_1 = 0.$$

The second step: subtract equation (46) from equation (45) in which $i = S + 1$. Instead of equation (46), we obtain the new equation

$$-2u_{S+1} + u_{S+2} + \gamma_2 u_{N-S} + \lambda h^2 u_{S+1} = 0.$$

In this way, we get a new system

$$\begin{aligned} &-2u_1 + u_2 + \gamma_1 u_N + \lambda h^2 u_1 = 0, \\ &u_{i-1} - 2u_i + u_{i+1} + \lambda h^2 u_i = 0, \quad i = 2, \dots, N-1, \\ &-2u_{S+1} + u_{S+2} + \gamma_2 u_{N-S} + \lambda h^2 u_{S+1} = 0. \end{aligned} \tag{47}$$

By adding equation (44) this system is equivalent to initial system (44)–(46). Now system (47) can be written as generalised matrix eigenvalue problem (43), where B is singular matrix [16]. Next, we write expressions of N-order matrices A and B:

$$A = \begin{pmatrix} 2 & -1 & & & & & & & & -\gamma_1 \\ -1 & 2 & -1 & & & & & & & 0 \\ & -1 & 2 & -1 & & & & & & 0 \\ & & & \ddots & & & & & & \\ & & & & & & & -1 & 2 & -1 \\ 0 & \dots & 2 & -1 & \dots & -\gamma_2 & \dots & 0 & 0 & 0 \end{pmatrix},$$

$$B = h^2 \begin{pmatrix} 1 & & & & & & & & 0 \\ & 1 & & & & & & & 0 \\ & & 1 & & & & & & 0 \\ & & & \ddots & & & & & \\ & & & & & & & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Nonzero elements on the last row of matrix A are in columns with the numbers $S+1$, $S+2$, $N-S$. Nonzero element on the last row of matrix B is in the column with the number $S+1$. Matrix B is singular matrix, its last column consists only of zeros. The eigenvalues of eigenvalue problem (43) are the roots of a generalized characteristic equation

$$\det(A - \lambda B) = 0. \tag{48}$$

Since the order of matrix A and B is N, $\det(A - \lambda B)$ is not higher than the N-order polynomial. It would be not right assert that the characteristic polynomial is of N-order because that is possible only in the case when exists B^{-1}. Equation (48) yields the following assertion.

Corollary 12. *If the spectrum of difference problem* (14)–(15) *is not continuous* (*case* $\gamma_1 = \gamma_2 = \pm 1$)*, then there exist no more than* N *eigenvalues.*

6 Illustrative example

Let us take a concrete example: $\xi = 0.25$, $N = 4$ and γ_1, γ_2 are varying parameters. By means of this rather elementary example we illustrate the substance and variety of the spectrum of difference operator with nonlocal conditions.

So we get the following eigenvalue problem:

$$u_0 = \gamma_1 u_4, \tag{49}$$

$$u_0 - 2u_1 + u_2 + \lambda h^2 u_1 = 0, \tag{50}$$

$$u_1 - 2u_2 + u_3 + \lambda h^2 u_2 = 0, \tag{51}$$

$$u_2 - 2u_3 + u_4 + \lambda h^2 u_3 = 0, \tag{52}$$

$$u_1 = \gamma_2 u_3. \tag{53}$$

We transform this eigenvalue problem into a matrix form $Au = \lambda Bu$ in accordance with the methodology described in Section 5. To this end, we substitute u_0 from (49) into (50), and we change equation (53) by a difference of (51) and (53). Thus, we derive

$$\begin{aligned}
&-2u_1 + u_2 + \gamma_1 u_4 + \lambda h^2 u_1 = 0,\\
&u_1 - 2u_2 + u_3 + \lambda h^2 u_2 = 0,\\
&u_2 - 2u_3 + u_4 + \lambda h^2 u_3 = 0,\\
&-2u_2 + (1+\gamma_2)u_3 + \lambda h^2 u_2 = 0
\end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} -2 & 1 & 0 & \gamma_1\\ 1 & -2 & 1 & 0\\ 0 & 1 & -2 & 1\\ 0 & -2 & 1+\gamma_2 & 0 \end{pmatrix} \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{pmatrix} = \lambda h^2 \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{pmatrix}.$$

Next, we calculate the fourth-order determinant:

$$\begin{aligned}
&\det(A - \lambda B)\\
&= \left(-2+\lambda h^2\right) \begin{vmatrix} -2+\lambda h^2 & 1 & 0\\ 1 & -2+\lambda h^2 & 1\\ -2+\lambda h^2 & 1+\gamma_2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & \gamma_1\\ 1 & -2+\lambda h^2 & 1\\ -2+\lambda h^2 & 1+\gamma_2 & 0 \end{vmatrix}\\
&= \left(-2+\lambda h^2\right)\left(-2+\lambda h^2 - (1+\gamma_2)\left(-2+\lambda h^2\right)\right)\\
&\quad - \left(\gamma_1(1+\gamma_2) - \gamma_1\left(-2+\lambda h^2\right)^2 - (1+\gamma_2)\right)\\
&= \left(-2+\lambda h^2\right)^2(\gamma_1-\gamma_2) - (\gamma_1-1)(\gamma_2+1) = 0. \qquad (54)
\end{aligned}$$

In the general case, we obtain the second-order characteristic polynomials. To be precise, depending on the values of γ_1 and γ_2, we obtain not higher than the second-order polynomial (we remind, $N = 4$). Besides, note that, in the case $\gamma_1 \neq \gamma_2$, we can write equation (54) as follows:

$$\left(-2+\lambda h^2\right)^2 = \frac{\gamma_1\gamma_2 - 1}{\gamma_1 - \gamma_2} + 1$$

or

$$-2 + \lambda h^2 = \pm\sqrt{\tilde{\gamma} + 1}. \qquad (55)$$

Let us take several concrete values of γ_1 and γ_2.

Case 1. $\gamma_1 = \gamma_2 = \pm 1$. Characteristic equation (54) becomes an identity $0 = 0$ for all the values λ. Thus, the spectrum of difference operator (49)–(53) is continuous (see Corollaries 3, 4, 6).

Case 2. $\gamma_1 = \gamma_2 \neq \pm 1$. Characteristic equation (54) becomes as follows:

$$(\gamma_1 - 1)(\gamma_2 + 1) = 0,$$

what it is not true. So the spectrum of difference equation is an empty set (see Remark 2).

Case 3. Let us choose γ_1 and γ_2 so that

$$\tilde{\gamma} = \frac{\gamma_1\gamma_2 - 1}{\gamma_1 - \gamma_2} = \frac{1-\xi}{\xi}.$$

Since $\xi = 0.25$, we have $(1-\xi)/\xi = 3$. It follows from (55) that

$$-2 + \lambda h^2 = \pm 2$$

or $\lambda_1 = 0$, $\lambda_2 = 4/h^2$ (see Corollary 5). Note that there is an infinite set of points of parameters γ_1 and γ_2 with the property $\tilde{\gamma} = 3$.

Case 4. Let us choose γ_1 and γ_2 so that it were

$$\tilde{\gamma} = \frac{\gamma_1\gamma_2 - 1}{\gamma_1 - \gamma_2} > \frac{1-\xi}{\xi} = 3.$$

In this case, $\sqrt{\tilde{\gamma}+1} > 2$, and from (55) we derive

$$\lambda_1 = \frac{2}{h^2} + \frac{\sqrt{\tilde{\gamma}+1}}{h^2} > \frac{4}{h^2},$$
$$\lambda_2 = \frac{2}{h^2} - \frac{\sqrt{\tilde{\gamma}+1}}{h^2} < 0$$

(see Corollary 7).

Case 5. We take γ_1 and γ_2 such that $|\tilde{\gamma}| \leqslant 1$. It follows from (55) that both eigenvalues

$$\lambda_{1,2} = \frac{2}{h^2} \pm \frac{\sqrt{\tilde{\gamma}+1}}{h^2}$$

are real (see Theorem 9). Note that the condition $|\tilde{\gamma}| \leqslant 1$ is not necessary for eigenvalues to be real. In our particular case, the necessary and sufficient condition for both eigenvalues to be real is $\tilde{\gamma} \geqslant -1$.

Case 6. $\tilde{\gamma} < -1$. It follows from (55) that both eigenvalues are complex conjugate numbers:

$$\lambda_{1,2} = \frac{2}{h^2} \pm \frac{\sqrt{\tilde{\gamma}+1}}{h^2} = \frac{2 \pm \mathrm{i}\sqrt{|\tilde{\gamma}|-1}}{h^2}.$$

7 Remarks and generalization

In this paper, the spectrum structure of differential and difference operators with nonlocal condition has been explored. The main aim was research of the eigenvalue problem of difference operator.

It has been proved that the spectrum structure of both differential and difference operators depends not only on the type on nonlocal conditions, but particularly on the parameter value under nonlocal conditions. Besides, the dependence is more clearly defined not by the values of separate parameters γ_1, γ_2 and ξ, but using generalized parameter values

$$\tilde{\gamma} = \frac{\gamma_1\gamma_2 - 1}{\gamma_1 - \gamma_2}, \qquad \tilde{\xi} = \frac{1-\xi}{\xi}$$

as well as interdependence of these generalized values. The conditions (equalities and inequalities) that include these generalized parameters in the coordinates plane (γ_1, γ_2) are related with a hyperbola.

By comparing the published results of the considered subject, we have to note that, perhaps for the first time, it has been proved that the spectrum of difference operators with nonlocal conditions can be continuous or coincident with an empty set. Namely, if the parameters γ_1, γ_2 satisfy the condition $\gamma_1 = \gamma_2 = 1$ or $\gamma_1 = \gamma_2 = -1$, any real or complex number is an eigenvalue of both the difference and differential operator. Meanwhile, the spectrum of the difference operator as an empty set can be in case $\xi = h$. In addition, the number of eigenvalues of difference operator (14)–(15) depends on the parameters γ_1, γ_2 and ξ rather in a complicated way.

The eigenvalue problem of a difference operator with nonlocal conditions investigated in our paper differs by many properties from the eigenvalue problems of a differential operator and that of matrix.

The fact that the spectrum of a difference operator with concrete values of parameters γ_1, γ_2 has some unusual properties is conditioned by form of nonlocal conditions. Without an exhaustive examination of this issue, we refer to one typical property of nonlocal conditions analyzed in this paper.

Boundary interval points $x = 0$ and $x = 1$ are included only in one nonlocal condition. Another nonlocal condition is defined only at interior points of the interval under consideration. In essence, that is the main reason why the eigenvalue problem of difference operator cannot be expressed in the matrix form $Au = \lambda u$ with the $(N-1)$-order matrix. Note that this property is typical not only of nonlocal conditions (15). Nonlocal conditions of different type can have such property, for example, integral conditions

$$\int\limits_0^1 u(x)\,\mathrm{d}x = 0, \quad \int\limits_{\xi_1}^{\xi_2} u(x)\,\mathrm{d}x = 0, \quad 0 < \xi_1 < \xi_2 < 1.$$

If we exchange nonlocal condition (3) by a more general one

$$u(\xi) = \gamma_2 u(\eta), \quad \frac{1}{2} < \eta < 1,$$

many propositions, proved in this paper, will be correct. Taking apart, conclusions about a continuous spectrum will be true also. Without doubt, in each concrete case of new nonlocal condition, more exhaustive researches are necessary.

The spectrum structure of difference operators is rather important in the investigation of stability and convergence of difference schemes as well as iterative methods for systems of difference equations. These issues are not the research object of this article as well as the spectrum structure of the corresponding two-dimensional differential operators. We left these problems for future research.

The main aim of our paper was to pay attention to some unusual properties of the spectrum structure of difference operators with nonlocal conditions. The research we performed allows us to make the following conclusion. The spectrum of the difference operator with nonlocal conditions may essentially differ from the spectrum of the differential operators or the one of the matrices. Therefore, the eigenvalue problem of the difference operator with nonlocal conditions is worth to be self-contained object of the investigation.

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