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On the standard Brownian motion. I

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1. Introduction

We consider the standard Brownian motion $\{B_s,\ 0 \leqslant s \leqslant 1\}$ in the *probability space* $(\Omega, \mathfrak{F}, \mathbf{P})$. Let $B_0 = 0$, $\mathbf{E}B_s = 0$ and $\mathbf{E}B_s^2 = s$.

The standard Brownian motion has been investigated by many authors and a lot of expressions and different results have been obtained, which describe that process. Some results are presented without any demonstration or they are published in editions that are difficult to find. A thorough analysis displays that some presented results are erroneous, e.g. the formula (5.10) in W. Feller's book (Chapter X) [5] or formula (12) in R. Douady, M. Yor, A.N. Shiryaev [3].

We denote a random variable $B = \sup\limits_{0 \leqslant s \leqslant 1} |B_s|$ and define the probability distribution functions of the maximum of Brownian motion:

$$F(x) = \mathbf{P}\{\mathbf{B} < \mathbf{x}\} = \mathbf{P}\{\sup_{0 \leqslant s \leqslant 1} |\mathbf{B_s}| < \mathbf{x}\}$$

and

$$F(X|y,t) = \mathbf{P}\{\mathbf{B_s} + \mathbf{y} \in \mathbf{X},\ \mathbf{0} \leqslant \mathbf{s} \leqslant \mathbf{t}\},$$

where $X = (-x, x)$ is an open interval. Note that in this case $F(x) = F(X|0,1)$.

This probability distribution function $F(x)$ has a complicated expression and different authors obtained several forms of $F(x)$ in their works [2, 3, 4, 5, 6, 7, 9, 10, 11].

My purpose is to present a survey of different expressions and results, to make certain that they are truthful. Another aim is to present some new results.

2. Review of the function $F(x)$ forms

In their paper (now classical) P. Erdös and M. Kac [4] used such a probability distribution function

$$F(x) = \frac{4}{\pi} \sum_{0 \leqslant k \leqslant \infty} \frac{(-1)^k}{1+2k} \exp\left\{-\frac{(1+2k)^2\pi^2}{8x^2}\right\}. \tag{1}$$

A. Rosenkrantz [9], F. Spitzer [11] and W. Feller [5] also used this formula (1), too. To complete the presentation we give a derivation of formula (1).

For simplicity, to prove formula (1) we shall need the following lemma.

Lemma 2. 1. *The probability measure*

$$F(X|y,t) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left(-\frac{(2k+1)^2\pi^2 t}{8x^2}\right)\cos\left(\frac{(2k+1)\pi y}{2x}\right) \tag{2}$$

for every open interval $X = (-x, x)$ is the solution of the classical diffusion equation (see, e.g., [5, 8])

$$\frac{\partial F}{\partial t} = \frac{1}{2}\frac{\partial^2 F}{\partial y^2}, \tag{3}$$

which satisfies the initial condition $F(X|y,0) = 1$ and boundary conditions

$$F(X|-x,t) = F(X|x,t) = 0.$$

Proof. We find the solution to this differential diffusion equation (3) by the standard Fourier method. We try to find a solution of the form

$$F(X|y,t) = Y(y)T(t).$$

We can separate this equation so that all the t's are on the one side and the y's on the other:

$$2\frac{T'(t)}{T(t)} = \frac{Y''(y)}{Y(y)} = -\lambda^2.$$

Evidently, the general formal solution of partial differential equation (3) (see e.g., [1]) is

$$F(X|y,t) = \sum_{k=0}^{\infty} c_k \exp\left(-\frac{\lambda_k^2 t}{2}\right)\cos(\lambda_k y).$$

Here

$$\lambda_k = \frac{(2k+1)\pi}{2x}$$

and

$$c_k = \frac{1}{x}\int_{-x}^{x}\cos(\lambda_k y)dy = \frac{4(-1)^k}{\pi(2k+1)}.$$

Finally, we have got the solution (2) of equation (3).

Thus, formula (1) follows from this Lemma 1.1, when $y = 0$ and $t = 1$.

Note that W. Feller [5] used the probability measure

$$F(X|y,t) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left(-\frac{(2k+1)^2\pi^2 t}{2x^2}\right)\sin\left(\frac{(2k+1)\pi y}{x}\right). \tag{4}$$

when y lies in the open interval $X = (0, x)$. In this way, when $X = (0, 2x), y = x$ and $t = 1$ formula (1) also follow from equality (4).

We enumerate these expressions of the probability distribution function $F(x)$ for all $x > 0$, obtained by different authors:

$$F(x) \equiv$$

$$\equiv 2 \sum_{k=-\infty}^{\infty} \{\Phi((4k+1)x) - \Phi((4k-1)x)\} - 1 \tag{5}$$

$$\equiv \sum_{k=-\infty}^{\infty} \{2\,\Phi(4kx + x) - \Phi(4kx - x) - \Phi(4kx + 3x)\} \tag{6}$$

$$\equiv \frac{1}{\sqrt{2\pi}} \sum_{-\infty \leqslant k \leqslant \infty} \int_{-x}^{x} \left\{ \exp\left(-\frac{(y-4kx)^2}{2}\right) - \exp\left(-\frac{(y-2x+4kx)^2}{2}\right) \right\} dy \tag{7}$$

$$\equiv \frac{1}{\sqrt{2\pi}} \sum_{-\infty \leqslant k \leqslant \infty} \int_{-x}^{x} \left\{ \exp\left(-\frac{(y+4kx)^2}{2}\right) - \exp\left(-\frac{(y+2x+4kx)^2}{2}\right) \right\} dy \tag{8}$$

$$\equiv \frac{1}{\sqrt{2\pi}} \sum_{-\infty \leqslant k \leqslant \infty} \int_{-x}^{x} \left\{ \exp\left(-\frac{(y+4kx)^2}{2}\right) - \exp\left(-\frac{(y-2x+4kx)^2}{2}\right) \right\} dy \tag{9}$$

$$\equiv \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{(u-2kx)^2}{2}\right) du \tag{10}$$

$$\equiv 1 - \frac{4}{\sqrt{2\pi}} \sum_{0 \leqslant k \leqslant \infty} \int_{(4k+1)x}^{(4k+3)x} \exp\left(-\frac{u^2}{2}\right) du \tag{11}$$

$$\equiv \frac{1}{\sqrt{2\pi}} \sum_{h=-\infty}^{\infty} (-1)^h \int_{-x+2xh}^{x+2xh} \exp\left(-\frac{u^2}{2}\right) du, \tag{12}$$

where $\Phi(x)$ is a *standard normal* distribution function. We denote these different formulas by $F_1(x), F_5(x), F_6(x) \ldots$ and etc.

W. Feller [5] (see Chapter X, formulas (5.8) and (5.11)) derived expressions (1), (5) and (6), P. Levy [7] – (7), R. Douady, M. Yor, A.N. Shiryaev [3] – (8),I.I. Gihman and A.V. Skorokhod [6, 10] – (9) and (10), from a more general result in A.V. Skorokhod's book ([10], p.p. 180–186) – (11), and L. Beghin, E. Orsingher [2] derived the usual distribution (12), which is similar to expression (10).

We can see that the same function $F(x)$ has many absolutely different expressions (1) and (5)–(12). Evidently, we can reform the expression (5) to the formula:

$$F_5(x) \equiv \frac{4}{\sqrt{2\pi}} \left[\int_0^x \exp\left(-\frac{u^2}{2}\right) + \sum_{1 \leqslant k \leqslant \infty} \int_{(4k-1)x}^{(4k+1)x} \exp\left(-\frac{u^2}{2}\right) du \right] - 1.$$

Hence it follows that for all $x > 0$

$$F_{11}(x) = 1 - \frac{4}{\sqrt{2\pi}} \sum_{0 \leqslant k \leqslant \infty} \int_{(4k+1)x}^{(4k+3)x} \exp\left(-\frac{u^2}{2}\right) du \equiv F_5(x).$$

Also by I.I. Gihman's and A.V. Skorokhod's [6, 10] formula (10) we have identities

$$\begin{aligned} F_{10}(x) &\equiv \frac{2}{\sqrt{2\pi}} \left[\int_0^x \exp\left(-\frac{u^2}{2}\right) + \sum_{1 \leqslant k \leqslant \infty} (-1)^k \int_{(2k-1)x}^{(2k+1)x} \exp\left(-\frac{u^2}{2}\right) du \right] \\ &\equiv \frac{F_5(x) + F_{11}(x)}{2} \equiv F_5(x) \equiv F_{11}(x). \end{aligned}$$

So, it is easy to prove that all the expressions (5)–(12) are equivalent. It remains to prove that (1) is equivalent to (5) for all $x > 0$. W. Feller ([5], Chapter XIX, paragraph 5) has proved that. He used the well-known Poisson summation formula:

$$\sum_{k=-\infty}^{\infty} f((1+2k)\lambda) \cos((1+2k)\lambda s)) = \frac{\pi}{\lambda} \sum_{n=-\infty}^{\infty} (-1)^n \varphi\left(\frac{\pi n}{\lambda} + s\right),$$

where f is a characteristic function and φ is probability density.

In a separate case, where $\varphi(x) = \Phi'(x)$, there is a mistake in formula (5.10) ([5], Chapter XIX). We rewrite this formula for the normal density $\varphi((x)$, where this mistake is corrected:

$$\sum_{k=-\infty}^{\infty} \exp\left(-\frac{(1+2k)^2\lambda^2}{2}\right) \cos((1+2k)\lambda s) = \frac{\pi}{\lambda} \sum_{n=-\infty}^{\infty} (-1)^n \varphi\left(\frac{\pi n}{\lambda} + s\right).$$

Furthermore, according to W. Feller [5], we can see that expressions (1) and (5) are equivalent for all $x > 0$.

Corollary 2. 2. *This expression also is equivalent to $F(x)$ for $x > 0$*

$$F(x) \equiv 1 - 2\sum_{k=0}^{\infty} (-1)^k \operatorname{erf c}\left(\frac{(2k+1)x}{\sqrt{2}}\right). \tag{13}$$

The following corollary evidently follows from the expressions (10) or (12), where function erf c(x) = 1 − erf (x) is a *complementary error* function.

3. A generalization of the function $F(x)$

P. Levy ([7], (1948)) and I.I. Gihman and A.V. Skorokhod ([6],(1965)) have obtained general formulas for the joint distribution of minimum and maximum of Brownian motion.

If $x_1 < 0 < x_2$ and $d = x_2 - x_1$, then they derived such equalities:

$$\mathbf{P}(\min_{0\leqslant s\leqslant t} \mathbf{B}_s > \mathbf{x}_1, \max_{0\leqslant s\leqslant t} \mathbf{B}_s < \mathbf{x}_2)$$

$$= \frac{1}{\sqrt{2\pi t}} \sum_{-\infty\leqslant k\leqslant\infty} \int_{x_1}^{x_2} \left\{ \exp\left(-\frac{(y-2kd)^2}{2t}\right) - \exp\left(-\frac{(y-2x_1+2kd)^2}{2t}\right) \right\} dy \quad (14)$$

$$\equiv \frac{1}{\sqrt{2\pi t}} \sum_{-\infty\leqslant k\leqslant\infty} \int_{x_1}^{x_2} \left\{ \exp\left(-\frac{(y+2kd)^2}{2t}\right) - \exp\left(-\frac{(y-2x_2+2kd)^2}{2t}\right) \right\} dy. \quad (15)$$

Expression (14) was obtained by P. Levy and (15) – by I.I. Gihman and A.V. Skorokhod. We derive an equality equivalent to (14) and (15), but similar to expression (1).

Theorem 3. 1. *If $x_1 < 0 < x_2$ then we have the equalities:*

$$\mathbf{P}\ (\min_{0\leqslant s\leqslant t} B_s > x_1, \max_{0\leqslant s\leqslant t} B_s < x_2)$$

$$= \frac{4}{\pi} \sum_{0\leqslant k\leqslant\infty} \frac{(-1)^k}{1+2k} \exp\left(-\frac{(1+2k)^2\pi^2 t}{2(x_2-x_1)^2}\right) \cos\left(\frac{(2k+1)\pi(x_1+x_2)}{2(x_2-x_1)}\right) \quad (16)$$

$$\equiv \frac{4}{\pi} \sum_{0\leqslant k\leqslant\infty} \frac{1}{1+2k} \exp\left(-\frac{(1+2k)^2\pi^2 t}{2(x_2-x_1)^2}\right) \sin\left(\frac{(2k+1)\pi x_1}{x_1-x_2}\right). \quad (17)$$

Proof. The expression (16) evidently follows from the Lemma 2.1 and (17) – from W. Feller's formula (4). These expressions (14) and (15) are also convenient to use for $t < 1$, while expressions (16) and (17) then $t \geqslant 1$. It is easy to see that the formula (1) follows from these formulas (16) or (17), when $x_1 = -x$, $x_2 = x$ and $t = 1$.

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Apie standartinį Brauno judesį. I

S. Steišūnas

Tegul turime standartinį Brauno judesį $\{B_s,\ 0 \leqslant s \leqslant 1\}$ tikimybinėje erdvėje $(\Omega, \mathfrak{F}, \mathbf{P})$. Tegul $B_0 = 0$, $\mathbf{E}B_s = 0$ ir $\mathbf{E}B_s^2 = s$. Nagrinėjame atsitiktinį dydį $B = \sup\limits_{0\leqslant s\leqslant 1} |B_s|$ ir jo pasiskirstymo funkciją

$$F(x) = \mathbf{P}\{\sup_{0\leqslant s\leqslant 1} |\mathbf{B_s}| < \mathbf{x}\}.$$

Darbe išnagrinėtos tos pasiskirstymo funkcijos $F(x)$ skirtingos formulės išraiškos ir gautos naujos formulės tikimybiniam matui:

$$\begin{aligned}
&\mathbf{P}\ (\min_{0\leqslant s\leqslant t} B_s > x_1, \max_{0\leqslant s\leqslant t} B_s < x_2)\\
&= \frac{4}{\pi}\sum_{0\leqslant k\leqslant\infty} \frac{(-1)^k}{1+2k} \exp\left(-\frac{(1+2k)^2\pi^2 t}{2(x_2-x_1)^2}\right)\cos\left(\frac{(2k+1)\pi(x_1+x_2)}{2(x_2-x_1)}\right)\\
&\equiv \frac{4}{\pi}\sum_{0\leqslant k\leqslant\infty} \frac{1}{1+2k} \exp\left(-\frac{(1+2k)^2\pi^2 t}{2(x_2-x_1)^2}\right)\sin\left(\frac{(2k+1)\pi x_1}{x_1-x_2}\right)
\end{aligned}$$

visiems x_1 ir x_2, kai $x_1 < 0 < x_2$.