Nonlinear Analysis: Modelling and Control, Vol. 24, No. 3, 407–432 ISSN: 1392-5113
https://doi.org/10.15388/NA.2019.3.6 eISSN: 2335-8963

Ekeland-type variational principle with applications to nonconvex minimization and equilibrium problems

Iram Iqbal[a], **Nawab Hussain**[b]

[a]University of Sargodha, Sargodha, Pakistan
iram.iqbal@uos.edu.pk

[b]King Abdulaziz University, Jeddah, Saudi Arabia
nhusain@kau.edu.sa

Received: April 6, 2018 / **Revised:** September 28, 2018 / **Published online:** April 19, 2019

Abstract. The aim of the present paper is to establish a variational principle in metric spaces without assumption of completeness when the involved function is not lower semicontinuous. As consequences, we derive many fixed point results, nonconvex minimization theorem, a nonconvex minimax theorem, a nonconvex equilibrium theorem in noncomplete metric spaces. Examples are also given to illustrate and to show that obtained results are proper generalizations.

Keywords: variational principle, T-orbitally lower semicontinuous function, fixed points, minimization problem.

1 Introduction and preliminaries

Ekeland [17, 18] formulated a variational principle, which is considered the basis of modern calculus of variations. Since its discovery, there are many generalizations and equivalent formulations of it (see [19, 20, 28, 37, 40, 45] and references therein). This principle says that when a function is not guaranteed to have a minimum, there is a "good" approximate substitute. Under the conditions of lower semicontinuity and boundedness below, the best we can get is an approximate minimum. These results are essential in areas of nonlinear analysis and optimization theory. A visualization of Ekeland's variational principle is shown in Fig. 1.

In Fig. 1, by taking $\lambda = 1$ we draw a line with slope equal to $-\epsilon = -\tan\theta$. Then the theorem guarantees that for any given ϵ, there is a point $(v, F(v))$ such that if we create an open downwards cone with that point as its vertex and having angle 2θ, the function values for all other inputs will stay above the cone. Here $F : X \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous function, which is bounded below.

Ekeland's variational principle has been widely used to prove the existence of approximate solutions of minimization problems for lower semicontinuous functions on

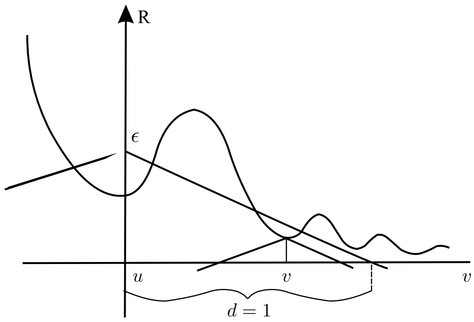

Figure 1. Ekeland's variational principle.

complete metric spaces (see, for instance, [5]). Since minimization problems are particular cases of equilibrium problems, many authors deal to derive existence results for solutions of equilibrium problems using Ekeland-type variational principles. There are many improvements and generalizations of Takahashi's nonconvex minimization theorem, Caristi's fixed point theorem and Ekeland's variational principle in complete metric spaces by using generalized distances: for example, w-distances, τ-distances, τ-functions, weak τ-functions and Q-functions (see [2, 22, 30, 42, 44]).

Meanwhile, the lower semicontinuous condition plays a key role in finding the solution of $\min_{x\in X} f(x)$, but it is not essential for solving some minimization problems. A function, which may not be necessarily lower semicontinuous, can still obtain its infimum. By getting motivation from above mentioned work, in present paper, we prove some generalizations of Ekeland's variational principle in the setting of T-orbitally complete metric spaces for functions, which are not necessarily lower semicontinuous, by introducing T-orbitally lower semicontinuity. As an application, we deduce some fixed point results, Takahashi's nonconvex minimization theorem, a nonconvex minimax theorem, a nonconvex equilibrium theorem in the setting of T-orbitally complete metric spaces. Our result generalizes the results of [9, 10, 12–14, 17, 23, 32, 34, 37].

In the sequel, let (X,d) be a metric space and $T : X \to X$ be a self mapping. For $A \subseteq X$, the diameter of A is $\mathrm{diam}(A) = \sup\{d(a,b)\colon a,b \in A\}$, and for each $x \in X$, orbit of T is $O(x;n) = \{x, Tx, T^2x, T^3x, \dots, T^nx\}$, $n = 1,2,3,\dots$, and $O(x) = O(x;\infty) = \{x, Tx, T^2x, \dots\}$. A metric space X is said to be T-orbitally complete if every Cauchy sequence, which is contained in $O(x)$ for some $x \in X$, converges in X [13]. Note that every complete metric space is T-orbitally complete space, but converse is not true in general. For example, $X = (0,1)$ with usual metric is not a complete metric space but a T-orbitally complete, where $T : X \to X$ is defined by $Tx = 1/2$ for all $x \in X$.

Definition 1. (See [7].) Let (X,d) be a metric space and $T : X \to X$ be a self mapping. T is said to be orbitally continuous at a point z in X if for any sequence $\{x_n\} \subseteq O(x)$ for some $x \in X$, $x_n \to z$ as $n \to \infty$ implies $Tx_n \to Tz$ as $n \to \infty$.

The Banach contraction principle [8] is an elementary result in metric fixed point theory. Many interesting generalizations of this golden principle have been obtained by considering several conditions on metric spaces (see [1, 6, 12, 15, 21, 23, 24, 31, 32, 34–36, 43]). An interesting generalization is given by Berinde [10] by introducing a comparison function.

Definition 2. A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be comparison function if it satisfies the following:

(ϕ1) ϕ is monotone increasing;
(ϕ2) $\{\phi^n(t)\}$ converges to 0 as $n \to \infty$ for all $t \geqslant 0$.

Denote the set all comparison functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by Φ.

Lemma 1. *If $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then $\phi(t) < t$ for all $t > 0$.*

2 Auxiliary results

We begin with the following definitions.

Definition 3. Let (X, d) be a metric space and $T : X \to X$ be a self mapping. A function $f : X \to \mathbb{R}$ is said to be T-orbitally continuous at a point z in X if for any sequence $\{x_n\} \subseteq O(x)$ for some $x \in X$, $x_n \to z$ as $n \to \infty$ implies $fx_n \to fz$ as $n \to \infty$.

Lemma 2. *Let (X, d) be a metric space, $T : X \to X$ and $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx)$ for all $x \in X$. If T is orbitally continuous function at $z \in X$, then f is T-orbitally continuous function at $z \in X$.*

Proof. Suppose that T is orbitally continuous function at $z \in X$, then there exists a sequence $\{x_n\}$ contained in $O(x)$ for some $x \in X$ such that $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$. This implies that $d(x_n, Tx_n) \to d(z, Tz)$ as $n \to \infty$, consequently, $f(x_n) \to f(z)$ as $n \to \infty$. Hence f is T-orbitally continuous function at $z \in X$. $\square$

Definition 4. Let (X, d) be a metric space and T be a self mapping on X. A function $f : X \to \mathbb{R}$ is said to be T-orbitally lower semicontinuous mapping if

$$x_n \to z \quad \text{implies} \quad f(z) \leqslant \liminf_n f(x_n),$$

where $\{x_n\}$ is a sequence contained in $O(x)$ for some $x \in X$.

Definition 5. Let (X, d) be a metric space and T be a self mapping on X. A function $f : X \to \mathbb{R}$ is said to be T-orbitally upper semicontinuous mapping if

$$x_n \to z \quad \text{implies} \quad f(z) \geqslant \limsup_n f(x_n),$$

where $\{x_n\}$ is a sequence contained in $O(x)$ for some $x \in X$.

Remark 1. Every lower semicontinuous (upper semicontinuous) mapping is T-orbitally lower semicontinuous (upper semicontinuous) mapping, but converse needs not to be true as shown in example below.

Example 1. Let $X = (1,2] \cup \{-1,0\}$ with usual metric d and $T : X \to X$ be a mapping defined as

$$Tx = \begin{cases} -1 & \text{if } x = 2, \\ 0 & \text{if } x \neq 2. \end{cases}$$

Then $O(2) = \{2,-1,0,0,0,\dots\}$, and for $2 \neq x \in X$, $O(x) = \{x,0,0,0,\dots\}$. Define $f : X \to \mathbb{R}$ by

$$fx = \begin{cases} -x & \text{if } x \in (1,\frac{3}{2}) \cup (\frac{3}{2},2], \\ 0 & \text{if } x \in \{0,\frac{3}{2},-1\}. \end{cases}$$

Note that every sequence $\{x_n\}$ contained in $O(x)$ converges to $0 \in X$. So, for $\{x_n\} \subset O(x)$, we have

$$\lim_n \inf f(x_n) = 0 = f(0).$$

Hence f is T-orbitally lower semicontinuous mapping. On the other hand, $x_n = 3/2 + 1/n$ is a sequence in X and converges to $3/2 \in X$, but

$$\lim_n \inf f(x_n) = \lim_n \inf f\left(\frac{3}{2} + \frac{1}{n}\right) = \lim_n \inf \left(-\frac{3}{2} - \frac{1}{n}\right) = -\frac{3}{2} \ngeqslant 0 = f\left(\frac{3}{2}\right).$$

Hence f is not lower semicontinuous mapping.

Lemma 3. *Let T be a self mapping on a metric space (X,d) and $f : X \to \mathbb{R}$ be a function. Then f is T-orbitally continuous if and only if f is T-orbitally lower semicontinuous and T-orbitally upper semicontinuous.*

Proof. Suppose that f is T-orbitally lower semicontinuous and T-orbitally upper semicontinuous, then for a sequence $\{x_n\} \subseteq O(x)$ for some $x \in X$ such that $x_n \to z \in X$, we have

$$f(z) \leqslant \lim_n \inf f(x_n) \tag{1}$$

and

$$f(z) \geqslant \lim_n \sup f(x_n). \tag{2}$$

Combining (1) and (2), we get $fx_n \to fz$ as $n \to \infty$. Hence f is T-orbitally continuous.

Conversely, suppose that f is T-orbitally continuous, then there exists a sequence $\{x_n\} \subseteq O(x)$ for some $x \in X$ such that $x_n \to z \in X$ implies

$$\lim_n f(x_n) = f(z), \tag{3}$$

which further implies $f(z) \leqslant \lim_n \inf f(x_n)$ and $f(z) \geqslant \lim_n \sup f(x_n)$. Hence f is T-orbitally lower semicontinuous and T-orbitally upper semicontinuous. $\square$

Definition 6. Let (X,d) be a metric space and T be a self mapping on X. A subset Y of X is said to be T-orbitally complete if every Cauchy sequence, which is contained in $O(y)$ for some $y \in Y$, converges in Y.

Lemma 4. *Let (X, d) be a metric space and T be a self mapping on X. Assume that X is T-orbitally complete and Y is closed subset of X. Then Y is T-orbitally complete.*

Proof. Suppose $\{y_n\}$ is a Cauchy sequence contained in $O(y)$ for some $y \in Y \subseteq X$. Since X is T-orbitally complete, so $y_n \to z \in X$. But Y is closed, so $z \in Y$. This completes the proof. □

3 Variational principle

Let (X, d) be a metric space and T be a self mapping on X such that $O(x) = \{x, Tx, T^2x, T^3x, \dots\}$. Assume the following:

(A) There are two functions $\alpha : X \times X \to [0, \infty)$, $\rho : X \times X \to [0, \infty) \cup \{+\infty\}$ satisfying the following, respectively:

(α1) If $\{x_n\}$ is any sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $\alpha(y, x_n) \to \alpha(y, x)$ for all $y \in X$;
(ρ1) $\rho(x, x) = 0$ for all $x \in X$;
(ρ2) For each $(x_n, y_n) \in X \times X$ as $n \to \infty$, $\rho(x_n, y_n) \to 0$ implies $d(x_n, y_n) \to 0$;
(ρ3) The function $y \to \rho(y, z)$ is T-orbitally lower semicontinuous for each $z \in X$;

and

(O1) If x_n is a sequence contained in orbit of T such that $x_n \to z$, then z belongs to some orbit of T.

Denote the collection of all functions $\rho : X \times X \to [0, \infty) \cup \{+\infty\}$, which satisfy ($\rho$1), ($\rho$2) and ($\rho$3), by Ω.

In addition, if $f : X \to \mathbb{R} \cup \{+\infty\}$, $\delta_0 > 0$, $\delta_n \geqslant 0$, $n \in \mathbb{N}$, is a sequence of nonnegative integers and $u_0 \in O(x_0)$ for some $x_0 \in X$, then for all $w \in X$, denote

$$\Delta(w, m) = f(w) + \alpha(u_0, w) \sum_{n=0}^{m} \delta_n \rho(w, u_n); \quad m \in \mathbb{N} \cup \{\infty\}.$$

Theorem 1. *Let (X, d) be a metric space, T be a self mapping on X and $f : X \to \mathbb{R} \cup \{+\infty\}$. Assume that $\rho \in \Omega$, (α1) is satisfied and for $u_0 \in O(x_0)$ for some $x_0 \in X$ and $\epsilon > 0$, following assumptions hold*:

(A1) $f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon$;
(A2) *the set $S_0 = \{w \in O(x) \mid \Delta(w, 0) \leqslant f(u_0)\}$ is nonempty and T-orbitally complete for some $x \in X$*;
(A3) *for each $m \in \mathbb{N}$, the set $S_m = \{w \in S_{m-1} \mid \Delta(w, m) \leqslant \Delta(y, m-1)\}$ for $y \in S_{m-1}$ is closed.*

Then there exists a sequence u_n in X and $u_\epsilon \in O(x)$ such that

(i) $u_n \to u_\epsilon$ *as* $n \to \infty$;
(ii) $\alpha(u_0, u_\epsilon)\rho(u_n, u_\epsilon) \leqslant \epsilon/(2^n \delta_0)$, $n \in \mathbb{N}$;

when for infinitely many n, $\delta_n > 0$,

(iii) $\varDelta(u_\epsilon, \infty) \leqslant f(u_0) \leqslant \inf_{x\in X} f(x) + \epsilon$;
(iv) $\varDelta(w, \infty) > \varDelta(u_\epsilon, \infty)$ *for every* $w \neq u_\epsilon$;

and when $\delta_k > 0$ and $\delta_j = 0$ for all $j > k \geqslant 0$, conclusion (iv) *is replaced by*

(v) *for all $w \neq u_\epsilon$, there exists $m \geqslant k$ such that*
$\varDelta(w, k-1) + \alpha(u_0, w)\delta_k\rho(w, u_m) > \varDelta(u_\epsilon, k-1) + \alpha(u_0, u_\epsilon)\delta_k\rho(u_\epsilon, u_m)$.

Proof. There arises two cases for δ_n:

Case 1: Infinitely many $\delta_n > 0$.

In this case, without loss of generality, we assume that $\delta_n > 0$ for all n. Then for $u_0 \in O(x_0)$ for some $x_0 \in X$, consider

$$G(u_0) = \bigl\{w \in O(x) \text{ for some } x \in X \bigm| \varDelta(w, 0) \leqslant f(u_0)\bigr\}. \tag{4}$$

Since $u_0 \in G(u_0)$, so $G(u_0)$ is nonempty. For every $y \in G(u_0)$, by using assumption (A1), we have

$$\alpha(u_0, y)\delta_0\rho(y, u_0) \leqslant f(u_0) - f(y) \leqslant f(u_0) - \inf_{x\in X} f(x) \leqslant \epsilon. \tag{5}$$

Choose $u_1 \in G(u_0)$ such that

$$\varDelta(u_1, 0) \leqslant \inf_{x\in G(u_0)} \varDelta(x, 0) + \frac{\delta_1\epsilon}{2\delta_0}$$

and let

$$G(u_1) = \bigl\{w \in G(u_0) \bigm| \varDelta(w, 1) \leqslant \varDelta(u_1, 0)\bigr\}, \tag{6}$$

similarly as above, $G(u_1)$ is nonempty. Continuing in this process, we can choose $u_{n-1} \in G(u_{n-2})$ and consider

$$G(u_{n-1}) = \bigl\{w \in G(u_{n-2}) \bigm| \varDelta(w, n-1) \leqslant \varDelta(u_{n-1}, n-2)\bigr\}. \tag{7}$$

Let us choose $u_n \in G(u_{n-1})$ such that

$$\varDelta(u_n, n-1) \leqslant \inf_{x\in G(u_{n-1})} \varDelta(x, n-1) + \frac{\delta_n\epsilon}{2^n\delta_0} \tag{8}$$

and define

$$G(u_n) = \bigl\{w \in G(u_{n-1}) \bigm| \varDelta(w, n) \leqslant \varDelta(u_n, n-1)\bigr\}, \tag{9}$$

which is nonempty. From (8) and (9), for each $y \in G(u_n)$, we get

$$\begin{aligned}
&\alpha(u_0, y)\delta_n\rho(y, u_n) \\
&\quad\leqslant \left(f(u_n) + \alpha(u_0, u_n)\sum_{i=0}^{n-1}\delta_i\rho(u_i, u_n)\right) - \left(f(y) + \alpha(u_0, y)\sum_{i=0}^{n-1}\delta_i\rho(u_i, y)\right) \\
&\quad\leqslant \varDelta(u_n, n-1) - \inf_{x\in G(u_{n-1})} \varDelta(x, n-1) \leqslant \frac{\delta_n\epsilon}{2^n\delta_0}.
\end{aligned} \tag{10}$$

This implies that for all $y \in G(u_n)$, we have

$$\alpha(u_0, y)\rho(y, u_n) \leqslant \frac{\epsilon}{2^n \delta_0}. \tag{11}$$

Now we have sequence of nonempty sets $\{G(u_n)\}$ such that

$$G(u_0) \supseteq G(u_1) \supseteq G(u_2) \supseteq \cdots \supseteq G(u_{n-1}) \supseteq G(u_n) \supseteq \cdots.$$

Therefore, $u_n \in G(u_{n-1})$ implies $u_n \in G(u_0)$. From assumption (A2), $G(u_0)$ is T-orbitally complete, so $u_n \to u_\epsilon \in X$. Also, by assumption (A3), $G(u_n)$ is closed for each $n \in \mathbb{N}$, so $u_\epsilon \in \bigcap_{n=0}^{\infty} G(u_n)$. Letting limit as $n \to \infty$ in (11), we get $\rho(y, u_n) \to 0$, and since $\rho \in \Omega$, we get $d(y, u_n) \to 0$. Therefore, $\operatorname{diam}(G(u_n)) \to 0$ as $n \to \infty$. If $y \in \bigcap_{n=0}^{\infty} G(u_n)$ and $y \neq u_\epsilon$, then $d(y, u_\epsilon) = \beta > 0$. There exists $n \in \mathbb{N}$ large enough such that $\operatorname{diam}(G(u_n)) < \beta$, which ensures that $y \notin G(u_n)$. Hence y cannot be in $\bigcap_{n=0}^{\infty} G(u_n)$. Thus, the intersection contains only one point. From (5)–(11) we obtain that u_ϵ satisfies (ii) and $u_n \to u_\epsilon$ as $n \to \infty$. Further, for all $w \neq u_\epsilon$, we have $w \notin \bigcap_{n=0}^{\infty} G(u_n)$, so there exists $m \in \mathbb{N}$ such that

$$\Delta(w, m) > \Delta(u_m, m-1). \tag{12}$$

From (4), (7) and (8), for every $q \geqslant m$, we obtain

$$f(u_0) \geqslant \Delta(u_m, m-1) \geqslant \Delta(u_q, q-1) \geqslant \Delta(u_\epsilon, q-1). \tag{13}$$

Combining (12) and (13) gives

$$\Delta(w, m) > \Delta(u_\epsilon, q-1).$$

Letting $q, m \to \infty$ gives (iii) and (iv).

Case 2: Finitely many $\delta_n > 0$.

Assume that $\delta_k > 0$ and $\delta_j = 0$ for all $j > k \geqslant 0$. Without loss of generality, we suppose that $\delta_i > 0$ for all $i \leqslant k$. Thus, when $n \leqslant k$, we choose the same u_n and $G(u_n)$ as in Case 1. When $n > k$, we take $u_n \in G(u_{n-1})$ such that

$$\Delta(u_n, k-1) \leqslant \inf_{x \in G(u_{n-1})} \Delta(x, k-1) + \frac{\delta_k \epsilon}{2^n \delta_0} \tag{14}$$

and put

$$G(u_n) = \bigl\{ w \in G(u_{n-1}) \bigm| \Delta(w, k-1) + \alpha(u_0, w)\delta_k \rho(u_n, w) \leqslant \Delta(u_n, k-1) \bigr\}. \tag{15}$$

Then by following the same steps as in Case 1, hypotheses (ii)–(iv) hold. When $w \neq u_\epsilon$, there exists an $m > k$ such that

$$\begin{aligned} &\Delta(w, k-1) + \alpha(u_0, w)\delta_k \rho(u_m, w) \\ &\quad > \Delta(u_m, k-1) \geqslant \Delta(u_\epsilon, k-1) + \alpha(u_0, u_\epsilon)\delta_k \rho(u_m, u_\epsilon). \end{aligned}$$

This completes the proof. □

Now, we illustrate Theorem 1 by following example.

Example 2. Let $X = C[0,1]$ is the space of all real-valued continuous functions on $[0,1]$ with metric d defined as

$$d(g,h) = \left[\int_0^1 \left|h(t) - g(t)\right|^2 \mathrm{d}t \right]^{1/2}.$$

Then (X,d) is metric space. Define $T : X \to X$ by

$$T(g) = \begin{cases} \mathbf{I} & \text{if } g = \mathbf{I}', \\ \mathbf{0} & \text{otherwise}, \end{cases}$$

where $\mathbf{0}(t) = 0$, $\mathbf{I}(t) = 1$ and $\mathbf{I}'(t) = -1$ for all $t \in [0,1]$. Also, define $f : X \to \mathbb{R} \cup \{+\infty\}$, $\alpha : X \times X \to [0,\infty)$ and $\rho : X \times X \to [0,\infty) \cup \{+\infty\}$ as

$$f(g) = \begin{cases} 0 & \text{if } g(t) = 0, \\ g(t) & \text{if } g(t) > 0 \\ -g(t) & \text{if } g(t) < 0, \end{cases}$$

$\alpha(x,y) = 1$ and $\rho(g,h) = |h(t) - g(t)|$ for all $g,h \in X$, respectively. Then $O(\mathbf{I}') = \{\mathbf{I}', \mathbf{I}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \dots\}$, and for $\mathbf{I}' \neq g \in X$, $O(g) = \{g, \mathbf{0}, \mathbf{0}, \mathbf{0}, \dots\}$, $\rho \in \varOmega$ and $(\alpha 1)$ holds. There exists $u_0 = \mathbf{0} \in O(\mathbf{I}')$ and $\epsilon = 2^{n+3}$ for $n \in \mathbb{N}$ such that

$$\inf_{g \in X} f(g) + \epsilon = 0 + 2^{n+3} > 0 = f(u_0).$$

Take $\delta_0 = 1$ and $\delta_n = 0$ for all $n \in \mathbb{N}$. Since $S_m = \{\mathbf{0}\}$ for each $m \in \mathbb{N} \cup \{0\}$, therefore S_0 is nonempty and T-orbitally complete, also for each $m \in \mathbb{N}$, S_m is closed. Hence hypothesis (A), (A1), (A2) and (A3) of Theorem 1 hold true. Further, there exists a sequence g_n in X such that $g_n(t) = 1/n$ for all $t \in [0,1]$ and

$$\Bigl(\lim_{n\to\infty} g_n\Bigr)(t) = \lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} \frac{1}{n} = 0 = \mathbf{0}(t).$$

This implies that $u_n \to \mathbf{0} = u_\epsilon \in O(g)$ as $n \to \infty$ with

$$\alpha(\mathbf{0},\mathbf{0})\rho(u_n,\mathbf{0}) = \left|\mathbf{0}(t) - g_n(t)\right| = \frac{1}{n} \leqslant 8 = \frac{\epsilon}{2^n \delta_0}$$

and

$$\varDelta(\mathbf{0},\infty) = 0 = f(\mathbf{0}).$$

Hence conclusions (i), (ii) and (iii) of Theorem 1 hold. Since $\delta_0 = 1$ and $\delta_j = 0$ for all $j > 0$, so we have to satisfy conclusion (v). For this, there arise two cases.

Case 1. For $\mathbf{0} \neq g \in X$ such that $g(t) > 0$, there exists $m \geqslant 0$ such that

$$\begin{aligned} &f(g) + \alpha(u_0,g)\delta_k\rho(g,u_m) \\ &\quad = g(t) + \alpha(\mathbf{0},\mathbf{0})\delta_0\rho(g,u_m) = g(t) + \left|u_m(t) - g(t)\right|. \end{aligned} \tag{16}$$

(a) If $u_m(t) > 0$ and $u_m(t) > g(t)$ for all t, then (16) gives

$$f(g) + \alpha(u_0, g)\delta_k\rho(g, u_m) = \left|u_m(t)\right| = f(u_\epsilon) + \alpha(u_0, u_\epsilon)\delta_k\rho(u_\epsilon, u_m).$$

(b) If $u_m(t) > 0$ and $u_m(t) < g(t)$ for all t, then (16) gives

$$f(g) + \alpha(u_0, g)\delta_k\rho(g, u_m) > \left|u_m(t)\right| = f(u_\epsilon) + \alpha(u_0, u_\epsilon)\delta_k\rho(u_\epsilon, u_m).$$

(c) If $u_m(t) < 0$, then (16) gives

$$\begin{aligned} f(g) + \alpha(u_0, g)\delta_k\rho(g, u_m) &= g(t) + \left|g(t) + u_m(t)\right| > \left|g(t) + u_m(t)\right| \\ &> \left|u_m(t)\right| = f(u_\epsilon) + \alpha(u_0, u_\epsilon)\delta_k\rho(u_\epsilon, u_m). \end{aligned}$$

Case 2. For $\mathbf{0} \neq g \in X$ such that $g(t) < 0$, there exists $m \geqslant 0$ such that

$$\begin{aligned} f(g) + \alpha(u_0, g)\delta_k\rho(g, u_m) &= -g(t) + \alpha(\mathbf{0}, \mathbf{0})\delta_0\rho(g, u_m) \\ &= -g(t) + \left|u_m(t) - g(t)\right|. \end{aligned} \tag{17}$$

(a) If $u_m(t) > 0$ for all t, then (17) gives

$$f(g) + \alpha(u_0, g)\delta_k\rho(g, u_m) > \left|u_m(t)\right| = f(u_\epsilon) + \alpha(u_0, u_\epsilon)\delta_k\rho(u_\epsilon, u_m).$$

(b) If $u_m(t) < 0$ and $u_m(t) < g(t)$ for all t, then (17) gives

$$f(g) + \alpha(u_0, g)\delta_k\rho(g, u_m) = \left|u_m(t)\right| = f(u_\epsilon) + \alpha(u_0, u_\epsilon)\delta_k\rho(u_\epsilon, u_m).$$

(c) If $u_m(t) < 0$ and $u_m(t) > g(t)$ for all t, then (17) gives

$$f(g) + \alpha(u_0, g)\delta_k\rho(g, u_m) > \left|u_m(t)\right| = f(u_\epsilon) + \alpha(u_0, u_\epsilon)\delta_k\rho(u_\epsilon, u_m).$$

Thus, conclusion (v) of Theorem 1 holds for all $g \neq \mathbf{0} = u_\epsilon$.

Remark 2. Note that, in Example 2, metric space (X, d) is not complete. Indeed, consider the sequence $\{g_n\}$ in Fig. 2. Since $d(g_n, g_m) < \epsilon$ for $n, m > 1/\epsilon$, therefore $\{g_n\}$ is a Cauchy sequence such that for every $g \in X$

$$d(g_n, g) = \left[\int_0^{1/2} \left|g(t)\right|^2 \mathrm{d}t + \int_{1/2}^{1/2+1/n} \left|g_n(t) - g(t)\right|^2 \mathrm{d}t + \int_{1/2+1/n}^{1} \left|1 - g(t)\right|^2 \mathrm{d}t\right]^{1/2}.$$

$d(g_n, g) \to 0$ as $n \to \infty$ when

$$g(t) = \begin{cases} 0 & \text{if } t < 1/2 \\ 1 & \text{if } t > 1/2, \end{cases}$$

which is not continuous. Hence $\{g_n\}$ is not convergent. Therefore, Theorem 3.1 of [3], Theorem 1.1 of [17], Theorem 4.2 of [16] and Theorem 1 of [37] cannot be applied for this example.

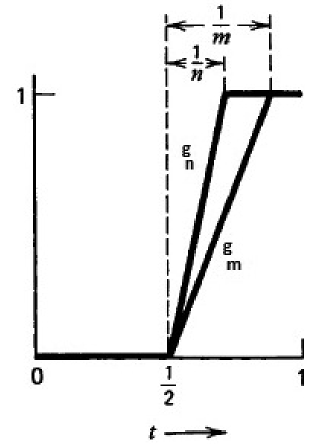

Figure 2. A sequence in C[0,1].

Lemma 5. *Let* (X, d) *be a metric space,* T *be a self mapping on* X *such that* X *is* T*-orbitally complete. Assume that* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is a* T*-orbitally lower semicontinuous function, bounded from below,* $\rho \in \Omega$*,* $(\alpha 1)$ *and* (O1) *hold true. Then assumptions* (A2) *and* (A3) *in Theorem* 1 *hold.*

Proof. Let $u_0 \in O(x_0)$ for some $x_0 \in X$ and

$$S_0 = \bigl\{ w \in O(x) \text{ for some } x \in X \bigm| \Delta(w, 0) \leqslant f(u_0) \bigr\}.$$

Since $u_0 \in S_0$, so $G(u_0)$ is nonempty. Now, if z_n is a sequence in S_0 that converges to $z \in X$, then z belongs to orbit of T and $\Delta(z_n, 0) \leqslant f(u_0)$. From T-orbitally lower semicontinuity of f, T-orbitally lower semicontinuity of $\rho(., u_0)$ and $(\alpha 1)$ we get

$$f(z) \leqslant \liminf_n f(z_n) \leqslant f(u_0) - \alpha(u_0, z)\delta_0 \rho(z, u_0).$$

This shows that $z \in S_0$ and S_0 is closed subset of X. Since X is T-orbitally complete, so from Lemma 4, S_0 is T-orbitally complete. Next, suppose that for $m \in \mathbb{N}$, $y \in S_{m-1}$, w_n is a sequence contained in the set $S_m = \{w \in S_{m-1} \mid \Delta(w, m) \leqslant \Delta(y, m-1)\}$ such that $w_n \to w^* \in X$. Then w^* belongs to some orbit of T, and $\Delta(w_n, m) \leqslant \Delta(y,\, m-1)$. From T-orbitally lower semicontinuity of f, T-orbitally lower semicontinuity of $\rho(., u_0)$ and $(\alpha 1)$ we get

$$\begin{aligned}
f(w^*) &\leqslant \liminf_n f(w_n) \\
&\leqslant \liminf_n \left\{ f(y) + \alpha(u_0, y) \sum_{i=0}^{m-1} \delta_i \rho(y, u_i) - \alpha(u_0, w_n) \sum_{i=0}^{m} \delta_i \rho(w_n, u_i) \right\} \\
&\leqslant f(y) + \alpha(u_0, y) \sum_{i=0}^{m-1} \delta_i \rho(y, u_i) - \alpha(u_0, w^*) \sum_{i=0}^{m} \delta_i \rho(w^*, u_i),
\end{aligned}$$

that is, $\Delta(w^*, m) \leqslant \Delta(y,\, m-1)$. Consequently, S_m is closed. □

From Lemma 5 and Theorem 1 we obtain the following:

Theorem 2. *Let (X, d) be a metric space, T be a self mapping on X and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a T-orbitally lower semicontinuous function, bounded from below. Assume that X is T-orbitally complete and* (A) *holds. If for $u_0 \in O(x_0)$ for some $x_0 \in X$ and $\epsilon > 0$,*

$$f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon.$$

Then there exists $u_\epsilon \in O(x)$ such that all conclusions (i)–(v) *of Theorem* 1 *hold.*

Taking $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2, we get the following:

Corollary 1. *Let (X, d) be a metric space, T be a self mapping on X and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a T-orbitally lower semicontinuous function, bounded from below. Assume that X is T-orbitally complete and* (A) *holds. If for $u_0 \in O(x_0)$ for some $x_0 \in X$ and $\epsilon > 0$,*

$$f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon. \tag{18}$$

Then there exists $u_\epsilon \in O(x)$ such that

(i) $u_n \to u_\epsilon$ *as* $n \to \infty$;
(ii) $\rho(u_n, u_\epsilon) \leqslant \epsilon/(2^n \delta_0)$, $n \in \mathbb{N}$;

when for infinitely many n, $\delta_n > 0$,

(iii) $\chi(u_\epsilon, \infty) \leqslant f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon$;
(iv) $\chi(w, \infty) > \chi(u_\epsilon, \infty)$, *for every* $w \neq u_\epsilon$;

and when $\delta_k > 0$ and $\delta_j = 0$ for all $j > k \geqslant 0$, conclusion (iv) *is replaced by*

(v) *for all $w \neq u_\epsilon$ there exists $m \geqslant k$ such that* $\chi(w, k-1) + \delta_k \rho(w, u_m) > \chi(u_\epsilon, k-1) + \delta_k \rho(u_\epsilon, u_m)$,

where

$$\chi(w, m) = f(w) + \sum_{i=0}^{m} \delta_n \rho(w, u_i); \quad m \in \mathbb{N} \cup \{\infty\}.$$

Since there is always a some point x with $f(x) \leqslant \inf F + \epsilon$ and also $x \in O(x)$, therefore form Corollary 1 we have the following:

Corollary 2. *Let (X, d) be a metric space, T be a self mapping on X and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a T-orbitally lower semicontinuous function, bounded from below. Assume that X is T-orbitally complete and* (A) *holds. Then there exists $u_\epsilon \in O(x)$ such that conclusions* (i)–(v) *of Corollary* 1 *hold.*

If we consider $\rho(y, z) = (\epsilon/\lambda) d(y, z)$, $\delta_0 = 1$ and $\delta_n = 0$ for all $n > 0$ in Corollary 1, then we get the following corollary, which is the Ekeland's ϵ-variational principle in the context of T-orbitally complete metric spaces.

Corollary 3 [Ekeland's ϵ-variational principle]. *Let (X, d) be a metric space, T be a self mapping on X and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a T-orbitally lower semicontinuous function, bounded from below. Assume that X is T-orbitally complete and* (O1) *holds. If for $u_0 \in O(x_0)$ for some $x_0 \in X$ and $\epsilon, \lambda > 0$,*

$$f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon,$$

then there exists $u_\epsilon \in O(x)$ such that

(i) $u_n \to u_\epsilon$ *as* $n \to \infty$;
(ii) $d(u_n, u_\epsilon) \leqslant \lambda/2^n$, $n \in \mathbb{N}$;
(iii) $f(u_\epsilon) + (\epsilon/\lambda) d(u_0, u_\epsilon) \leqslant f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon$;
(iv) $f(w) > f(x_\epsilon) - (\epsilon/\lambda) d(w, u_\epsilon)$ *for every* $w \neq u_\epsilon$.

Taking $\rho(y, z) = \sqrt{\epsilon} d(y, z)$, $\delta_0 = 1$ and $\delta_n = 0$ for all $n > 0$ in Corollary 1, we get the following:

Corollary 4. *Let (X, d) be a metric space, T be a self mapping on X and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a T-orbitally lower semicontinuous function bounded from below. Assume that X is T-orbitally complete and* (O1) *holds. If for $u_0 \in O(x_0)$ for some $x_0 \in X$ and $\epsilon > 0$,*

$$f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon,$$

then there exists $u_\epsilon \in O(x)$ such that

(i) $u_n \to u_\epsilon$ *as* $n \to \infty$;
(ii) $d(u_n, u_\epsilon) \leqslant \sqrt{\epsilon}/2^n$, $n \in \mathbb{N}$;
(iii) $f(u_\epsilon) + \sqrt{\epsilon} d(u_0, u_\epsilon) \leqslant f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon$;
(iv) $f(w) > f(x_\epsilon) - \sqrt{\epsilon} d(w, u_\epsilon)$, *for every* $w \neq u_\epsilon$.

Remark 3. Since every metric space is T-orbitally complete metric space, therefore, in the light of Remark 1, Corollary 1 is the generalization of Theorem 1 of [37], and Corollary 3 is the generalization of Theorem 1.1 of [17].

Theorem 3. *Let (X, d) be a metric space and T be a self mapping on X. Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous and there exists $\phi \in \Phi$ such that*

$$f(Tx) \leqslant \phi\big(f(x)\big) \tag{19}$$

for all $x \in X$. Then for every $\epsilon > 0$, there exists $u_0 \in O(x_0)$ for some $x_0 \in X$ such that

$$f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon.$$

Proof. For $n \in \mathbb{N}$ and $x \in X$, from (19) and Lemma 1 we get

$$f\big(T^n x\big) = f\big(T\big(T^{n-1} x\big)\big) \leqslant \phi\big(f\big(T^{n-1} x\big)\big) < f\big(T^{n-1} x\big).$$

This shows that $\{f(T^n x)\}$ is a decreasing sequence of nonnegative integers and bounded below by 0. Also,

$$f\bigl(T^n x\bigr) = f\bigl(T\bigl(T^{n-1}x\bigr)\bigr) \leqslant \phi\bigl(f\bigl(T^{n-1}x\bigr)\bigr) \leqslant \phi^2\bigl(f\bigl(T^{n-2}x\bigr)\bigr) \leqslant \cdots \leqslant \phi^n\bigl(f(x)\bigr). \tag{20}$$

Letting limit as $n \to \infty$ in (20) and using ($\phi 2$), we obtain

$$f(T^n x) \to 0 \quad \text{as } n \to \infty.$$

It follows that there exists $\epsilon > 0$ such that

$$f\bigl(T^n x\bigr) \leqslant \epsilon = 0 + \epsilon = \inf\bigl\{f(x)\colon x \in X\bigr\} + \epsilon.$$

This completes the proof. □

Proposition 1. *Let (X, d) be a metric space and T be a self mapping on X. Assume that X is T-orbitally complete and, for $\phi \in \Phi$, T satisfy* (19) *for all $x \in X$. If $\{T^n x\}$ is any sequence contained in $O(x)$ for some $x \in X$ and converges to $z \in X$, then z is a fixed point of T, provided that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous.*

Proof. For $n \in \mathbb{N}$ and $x \in X$, as in Theorem 2, we get

$$f\bigl(T^n x\bigr) \to 0 \quad \text{as } n \to \infty.$$

Since f is T-orbitally semicontinuous, so from above inequality we obtain

$$f(z) \leqslant \liminf_{n} f\bigl(T^n x\bigr) = 0.$$

This implies that $f(z) = 0$, consequently, z is a fixed point of T. □

From Theorem 2, Theorem 3 and Proposition 1 we get the following:

Corollary 5. *Let (X, d) be a metric space and T be a self mapping on X. Assume that X is T-orbitally complete, $\alpha : X \times X \to [0, \infty)$ satisfy ($\alpha 1$), $\rho \in \Omega$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous. If there exists $\phi \in \Phi$ such that*

$$f(Tx) \leqslant \phi\bigl(f(x)\bigr)$$

for all $x \in X$, then there exists $u_\epsilon \in O(x)$ such that

(i) $u_n \to u_\epsilon$ *as* $n \to \infty$;
(ii) $\alpha(u_0, u_\epsilon)\rho(u_n, u_\epsilon) \leqslant \epsilon/(2^n \delta_0)$, $n \in \mathbb{N}$;

when for infinitely many n, $\delta_n > 0$,

(iii) $\Delta(u_\epsilon, \infty) \leqslant f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon$;
(iv) $\Delta(w, \infty) > \Delta(u_\epsilon, \infty)$, *for every* $w \neq u_\epsilon$;

and when $\delta_k > 0$ *and* $\delta_j = 0$ *for all* $j > k \geqslant 0$*, conclusion* (iv) *is replaced by*

(v) *for all* $w \neq u_\epsilon$*, there exists* $m \geqslant k$ *such that* $\Delta(w, k-1) + \alpha(u_0, w)\delta_k \rho(w, u_m) > \Delta(u_\epsilon, k-1) + \alpha(u_0, u_\epsilon)\delta_k \rho(u_\epsilon, u_m)$.

Taking $\alpha(x, y) = 1$ for all $x, y \in X$ in Corollary 5, we get the following:

Corollary 6. *Let* (X, d) *be a metric space and* T *be a self mapping on* X*. Assume that* X *is* T*-orbitally complete,* $\rho \in \Omega$ *and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *such that* $f(x) = d(x, Tx)$ *is* T*-orbitally lower semicontinuous. If there exists* $\phi \in \Phi$ *such that*

$$f(Tx) \leqslant \phi\big(f(x)\big)$$

for all $x \in X$*, then there exists* $u_\epsilon \in O(x)$ *such that conclusions* (i)–(iv) *of Corollary* 1 *hold.*

If we consider $\rho(y, z) = (\epsilon/\lambda)d(y, z)$, $\delta_0 = 1$ and $\delta_n = 0$ for all $n > 0$ in Corollary 6, then we get the following:

Corollary 7. *Let* (X, d) *be a metric space and* T *be a self mapping on* X*. Assume that* X *is* T*-orbitally complete and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *such that* $f(x) = d(x, Tx)$ *is* T*-orbitally lower semicontinuous. If there exists* $\phi \in \Phi$ *such that*

$$f(Tx) \leqslant \phi\big(f(x)\big)$$

for all $x \in X$*, then for* $\epsilon, \lambda > 0$*, there exists a sequence* $\{u_n\}$ *and* $u_\epsilon \in O(x)$ *such that*

(i) $u_n \to u_\epsilon$ *as* $n \to \infty$;
(ii) $d(u_n, u_\epsilon) \leqslant \lambda/2^n$, $n \in \mathbb{N}$;
(iii) $f(u_\epsilon) + (\epsilon/\lambda)d(u_0, u_\epsilon) \leqslant f(u_0) \leqslant \inf_{x \in X} f(x) + \epsilon$;
(iv) $f(w) > f(u_\epsilon) - (\epsilon/\lambda)d(w, u_\epsilon)$ *for every* $w \neq u_\epsilon$.

4 Fixed point results

In this section, we derive some fixed point results from Section 3.

Theorem 4 [Suzuki-type fixed point theorem]. *Let* (X, d) *be a metric space and* $T : X \to X$ *be a mapping. Define a nonincreasing function* θ *from* $[0, 1)$ *onto* $(1/2, 1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant (\sqrt{5} - 1)/2, \\ (1 - r)r^{-2} & \text{if } (\sqrt{5} - 1)/2 \leqslant r \leqslant 2^{-1/2}, \\ (1 + r)^{-1} & \text{if } 2^{-1/2} \leqslant r < 1. \end{cases}$$

Assume that (X, d) *is* T*-orbitally complete and there exists* $r \in [0, 1)$ *such that*

$$\theta(r)d(x, Tx) \leqslant d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leqslant rd(x, y) \tag{21}$$

for all $x, y \in X$*. Then there exists a unique fixed point of* T*, provided that* $f : X \to \mathbb{R} \cup \{+\infty\}$ *such that* $f(x) = d(x, Tx)$ *is* T*-orbitally lower semicontinuous.*

Proof. Since $\theta(r) \leqslant 1$, so $\theta(r)d(x,Tx) \leqslant d(x,Tx)$ for all $x,y \in X$. By taking $y = Tx$ in (21), we get

$$f(Tx) = d\bigl(Tx,T^2x\bigr) \leqslant rd(x,Tx) = rf(x), \tag{22}$$

which shows that (19) holds true for $\phi(t) = rt$, $r \in [0,1)$. Therefore, by using Corollary 7, there exists a sequence $\{u_n\}$ and $\epsilon, \lambda > 0$ such that $u_n \to u_\epsilon \in O(x)$ and

$$f(w) > f(u_\epsilon) - \frac{\epsilon}{\lambda}d(w,u_\epsilon) \tag{23}$$

for every $w \in X$ with $w \neq u_\epsilon$. Now we claim that $f(u_\epsilon) = 0$. If not, then choose ϵ and λ such that $\epsilon/\lambda \in (0, 1-r)$, $r \in [0,1)$. Considering $w = Tu_\epsilon$ in (23), we obtain

$$f(u_\epsilon) < f(Tu_\epsilon) + \frac{\epsilon}{\lambda}d(u_\epsilon,Tu_\epsilon) < rf(u_\epsilon) + \frac{\epsilon}{\lambda}f(u_\epsilon),$$

which implies $(1 - r - \epsilon/\lambda)f(u_\epsilon) < 0$, this leads to contradiction because $(\epsilon/\lambda) \in (0,\, 1-r)$. Hence $f(u_\epsilon) = 0$, and thus, u_ϵ is a fixed point of T in X.

For uniqueness, suppose that $x \neq u_\epsilon$ is another fixed point of T, then $0 = \theta(r) \times d(u_\epsilon,Tu_\epsilon) \leqslant d(u_\epsilon,x)$. Hence (21) implies $d(u_\epsilon,x) \leqslant rd(u_\epsilon,x)$ for $r \in [0,1)$, a contradiction. Thus, $x = u_\epsilon$. □

Remark 4. Theorem 4 generalizes Theorem 2 of [39] from metric spaces to orbitally complete metric spaces.

Theorem 5 [Ćirić-type fixed point theorem]. *Let (X,d) be a metric space and $T: X \to X$ be a mapping. Assume that (X,d) is T-orbitally complete and there exists $q \in (0,1)$ such that the following holds*:

$$d(Tx,Ty) \leqslant qM(x,y), \tag{24}$$

where

$$M(x,y) = \max\biggl\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\biggr\}.$$

Then T has a unique fixed point in X, provided that $f: X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x,Tx)$ is T-orbitally lower semicontinuous.

Proof. Put $y = Tx$ in (24), we get

$$d\bigl(Tx,T^2x\bigr) \leqslant qM(x,Tx), \tag{25}$$

where

$$M(x,Tx) = \max\biggl\{d(x,Tx), d(Tx,T^2x), \frac{d(x,T^2x)}{2}\biggr\},$$

which implies

$$f(Tx) \leqslant q\max\biggl\{d(x,Tx), d(Tx,T^2x), \frac{d(x,T^2x)}{2}\biggr\}. \tag{26}$$

Now, if

$$\max\left\{d(x,Tx), d(Tx,T^2x), \frac{d(x,T^2x)}{2}\right\} = d\bigl(Tx,T^2x\bigr),$$

then (26) gives

$$f(Tx) \leqslant qd\bigl(Tx,T^2x\bigr) = qf(Tx),$$

which leads to contradiction. Therefore,

$$f(Tx) \leqslant q\max\left\{d(x,Tx), \frac{d(x,T^2x)}{2}\right\}. \tag{27}$$

Here arises two cases:

Case 1. If $\max\{d(x,Tx), d(x,T^2x)/2\} = d(x,T^2x)/2$, then from (27) we get

$$\begin{aligned} f(Tx) &\leqslant q\frac{d(x,T^2x)}{2} \leqslant \frac{q}{2}\bigl\{d(x,Tx) + d\bigl(Tx,T^2x\bigr)\bigr\} \\ &= \frac{q}{2}d(x,Tx) + \frac{q}{2}d\bigl(Tx,T^2x\bigr). \end{aligned}$$

This implies that

$$f(Tx) \leqslant \frac{q}{2-q}d(x,Tx) = rf(x),$$

where $r = q/(2-q) \in (0,1)$.

Case 2. If $\max\{d(x,Tx), d(x,T^2x)/2\} = d(x,Tx)$, then from (27) we get

$$f(Tx) \leqslant qd(x,Tx) \leqslant rf(x),$$

where $r = q/(2-q) \in (0,1)$.

Hence, in each case, (19) holds true for $\phi(t) = rt$, where $r = q/(2-q) \in (0,1)$. Therefore, by using Corollary 7, there exists a sequence $\{u_n\}$ and $\epsilon, \lambda > 0$ such that $u_n \to u_\epsilon \in O(x)$ and

$$f(w) > f(u_\epsilon) - \frac{\epsilon}{\lambda}d(w,u_\epsilon) \tag{28}$$

for every $w \in X$ with $w \neq u_\epsilon$. Now we claim that $f(u_\epsilon) = 0$. If not, then choose ϵ and λ such that $\epsilon/\lambda \in (0,\, 1-r)$. Considering $w = Tu_\epsilon$ in (28), we obtain

$$f(u_\epsilon) < f(Tu_\epsilon) + \frac{\epsilon}{\lambda}d(u_\epsilon, Tu_\epsilon) < rf(u_\epsilon) + \frac{\epsilon}{\lambda}f(u_\epsilon),$$

which implies $(1-r-\epsilon/\lambda)f(u_\epsilon) < 0$, this leads to contradiction because $\epsilon/\lambda \in (0,\, 1-r)$. Hence $f(u_\epsilon) = 0$, and thus, u_ϵ is a fixed point of T in X.

For uniqueness, suppose that $x \neq u_\epsilon$ is another fixed point of T, then from (24) we obtain $d(u_\epsilon, x) \leqslant qd(u_\epsilon, x)$, a contradiction. Thus, $x = u_\epsilon$. □

Theorem 6. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Assume that (X, d) is T-orbitally complete and T satisfy the following*:

$$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx), \qquad (29)$$

where $\alpha, \beta, \gamma, \delta, L \in [0, \infty)$ with $\alpha + \beta + \gamma + \delta + L < 1$. Then T has a fixed point in X, provided that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous.

Proof. From (29) and by using Lemma of [34], there exists $0 < k < 1$ such that

$$f(Tx) = d\bigl(Tx, T^2x\bigr) \leqslant kd(x, Tx) = kf(x). \qquad (30)$$

Relation (30) implies that (19) holds true for $\phi(t) = kt$, $k \in (0, 1)$. Therefore, by using Corollary 7, there exists a sequence $\{u_n\}$ and $\epsilon, \lambda > 0$ such that $u_n \to u_\epsilon \in O(x)$ and

$$f(w) > f(u_\epsilon) - \frac{\epsilon}{\lambda} d(w, u_\epsilon) \qquad (31)$$

for every $w \in X$ with $w \neq u_\epsilon$. Now we claim that $f(u_\epsilon) = 0$. If not, then choose ϵ and λ such that $\epsilon/\lambda \in (0,\, 1 - k)$. Considering $w = Tu_\epsilon$ in (31), we obtain

$$f(u_\epsilon) < f(Tu_\epsilon) + \frac{\epsilon}{\lambda} d(u_\epsilon, Tu_\epsilon) \leqslant kf(u_\epsilon) + \frac{\epsilon}{\lambda} f(u_\epsilon),$$

which implies $(1-k-\epsilon/\lambda)f(u_\epsilon) < 0$, this leads to contradiction because $\epsilon/\lambda \in (0, 1-k)$. Hence $f(u_\epsilon) = 0$, and thus, u_ϵ is a fixed point of T in X. □

In similar fashion, we can deduce the following:

Theorem 7. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Assume that (X, d) is T-orbitally complete and T satisfy the following*:

$$d(Tx, Ty) \leqslant \beta d(x, Tx) + \gamma d(y, Ty),$$

where $\beta, \gamma \in [0, \infty)$ with $\beta + \gamma < 1$. Then T has a fixed point in X, provided that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous.

Theorem 8. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Assume that (X, d) is T-orbitally complete and T satisfy the following*:

$$d(Tx, Ty) \leqslant \frac{1}{2} d(x, Ty) + Ld(y, Tx),$$

where $L \in [0, \infty)$ with $L \leqslant 1/2$. Then T has a fixed point in X, provided that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous.

Theorem 9. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Assume that (X, d) is T-orbitally complete and T satisfy the following*:

$$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty),$$

where $\alpha, \beta, \gamma \in [0, \infty)$ with $\alpha + \beta + \gamma < 1$. Then T has a fixed point in X, provided that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous.

Theorem 10. *Let (X,d) be a metric space and $T : X \to X$ be a mapping. Assume that (X,d) is T-orbitally complete and there exists $\delta \in (0,1)$ and $L \geqslant 0$ such that following holds*:

$$d(Tx,Ty) \leqslant \delta(x,y) + Ld(y,Tx).$$

Then T has a unique fixed point in X, provided that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x,Tx)$ is T-orbitally lower semicontinuous.

Theorem 11. *Let (X,d) be a metric space and $T : X \to X$ be a mapping. Assume that (X,d) is T-orbitally complete and there exists $q \in [0,1)$ such that the following holds*:

$$\begin{aligned} &d(Tx,Ty) \\ &\quad < q\max\bigl\{d(x,y), \bigl(d(x,y)\bigr)^{-1}d(x,Tx)d(y,Ty), \tau(x,y)d(x,Ty)d(y,Tx)\bigr\} \end{aligned}$$

for all $x,y \in X$, where $\tau(x,y)$ is a nonnegative real function. Then T has a fixed point in X, provided that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x,Tx)$ is T-orbitally lower semicontinuous. In addition, if $\tau(x,y) \leqslant (d(x,y))^{-1}$, then T has a unique fixed point.

Remark 5. Theorems 6–9 generalize the main results of [12, 23] and [32]. Theorems 10 and 11 generalize Theorem 1 of [9] and [14], respectively.

Theorem 12. *Let (X,d) be a metric space and $T : X \to X$ be a mapping. Assume that (X,d) is T-orbitally complete and there exists $L \geqslant 0$ such that following holds*:

$$d(Tx,Ty) \leqslant \frac{d(x,y)}{1 + d(x,y)} + Ld(y,Tx).$$

Then T has a fixed point in X, provided that $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(x) = d(x,Tx)$ is T-orbitally lower semicontinuous.

Theorem 13 [Caristi-type fixed point theorem]. *Let (X,d) be a metric space, $T : X \to X$ be a mapping, $\rho : X \times X \to [0,\infty) \cup \{\infty\}$ be a T-orbitally continuous and $\alpha : X \times X \to [0,\infty)$ satisfy $(\alpha 1)$. Assume that X is T-orbitally complete, there exists $\phi \in \varPhi$ and a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, $f(x) = d(x,Tx)$, which is bounded from below such that for $u_0 \in O(x_0)$ for some $x_0 \in X$ satisfies the following*:

(i) $\alpha(u_0,Tw)\rho(Tw,y) - \alpha(u_0,w)\rho(w,y) \leqslant \rho(w,Tw)$;
(ii) $\sum_{n=0}^{\infty} \rho(u,Tu) \leqslant f(u) - f(Tu)$;
(iii) $f(Tx) \leqslant \phi(f(x))$.

Then there exists $\bar{w} \in X$ such that $\bar{w} = T\bar{w}$.

Proof. From Corollary 5 we have that for each $\epsilon > 0$, there exists a sequence δ_j of positive real numbers and a sequence u_n such that $u_n \to u_\epsilon \in O(x)$ as $n \to \infty$, and for every $w \in X$, $w \neq u_\epsilon$, we have

$$\varDelta(w,\infty) > \varDelta(u_\epsilon,\infty)$$

or

$$f(w) + \alpha(u_0, w) \sum_{n=0}^{\infty} \delta_n \rho(w, u_n) > f(u_\epsilon) + \alpha(u_0, u_\epsilon) \sum_{n=0}^{\infty} \delta_n \rho(u_\epsilon, u_n). \tag{32}$$

We suppose that $\bar{w} \neq T\bar{w}$ for all $\bar{w} \in X$, then there exists Tu_ϵ such that $Tu_\epsilon \neq u_\epsilon$, so (32) implies

$$f(Tu_\epsilon) + \alpha(u_0, Tu_\epsilon) \sum_{n=0}^{\infty} \delta_n \rho(Tu_\epsilon, u_n) > f(u_\epsilon) + \alpha(u_0, u_\epsilon) \sum_{n=0}^{\infty} \delta_n \rho(u_\epsilon, u_n). \tag{33}$$

Relation (33) with hypothesis (i) and (ii) gives

$$\begin{aligned} f(u_\epsilon) - f\bigl(Tu_\epsilon\bigr) &< \sum_{n=0}^{\infty} \delta_n \bigl\{\alpha(u_0, Tu_\epsilon)\rho(Tu_\epsilon, u_n) - \alpha(u_0, u_\epsilon)\rho(u_\epsilon, u_n)\bigr\} \\ &\leqslant \sum_{n=0}^{\infty} \delta_n \bigl\{\rho(u_\epsilon, Tu_\epsilon)\bigr\} \leqslant f(u_\epsilon) - f(Tu_\epsilon). \end{aligned} \tag{34}$$

This leads to contradiction. Thus, there exists $\bar{w} \in X$ such that $\bar{w} = T\bar{w}$. □

5 Nonconvex minimax theorems and equilibrium problem

In many existing general topological minimax theorems, the convexity assumptions on the sets or on the functions are essential. Some applications of Ekeland's variational principles to minimax problems in Banach spaces are obtained by McLinden [26]. Ansari et al. [38] and Lin [25] studied minimax theorems for a family of multivalued mappings in locally convex topological vector spaces. For detail in this direction, see [4]. Recall that equilibrium problem is to find $x \in X$ such that $F(x, y) \geqslant 0$ for all $y \in X$, where X is a metric space and $F : X \times X \to \mathbb{R}$. The equilibrium problem is a unified model of optimization problems, saddle point problems, Nash equilibrium problems, variational inequality problems, nonlinear complementarity problems and fixed point problems. Blum [27], Oettli and Thera [41] first gave the existence of a solution of an equilibrium problem in the setting of complete metric spaces.

In this section, we obtain minimax theorems in incomplete metric spaces without assumption of convexity and also obtain the existence of a solution of equilibrium problem in incomplete metric spaces. The obtained results also present the importance of the T-orbitally lower semicontinuous mappings by showing that a function, which is T-orbitally lower semicontinuous but may not be necessarily lower semicontinuous, can still obtain its infimum and play an important role to solve minimization problems and equilibrium problems.

Theorem 14 [Takahashi's-type nonconvex minimization theorem]. *Let (X, d) be a metric space, T be a self mapping on X and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a T-orbitally lower semicontinuous function bounded from below. Assume that X is T-orbitally complete,*

(A) *hold and for any* $u \in O(x)$ *for* $x \in X$ *with* $f(u) > \inf_{z\in X} f(z)$*, there exists* $y \in X$ *such that* $\chi(y,\infty) \leqslant \chi(u,\infty)$*. Then there exists* $v \in O(x)$ *for* $x \in X$ *such that* $f(v) = \inf_{z\in X} f(z)$*.*

Proof. From Corollary 2 there exists $u_\epsilon \in O(x)$ for $x \in X$ such that $u_n \to u_\epsilon$ and for all $w \neq u_\epsilon$,

$$\chi(w,\infty) > \chi(u_\epsilon,\infty). \tag{35}$$

We claim that for $u_\epsilon \in O(x)$, $f(u_\epsilon) = \inf_{z\in X} f(z)$. If not, then $f(u_\epsilon) > \inf_{z\in X} f(z)$. By assumption, we get there exists $y \in X$ such that $\chi(y,\infty) \leqslant \chi(u_\epsilon,\infty)$, then from (35) we get

$$\chi(y,\infty) \leqslant \chi(u_\epsilon,\infty) < \chi(y,\infty),$$

which leads to contradiction. □

Theorem 15 [Nonconvex minimax theorem]. *Let* (X,d) *be a metric space,* T *be a self mapping on* X *and* $F : X\times X \to \mathbb{R}\cup\{+\infty\}$ *be a* T*-orbitally lower semicontinuous function and bounded from below in the first argument. Assume that* X *is* T*-orbitally complete,* (A) *hold and for any* $u \in O(x)$ *for* $x \in X$ *with* $\{b \in X\colon F(u,b) > \inf_{a\in X} F(a,b)\} \neq \emptyset$*, there exists* $w = w(u) \in X$ *with* $w \neq u$ *such that*

$$\xi(w,r,\infty) \leqslant \xi(u,r,\infty) \quad \textit{for all } r \in \Bigl\{b \in X\colon F(u,b) > \inf_{a\in X} F(a,b)\Bigr\}, \tag{36}$$

where

$$\xi(w,r,\infty) = F(w,r) + \sum_{i=0}^{\infty} \delta_i \rho_i(w,u_i).$$

Then

$$\inf_{u\in X} \sup_{v\in X} F(u(v),v) = \sup_{v\in X} \inf_{u\in X} F(u,v).$$

Proof. It follows from Theorem 14 that for all for $v \in X$, there exists $u(v) \in O(x)$ for $x \in X$ such that

$$F(u(v),v) = \inf_{u\in X} F(u,v).$$

Taking the supremum over v on both sides gives

$$\sup_{v\in X} F(u(v),v) = \sup_{v\in X} \inf_{u\in X} F(u,v).$$

Replacing $u(v)$ by an arbitrary $u \in X$ and taking infimum, we obtain

$$\inf_{u\in X} \sup_{v\in X} F(u(v),v) = \sup_{v\in X} \inf_{u\in X} F(u,v). \qquad \square$$

The following result is a nonconvex equilibrium theorem in T-orbitally complete metric spaces.

Theorem 16 [Nonconvex equilibrium theorem]. *Let (X, d) be a metric space, T be a self mapping on X and $F : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a T-orbitally lower semicontinuous function and bounded from below in the first argument. Assume that X is T-orbitally complete,* (A) *hold and for each $u \in O(x)$ for $x \in X$ with $\{b \in X\colon F(u, b) < 0\} \neq \emptyset$, there exists $w = w(u) \in X$ with $w \neq u$ such that* (36) *holds for all $r \in X$. Then there exists $v \in O(x)$ such that $F(v, y) \geqslant 0$ for all $y \in X$.*

Proof. By using Corollary 2, for each $z \in X$, there exists $u_\epsilon(z) \in O(x)$ for $x \in X$ such that $u_n \to u_\epsilon$ and for all $a \neq u_\epsilon(z)$,

$$\xi(a, r, \infty) > \xi\bigl(u_\epsilon(z), r, \infty\bigr). \tag{37}$$

We claim that there exists $v \in O(x)$ such that $F(v, y) \geqslant 0$ for all $y \in X$. If not, then for each $u \in O(x)$, there exists $y \in X$ such that $F(u, y) < 0$. This implies that for each $u \in O(x)$, the set $\{b \in X\colon F(u, b) < 0\}$ is nonempty. From our assumption there exists $w = w(u_\epsilon(z)) \in X$, $w \neq u_\epsilon(z)$, such that

$$\xi(w, r, \infty) \leqslant \xi\bigl(u_\epsilon(z), r, \infty\bigr). \tag{38}$$

Combining (37) and (38) gives

$$\xi(w, r, \infty) \leqslant \xi\bigl(u_\epsilon(z), r, \infty\bigr) < \xi(w, r, \infty),$$

which leads to contradiction. □

Example 3. Let (X, d) be a metric space as in Example 2 and $T : X \to X$ be a mapping defined as

$$T(g) = \begin{cases} \mathbf{I} & \text{if } g = \mathbf{I}' \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where $\mathbf{0}(t) = 0$, $\mathbf{I}(t) = 1$ and $\mathbf{I}'(t) = -1$ for all $t \in [0, 1]$ with $O(\mathbf{I}') = \{\mathbf{I}', \mathbf{I}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \dots\}$, and for $\mathbf{I}' \neq g \in X$, $O(g) = \{g, \mathbf{0}, \mathbf{0}, \mathbf{0}, \dots\}$. Then (X, d) is T-orbitally complete metric space. Define $F : X \times X \to \mathbb{R}$ by

$$F(g, h) = \begin{cases} 2\mathbf{I}(t) & \text{if } g = \mathbf{0},\ h \in X, \\ \mathbf{I}'(t) & \text{if } g \in \{\mathbf{I}, \mathbf{I}'\},\ h \in X, \\ |g(t) - h(t)| & \text{otherwise,} \end{cases}$$

then F is proper, bounded below and T-orbitally lower semicontinuous in first argument because for every sequence $\{g_n\}$ contained in $O(g)$, which converges to $h \in X$, we have

$$\liminf_n F(g_n, y) = F(h, y)$$

for all $y \in X$. Note that, only for $f \in \{\mathbf{I}, \mathbf{I}'\} \subset O(\mathbf{I}')$, we have $F(f, b) = \mathbf{I}'(t) = -1$ for all $b \in X$. So, $\{b \in X\colon F(f, b) < 0\} \neq \emptyset$ only when $f \in \{\mathbf{I}, \mathbf{I}'\}$. Also, define $\rho(g, h) =$

$|h(t) - g(t)|$ for all $g, h \in X, t \in [0,1]$, $\delta_0 = 1$ and $\delta_i = 0$ for all $i = 1, 2, 3, \dots$, then (36) is equivalent to

$$F(g,r) - F(h,r) \leqslant \bigl|g(t) - h(t)\bigr|.$$

For $g = \mathbf{I}'$, there exists $h = \mathbf{0} \in X$ such that $h = g + \mathbf{I}$ and for all $r \in X$,

$$F(g,r) - F(h,r) = F(\mathbf{I}', r) - F(\mathbf{0}, r) = 0 < 1 = \bigl|\mathbf{I}'(t) - \mathbf{0}(t)\bigr|.$$

For $g = \mathbf{I}$, there exists $h = \mathbf{I}' \in X$ such that $w = -u + \mathbf{0}$ and for all $r \in X$,

$$F(g,r) - F(h,r) = F(\mathbf{I}, r) - F(\mathbf{I}', r) = 0 = \bigl|\mathbf{I}(t) - \mathbf{I}'(t)\bigr|.$$

Hence (36) holds true. All the hypothesis of Theorem 16 are satisfied, and there exists $\mathbf{0} \in O(g)$ such that $F(\mathbf{0}, h) \geqslant 0$ for all $h \in X$.

Example 4. Let $X = (1,2] \cup \{-1, 0\}$ with usual metric d and $T : X \to X$ be a mapping defined as

$$Tx = \begin{cases} -1 & \text{if } x = 2, \\ 0 & \text{if } x \neq 2 \end{cases}$$

with $O(2) = \{2, -1, 0, 0, 0, \dots\}$, and when $x \neq 2$, $O(x) = \{x, 0, 0, 0, \dots\}$. Then (X, d) is T-orbitally complete metric space. Define $F : X \times X \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} 1 & \text{if } x = 2,\ y \in X, \\ -5x + 3y & \text{if } x \in (1, 1.5) \cup (1.5, 2),\ y \in \{0, -1\}, \\ 3y & \text{if } x, y \in \{0, -1\}, \\ 0 & \text{otherwise,} \end{cases}$$

then F is proper, bounded below and T-orbitally lower semicontinuous in first argument. Indeed, for every sequence contained in $O(x)$, which converges to $x \in X$, we have for all $y \in X$,

$$\liminf_n F(x_n, y) = F(x, y).$$

Note that there are three cases for which $u \in O(x)$ with $\{b \in X\colon\ F(u,b) < 0\} \neq \emptyset$:

Case 1. $u = -1$:

$$F(-1, b) = \begin{cases} 3b & \text{if } b \in \{0, -1\}, \\ 0 & \text{if } b \in (1, 2], \end{cases}$$

so $F(-1, b) < 0$ when $b = -1$.

Case 2. $u = 0$:

$$F(0, b) = \begin{cases} 3b & \text{if } b \in \{0, -1\}, \\ 0 & \text{if } b \in (1, 2], \end{cases}$$

so $F(0, b) < 0$ when $b = -1$.

Case 3. $u \in (1, 1.5) \cup (1.5, 2)$:

$$F(u, b) = \begin{cases} -5u + 3b & \text{if } b \in \{0, -1\}, \\ 0 & \text{if } b \in (1, 2], \end{cases}$$

so $F(u, b) < 0$ when $b \in \{0, -1\}$.

Also, define $\rho(x, y) = d(x, y)$ for all $x, y \in X$, $\delta_0 = 1$ and $\delta_i = 0$ for all $i = 1, 2, 3, \dots$. Then (36) is equivalent to

$$F(u, r) - F(w, r) \leqslant d(w, u).$$

For Case 1, there exists $w = 0 \in X$ such that $w = u + 1$ and for all r,

$$F(u, r) - F(w, r) = F(-1, r) - F(0, r) = 0 < 1 = d(-1, 0) = d(u, w).$$

For Case 2, there exists $w = -1 \in X$ such that $w = u - 1$ and for all r,

$$F(u, r) - F(w, r) = F(0, r) - F(-1, r) = 0 < 1 = d(0, -1) = d(u, w).$$

For Case 3, there exists $w \in X$ such that $w < u$ and for all r and $w \in (1, 1.5) \cup (1.5, 2)$,

$$F(u, r) - F(w, r) = (-5u + 3r) - (-5w + 3r) = 5(-u + w) \leqslant 0 \leqslant d(u, w).$$

When $w \in \{0, -1\}$,

$$F(u, r) - F(w, r) = (-5u + 3r) - (3r) \leqslant 0 \leqslant d(u, w).$$

Hence in all cases, (36) holds true. All the hypothesis of Theorem 16 are satisfied, and there exists $2 \in O(x)$ such that $F(2, y) \geqslant 0$ for all $y \in X$.

Remark 6. In Example 4, (X, d) is not a complete metric space because $x_n = 1 + 1/n$ is a Cauchy sequence in X that converges to 1 as $n \to \infty$, but $1 \notin X$. Also, the function F is not lower semicontinuous in the first argument. Indeed, $x_n = 1.5 + 1/n$ is a sequence in X, which converges to $1.5 \in X$, but for $y \in \{0, -1\}$,

$$\begin{aligned} \liminf_n F(x_n, y) &= \liminf_n F\left(1.5 + \frac{1}{n}, y\right) = \liminf_n \left(-7.5 - \frac{5}{n} + 3y\right) \\ &= -7.5 + 3y < 0 = F(1.5, y). \end{aligned}$$

Therefore, equilibrium formulations of Ekeland's variational principles given in [11, 29, 33, 41, 42, 44] cannot be applied for this example.

6 Conclusion

A variational principle is obtained in metric spaces, which are not necessarily complete, by introducing the notion of T-orbitally lower semicontinuous functions. We also obtain a nonconvex minimization theorem, a nonconvex minimax theorem and a nonconvex equilibrium theorem in such metric spaces. The existence of a solution of an equilibrium problem is also proved from obtained results. As consequences, the fixed points for many contractions in the existing literature are explored.

References

1. J. Ahmad, N. Hussain, New Suzuki–Berinde type fixed point results, *Carpathian J. Math.*, **33**(1):59–72, 2017.
2. Q.H. Ansari, Vectorial form of Ekeland-type variational principle with applications to vector equilibrium problems and fixed point theory, *J. Math. Anal. Appl.*, **334**(1):561–575, 2007.
3. Q.H. Ansari, Ekeland's variational principle and its extensions with applications, in S. Almezel, Q.H. Ansari, M.A. Khamsi (Eds.), *Topics in Fixed Point Theory*, Springer, Cham, 2014, pp. 65–100.
4. Q.H. Ansari, L.J. Lin, Ekeland-type variational principle and equilibrium problems, in S.K. Mishra (Ed.), *Topics in Nonconvex Optimization, Theory and Applications*, Springer Optim. Appl., Vol. 50, Springer, New York, 2011, pp. 147–174.
5. J.P. Aubin, *Optima and Equilibira: An intorduction to Nonlinear Analysis*, Grad. Texts Math., Vol. 140, Springer, Berlin, Heidelberg, 1993.
6. A. Azam, J. Ahmad, N. Hussain, On Suzuki-Wardowski type fixed point theorems, *J. Nonlinear Sci. Appl.*, **8**:1095–1111, 2015.
7. G.V.R. Babu, Generalization of fixed point theorems relating to the diameter of orbits by using a control function, *Tamkang J. Math.*, **35**(2):159–168, 2004.
8. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3**:133–181, 1922.
9. V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum*, **9**(1):43–53, 2004.
10. V. Berinde, *Iterative Approximation of Fixed Points,* 2nd. ed., Lect Notes Math., Vol. 1912, Springer, Berlin, Heidelberg, 2007.
11. M. Bianchi, S. Schaible, Equilibrium problems under generalized convexity and generalized monotonicity, *J. Glob. Optim.*, **30**(2–3):121–134, 2004.
12. S.K. Chatterjea, Fixed point theorems, *C. R. Acad. Bulg. Sci.*, **25**:727–730, 1972.
13. L.B. Ćirić, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, **45**(2): 267–273, 1974.
14. L.B. Ćirić, A certain class of maps and fixed point theorems, *Publ. Inst. Math., Nouv. Sér.*, **20**(34):73–77, 1976.
15. L.B. Ćirić, M. Abbas, R. Saadati, N. Hussain, Common fixed points of almost generalized contractive mappings in ordered metric spaces, *Appl. Math. Comput.*, **217**(12):5784–5789, 2011.
16. W.S. Du, L.J. Lin, On maximal element theorems, variants of Ekeland's variational principle and their applications, *Nonlinear Anal., Theory Methods Appl.*, **68**(5):1246–1262, 2008.
17. I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47**(2):324–353, 1974.
18. I. Ekeland, Nonconvex minimization problems, *Bull. Am. Math. Soc.*, **1**(3):443–474, 1979.
19. P.G. Georgiev, The strong Ekeland variational principle, the strong drop theorem and applications, *J. Math. Anal. Appl.*, **131**(1):1–21, 1988.
20. P.G. Georgiev, Parametric Ekeland's variational principle, *Appl. Math. Lett.*, **14**(6):691–696, 2001.

21. N. Hussain, A. Latif, I. Iqbal, Fixed point results for generalized F-contractions in modular metric and fuzzy metric spaces, *Fixed Point Theory Appl.*, **2015**(158), 2015.
22. O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japon.*, **44**:381–391, 1996.
23. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60**:71–76, 1968.
24. M.A. Kutbi, N. Hussain, A. Latif, Applications of Caristi's fixed point results, *J. Inequal. Appl.*, **2012**(40), 2012.
25. L.J. Lin, System of coincidence theorems with applications, *J. Math. Anal. Appl.*, **285**(2):408–418, 2003.
26. L. McLinden, An application of Ekeland's theorem to minimax problems, *Nonlinear Anal., Theory Methods Appl.*, **6**(2):189–196, 1982.
27. W. Oettli, E. Blum, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63**(1–4):123–145, 1994.
28. M. Oveisiha, R. Mirzaei, A. Razani, E. Khakrah, Some metric characterizations of well-posedness for hemivariational-like inequalities, *J. Nonlinear Funct. Anal.*, **2017**:113–151, 2017.
29. S. Park, On generalizations of the Ekeland-type variational principles, *Nonlinear Anal., Theory Methods Appl.*, **39**(7):881–889, 2000.
30. D.N. Quy, P.Q. Khanh, A generalized distance and enhanced Ekeland's variational principle for vector functions, *Nonlinear Anal., Theory Methods Appl.*, **73**(7):2245–2259, 2010.
31. E. Rakotch, A note on contractive mappings, *Proc. Am. Math. Soc.*, **13**:459–465, 1962.
32. S. Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14**(1):121–124, 1971.
33. H. Riahi, Z. Chbani, O. Chadli, Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities, *J. Optim. Theory Appl.*, **105**(2):299–323, 2000.
34. T.D. Rogers, G.E. Hardy, A generalization of a fixed point theorem of Reich, *Can. Math. Bull.*, **16**(2):201–206, 1973.
35. P. Salimi, M. Hezarjaribi, N. Hussain, Suzuki type theorems in triangular and non-Archimedean fuzzy metric spaces with application, *Fixed Point Theory Appl.*, **2015**(134), 2015.
36. P. Salimi, N. Hussain, Suzuki–Wardowski type fixed point theorems for α-GF-contractions, *Taiwanese J. Math.*, **18**(6):1879–1895, 2014.
37. S. Shuzhong, L. Yongxin, A generalization of Ekeland's ϵ-variational principle and its Borwein–Preiss smooth variant, *J. Math. Anal. Appl.*, **246**(1):308–319, 2000.
38. L.B. Su, L.J. Lin, Q.H. Ansari, Systems of simultaneous generalized vector quasi-equilibrium problems and their applications, *J. Optim. Theory Appl.*, **127**(1):27–44, 2005.
39. T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Am. Math. Soc.*, **136**(5):1861–1869, 2008.
40. C. Tammer, B. Soleimani, A vector-valued Ekeland's variational principle in vector optimization with variable ordering structures, *J. Nonlinear Var. Anal.*, **2017**(1):89–110, 2017.
41. M. Théra, W. Oettli, Equivalents of Ekeland's principle, *Bull. Aust. Math. Soc.*, **48**(3):385–392, 1993.

42. L.-J. Lin W.-S. Du, Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces, *J. Math. Anal. Appl.*, **323**(1):360–370, 2006.
43. J.S.W. Wong, D.W. Boyd, On nonlinear contractions, *Proc. Am. Math. Soc.*, **20**:458–464, 1969.
44. J.C. Yao, Q.H. Ansari, S. Al-Homidana, Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory, *Nonlinear Anal., Theory Methods Appl.*, **69**(1):126–139, 2008.
45. A.J. Zaslavski, Structure of approximate solutions of autonomous variational problems, *Appl. Anal. Optim.*, **1**(1):113–151, 2017.