LMD mokslo darbai, III tomas, 317–321

Parallel deduction-search algorithm for the predicate logic formulas

Stasys NORGĖLA (VU)
e-mail: stasys.norgela@maf.vu.lt

1. Introduction

Considering the deduction of the first-order predicate logic formula $\neg F$ on the basis of inverse method. Formula F is in the prenex normal form and its matrix (quantifier-free formula) is in the disjunctive normal form. We form the set *M(i)* of such favorable unit disjunctions, that they are derivable at the moment i. We consider in parallel a derivation of F on the basis of the resolution method with the following resolution rule

$$\frac{\sigma C_1, \sigma C_2 : M(i)}{\sigma C}$$

M(i) plays the role of the set of the justifications formulas at the moment i. The application of the rule has been based on the restrictions on the choice of substitution. The rule is applicable if such substitution γ and formula $\rho C_1 \in M(i)$ that $(\gamma \circ \rho)C_1$ absorb σC do not exist at the moment.

The realization of some systems of the inverse method have shown that, if the derivation of an empty disjunct is obtained after the step n and m unit favorable disjuncts are obtained during the process, then the majority of unit favorable disjuncts are obtained not more than after steps $n/4$ for almost all formulas. So, most of the unit favorable disjuncts are obtained rather quickly. They are used in order to show which clauses are useless.

The deduction in sequential calculus with the following rules is simulated by the deduction based on the inverse method:

$$(\vdash \&) \qquad \frac{\vdash \Gamma, F_1, \Delta \quad \vdash \Gamma, F_2, \Delta \ldots \vdash \Gamma, F_n, \Delta}{\vdash \Gamma, \&_{i=1}^{n} F_i, \Delta}$$

$$(\vdash \vee) \qquad \frac{\vdash \Gamma, F_1, F_2, \ldots, F_n, \Delta}{\vdash \Gamma, \vee_{i=1}^{n} F_i, \Delta}$$

$$(\vdash \exists) \qquad \frac{\vdash \Gamma, \sigma F, \exists x_1 \ldots \exists x_n, \Delta}{\vdash \Gamma, \exists x_1 \ldots \exists x_n, \Delta}$$

σ is a substitution satisfying usually restrictions.

The inverse method is presented in many divers papers (see, for example [1]).

We can easy construct the deduction based on the inverse method of the empty disjunct □ in accordance with the deduction tree of the sequent $\vdash F$ in considerable sequential calculus and on the contrary (see [1]). We denote the unit favorable disjuncts $[i, B]$ by σC_i, where σ denotes the substitution corresponding to B.

2. Basic result

Theorem 1. *If $\sigma_j D_k$ is derivable by inverse method then it belongs to the set of superfluous clauses M.*

Proof. Let $\vdash \sigma C$ is derivable by inverse method. It means that if σC is in some branch of the proof-search tree, then we are able to extend this branch of the tree started from σC to the axioms. We consider only the saturated formulas of the form $F = D_1 \& D_2 \& \ldots \& D_s$ where D_j are disjuncts, i.e., such formulas as any atomic formula F or its negation occur in each disjunct. More precisely we consider the matrix of F which does not contain function symbols before Skolemization. Then $\neg F = F' = D' \& D'' \& \ldots \& D^p$. This is all of the remaining disjuncts, which we can obtain from atoms occurred in F, i.e., the formula $F \& F'$ contains all possible disjuncts formed from the initial formula. We construct the following formula according to the deduction tree in sequent calculus.

$$(\sigma_1 D_1 \& \sigma_1 D_2 \& \ldots \& \sigma_1 D_s) \vee \ldots \vee (\sigma_j D_1 \& \sigma_j D_2 \& \ldots \& \sigma_j D_s) \\ \vee \ldots\ldots \vee (\sigma_l D_1 \& \sigma_l D_2 \& \ldots \& \sigma_l D_s) \quad (1)$$

Each application of the rule $(\vdash \exists)$ is met in the tree correspond to any formula

$$(\sigma_u D_1 \& \sigma_u D_2 \& \ldots \& \sigma_u D_s)$$

Assume $\vdash \sigma_j D_k$ is derivable by inverse method. Then (1) is a valid formula if and only if

$$(\sigma_1 D_1 \& \sigma_1 D_2 \& \ldots \& \sigma_1 D_s) \vee \ldots \vee (\sigma_j D_1 \& \ldots \& \sigma_j D_{k-1} \& \sigma_j D_{k+1} \& \ldots \& \sigma_j D_s) \\ \vee \ldots \vee (\sigma_l D_1 \& \sigma_l D_2 \& \ldots \& \sigma_l D_s) \quad (2)$$

is valid.

It means the negation of (2)

$$\neg(\sigma_1 D_1 \& \sigma_1 D_2 \& \ldots \& \sigma_1 D_s) \& \ldots \& \neg(\sigma_j D_1 \& \ldots$$

$$\ldots \& \sigma_j D_{k-1} \& \sigma_j D_{k+1} \& \ldots \& \sigma_j D_s) \& \ldots$$

$$\ldots \& \neg(\sigma_l D_1 \& \sigma_l D_2 \& \ldots \& \sigma_l D_s)$$

is equivalent with

$$\sigma_1 F' \& \sigma_2 F' \& \ldots \& (\sigma_j F' \& \sigma_j D_k) \& \ldots \& \sigma_l F' \quad (3)$$

The previous formula is equivalent and with negation of (1), i.e.,

$$\sigma_1 F' \& \sigma_2 F' \& \ldots \& \sigma_j F' \& \ldots \& \sigma_l F' \quad (4)$$

We have obtained that (3) is a contradictory formula if and only if (4) is a contradictory formula.

Definition. *The favorable disjunct* $[i_1, i_2, \ldots, i_u, j_1, j_2, \ldots, j_l; B]$ *is decomposable into* $[i_1, i_2, \ldots, i_u; B']$ *and* $[j_1, j_2, \ldots, j_l; B'']$ *if the variables of disjuncts* $i_1, i_2, \ldots, i_u$ *have no occurences in any set of basis B with any variable of disjuncts* $j_1, j_2, \ldots, j_l$, *except the case when any constant belong to considerable set.*

Detailed description of the formation of basis B' and B'' is in [2].

Theorem 2. *If we obtain any disjunct* $[i_1, i_2, \ldots, i_u; B]$ *decomposable into unit favorable disjuncts in the proof search of* □ *by the inverse method from the set of initial disjuncts correponding to* $\neg F$ *and we obtain the clauses* $\sigma_l D_j (l = 1, \ldots, u; j = i_1, i_2, ..., i_u$; *the substitutions* σ_l *are correspondence of the unit favorable disjuncts) in the proof search of* □ *by resolution method from the set of initial clauses corresponding to F, then* □ *is derivable by resolution method not using any clause from* $\sigma_l D_j$.

Proof. We can use only such replacements of variables that transpose of rules, that become permissible (after rename of certain variables) in the deduction tree of the formula $\neg F$ in sequential calculus. It means we can obtain that the unit considerable disjunct $\sigma_l D_j$ becomes the final node in the branch. Because we are able to prolong considerable node until axioms, then we can delete the formula corresponding to final node in (2), i.e., $\sigma_l D_j$ belong to the set *M* of superfluous disjuncts. Theorem is proved.

We can enumerate certain decidable classes without functional symbols. Favorable disjuncts derivable from formulas of following classes decompose into unit favorable disjuncts.

1. Monadic class (see [2]).
2. Gődel class. This is the formulas in prenex normal form with prefix $\exists x_1 \exists x_2 \forall y_1 \forall y_2 \ldots \forall y_l$ (see[2]).
3. The formulas in prenex normal form with matrix $P_1 \vee P_2 \vee \ldots \vee P_n \vee M$, where P_j are literals containing no much as one negative variable and *M* is formula containing only positive occurrences of variables (see [3]).

Remember that favorable disjuncts decompose into unit of not all decidable classes.

3. The case of monadic class

Consider the closed formulas with monadic predicate variables in prenex normal form. Matrix is in conjunctive normal form, i.e., we consider the formulas of the form $Q_1x_1Q_2x_2\ldots Q_nx_nG$, where the $Q_i(i = 1, \ldots, n)$ are quantifiers, G is quantifier-free formula in conjunctive normal form.

We will show that there exists equivalent formula of the form $\exists x_1\ldots\exists x_k\forall y_1\ldots\forall y_lH$ (H is quantifier-free formula) for any formula F. Search of such formula require much time. I would like to accentuate that one can consider only such formulas and this is a general form of the case of formulas with monadic predicate variables.

Theorem 3. *Let F be a formula of monadic class. Then there exists equivalent formula of the form* $\exists x_1\ldots\exists x_k\forall y_1\ldots\forall y_lG$ *where G is in conjunctive normal form.*

Proof. Consider the prenex normal form of F $Q_1x_1Q_2x_2\ldots Q_mx_mH$, where the H is in conjunctive normal form, and $Q_i = \forall$ or $Q_i = \exists(i = 1, \ldots, m)$. Consider two cases, when $Q_m = \forall$ and when $Q_m = \exists$.

- Let $Q_m = \forall$. Bring $\forall x_m$ in the brackets using equivalence $\forall y_m(P\&R) \equiv \forall xP\&\forall xR$, i.e., $\forall x_m$ is only immediate before elementary disjunction. Then if formula does not contain x_m we use the equivalence $\forall x_m(P \vee R) \equiv \forall x_mP \vee R$.
- Let $Q_m = \exists$. Transform H into disjunctive normal form. Bring $\exists x_m$ in the brackets using equivalence $\exists x_m(P \vee R) \equiv \exists x_mP \vee \exists x_mR$, i.e., $\exists x_m$ is only immediate before elemetary conjunction. Then if formula R does not contain x_m we use equivalence $\exists x_m(P\&R) \equiv \exists x_mP\&R$.

We apply m time described method and a disjunction of the formulas of the following form is obtained in this way (may be still after the transformation into disjunctive normal form)

$$R_1^j\&R_2^j\&\ldots R_r^j$$

Where $R_l^j(l = 1, \ldots, r)$ is a formula of the form $\forall y(P_1^u(y) \vee P_2^u(y) \vee \ldots \vee P_r^u(y))$ (where $P_i^u(y)$ are literals) or $\exists x(S_1^j(x)\&S_2^j(x)\&\ldots\&S_l^j(x))$ (where $S_l^j(x)$ are literals). More forward quantifier $\exists$ (the number of quantifiers may be increase with respect to initial formula). We obtain the formula $\exists x_1\ldots\exists x_kL$. Transform L into conjunctive normal form and then bring all quantifiers $\forall$ in front of the formula (the number of quantifiers may be increase again). Because we transform using equivalent formula then the obtained formula $\exists x_1\ldots\exists x_k\forall y_1\ldots\forall y_lG$ is equivalent to initial F. Theorem is proved.

We want to verify a formula F of the form (5) is an unsatisfiable formula. For this reason we search deduction by resolution method of empty clause using the default rule from the set of clauses corresponding to F. We search at the same time parallel deduction by inverse method of empty clause from the set of clauses correponding to $\neg F$ (it means we try to verify the considerable formula is valid) and we form the set of justifications. G is in conjunctive normal form, i.e., of the form $D_1\&D_2\&\ldots\&D_s$ (where D_i are clauses). We obtain $\sigma\forall y_1\ldots\forall y_l(D_1\&D_2\&\ldots\&D_s$ after the Skolemization of (5). σ is the substitution $(a_1/x_1, a_2/x_2, ..., a_m/x_m)$, i.e., we will have a set of clauses $S = \{D'_1, D'_2, ..., D'_s\}$ (where $D'_i = \sigma D_i$). We will search a deduction of the formula $\neg F$, i.e., of the $\forall x_1...\forall x_k\exists y_1...\exists y_l\neg G$. It means we will search a deduction of the formula $\sigma\exists y_1...\exists y_l\neg G$ with the same substitution $\sigma = (a_1/x_1, a_2/x_2, ..., a_m/x_m)$.

References

[1] S.Yu. Maslov, The inverse method for establiching deducibility in the classical predicate calculus, *Soviet Mathematics Doklady*, **5**, 1420–1424 (1964).

[2] S.Yu. Maslov, The inverse method of establiching deducibility for logical calculi, *Proceedings of the Steklov Institute of Mathematics*, **98**, 25–96 (1971).

[3] S. Norgėla, On approximation of some classes of the classical predicate calculus by decidable classes II, *Lithuanian Journal of Mathematics*, **20**(4), 89–96 (1980).

Lygiagretus įrodymo paieškos algoritmas predikatų logikos formulėms

S. Norgėla

Pirmos eilės formulės išvedimas ieškomas rezoliucijų metodu, o jos neigimas lygiagrečiai atvirkštiniu metodu. Nagrinėjamas ryšys tarp gaunamų disjunktų.