LMD mokslo darbai, III tomas, 136–142

The existence and uniqueness of the solution of the integral equation driven by a bounded p-variation function

Kęstutis KUBILIUS (MII, VGTU)
e-mail: kubilius@ktl.mii.lt

1. Introduction

In this note we examine the nonlinear integral equation

$$x_t = \alpha + \int_{t_0}^{t} f(s, x_s)\,\mathrm{d}s + \int_{t_0}^{t} g(s, x_s)\,\mathrm{d}h_s, \qquad t_0 \leqslant t \leqslant T, \tag{1}$$

where h is a continuous function of bounded p-variation for some p, $1 < p < 2$, i.e. $h \in CW_p([t_0, T])$. The first integral in (1) is the Riemann integral and the second is the Riemann-Stieltjes (RS) integral which exist for functions f and g considered below.

Lyons [3] considered the integral equation

$$y_t = a + \int_0^t \sum_{i=1}^{d} f^i(y_s)\mathrm{d}h_s^i, \tag{2}$$

where h^i, $i = 1, \ldots, d$, are continuous functions which have bounded p-variation for some p, $1 \leqslant p < 2$. He proved that the equation (2) can be solved by Picard iteration whenever f^i has a derivative $(f^i)'$ satisfying a global Lipschitz condition of order α, where $p < 1 + \alpha \leqslant 2$. This solution is unique in the space of continuous functions of bounded p-variation. We extend this result to the equation (1) in the case $d = 1$.

Denote by $C^{0,1}([t_0, T] \times \mathbf{R})$ the space of all continuous functions $u(t, x)$ on $[t_0, T] \times \mathbf{R}$ such that the norm

$$\|u\|_T = \sup_{s,x} |u(s, x)| + \sup_{s,x} |\partial_x u(s, x)|.$$

is finite.

Let $H^{\ell/2,\ell}([t_0, T] \times \mathbf{R})$, $\ell = 1 + \alpha$, $0 < \alpha < 1$, be the space of all continuous functions u on $[t_0, T] \times \mathbf{R}$ possessing continuous partial derivative $\partial_x u$ such that the norm

$$\|u\|_T^{(\ell)} = \|u\|_T + \sup_{\substack{(s,x),(s,y)\\ x \neq y}} \frac{|\partial_x u(s, x) - \partial_x u(s, y)|}{|x - y|^\alpha}$$

$$+ \sup_{\substack{(s,x),(t,x)\\ s\neq t}} \frac{|u(s,x)-u(t,x)|}{|s-t|^{(1+\alpha)/2}} + \sup_{\substack{(s,x),(t,x)\\ s\neq t}} \frac{|\partial_x u(s,x)-\partial_x u(t,x)|}{|s-t|^{\alpha/2}}$$

is finite.

Our main result is the following:

Theorem. *Let $T > 0$ and let $h \in CW_p([t_0,T])$, $1 < p < 2$, $f \in C^{0,1}([t_0,T]\times \mathbf{R})$, $g \in H^{\ell/2,\ell}([t_0,T]\times \mathbf{R})$, $\ell = 1+\alpha$, $2(1-1/p) < \alpha < 1$. Then the equation (1) has a unique solution in $C\mathcal{W}_p([t_0,T])$.*

Note. If the functions f and g don't depend on the time variable t then the condition $\alpha > 2(1-1/p)$ can replaced by the condition $\alpha + 1 > p$ which yields Lyons' result in the case $d = 1$.

Further we list few facts concerning the p-variation. The proofs can be found in [1] and [4].

Let f be a real-valued function defined on a closed interval $[a,b]$. Let $Q([a,b])$ be the set of all partitions $\varkappa = \{x_i\colon i = 0,\dots,n\}$ of $[a,b]$ such that $a = x_0 < x_1 < \dots < x_n = b$. The p-variation, $0 < p < \infty$, on $[a,b]$ of f is defined by

$$v_p(f) := v_p(f;[a,b]) := \sup\{s_p(f;\varkappa)\colon \varkappa \in Q([a,b])\},$$

where $s_p(f;\varkappa) = \sum_{i=1}^n |f(x_i)-f(x_{i-1})|^p$ for $\varkappa = \{x_i\colon i=0,\dots,n\}$. If $v_p(f) < \infty$, f is said to have a bounded p-variation on $[a,b]$. If f is a Hölder function with exponent $0<\alpha\leqslant 1$, then it has a bounded $1/\alpha$-variation.

Denote by $\mathcal{W}_p([a,b])$ the class of all functions defined on $[a,b]$ with a bounded p-variation, that is

$$\mathcal{W}_p([a,b]) := \{f\colon [a,b]\to \mathbf{R}\colon v_p(f;[a,b]) < \infty\}.$$

Define $V_p(f) := V_p(f;[a,b]) = v_p^{1/p}(f)$ which is 0 if and only if f is a constant. If $p \geqslant 1$ and $f,g \in \mathcal{W}_p$ then

$$V_p(f+g) \leqslant V_p(f) + V_p(g). \tag{3}$$

For each f, $V_p(f)$ is a non-increasing function of p, i.e., if $q < p$ then $V_p(f) \leqslant V_q(f)$. This follows from the inequality

$$\left(\sum_{k=1}^n |a_k|^p\right)^{1/p} \leqslant \left(\sum_{k=1}^n |a_k|^q\right)^{1/q} \tag{4}$$

valid for $0<q<p<\infty$ and any $\{a_k\}$.

Let $p \geqslant 1$ and

$$V_{p,\infty}(f) = V_{p,\infty}(f;[a,b]) = V_p(f;[a,b]) + |f|_{\infty,[a,b]},$$

where $|f|_{\infty,[a,b]} = \sup_{a\leqslant x\leqslant b}|f(x)|$. Then $V_{p,\infty}(f)$ is the norm, and $\mathcal{W}_p([a,b])$ equip- ped with the p-variation norm is a Banach space.

Let $f \in C\mathcal{W}_p([a,b])$, $1 < p < \infty$, and $\varepsilon > 0$. Then there exists $\{x_i\colon i = 0,\dots,n\} \in Q([a,b])$ such that $\max_{1\leqslant i\leqslant n} v_p(f;[x_{i-1},x_i]) < \varepsilon$ ([1], Lemma 2.20, p.94).

The existence of the RS integral in (1) follows by the Love–Young inequality. Now we formulate it. Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$ with $p \geqslant 1$, $q \geqslant 1$, $1/p + 1/q > 1$. If f and h have no common discontinuities then the Riemann–Stieltjes integral $\int_a^b f\,dh$ exists and the inequality

$$\left|\int_a^b f\,dh - f(\xi)\big[h(b) - h(a)\big]\right| \leqslant C_{p,q}V_q(f;[a,b])V_p(h;[a,b]), \tag{5}$$

holds for any $\xi \in [a,b]$, where $C_{p,q} = \zeta(p^{-1}+q^{-1})$, $\zeta(s)$ denotes the Riemann zeta function, i.e., $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$.

Let $f \in \mathcal{W}_q([a,b])$ and $h \in C\mathcal{W}_p([a,b])$. From (5) it follows that

$$V_p\left(\int_a^{\cdot} f\,dh;[a,b]\right) \leqslant \big[C_{p,q}V_q(f;[a,b]) + |f|_{\infty,[a,b]}\big]V_p(h;[a,b]). \tag{6}$$

If the function $h \in C\mathcal{W}_p([a,b])$ then the indefinite integral $\int_a^y f\,dh$, $y \in [a,b]$, is a continuous function ([2], Lemma 3.23, p. 124).

2. Proof

Lemma 1 (see [2]). *Let $g \in H^{\ell/2,\ell}([a,b]\times\mathbf{R})$. Then for any $a < b$ and $q = 2/\alpha$, $0 < \alpha < 1$, $p \geqslant 1$ and $x, y \in \mathcal{W}_q$*

$$\begin{aligned}V_q\big(g(\cdot,x) - g(\cdot,y);[a,b]\big) \leqslant{}& \|g\|_T^{(\ell)}V_q(x-y;[a,b])\\ &+ \|g\|_T^{(\ell)}|x-y|_{\infty,[a,b]}\Big((b-a)^{\alpha/2} + V_2^{\alpha}(y;[a,b])\Big).\end{aligned}$$

Proof. By the mean value theorem, we have

$$\begin{aligned}&\big[g(t,x_t) - g(t,y_t)\big] - \big[g(s,x_s) - g(s,y_s)\big]\\ &\quad= \big[g\big(t,y_t + (x_t - y_t)\big) - g\big(t,y_t + (x_s - y_s)\big)\big]\\ &\qquad+\Big\{\big[g\big(t,y_t + (x_s - y_s)\big) - g(t,y_t)\big] - \big[g\big(s,y_t + (x_s - y_s)\big) - g(s,y_t)\big]\Big\}\\ &\qquad+\Big\{\big[g\big(s,y_t + (x_s - y_s)\big) - g(s,y_t)\big] - \big[g\big(s,y_s + (x_s - y_s)\big) - g(s,y_s)\big]\Big\}\\ &\quad= \big[g\big(t,y_t + (x_t - y_t)\big) - g\big(t,y_t + (x_s - y_s)\big)\big]\\ &\qquad+\int_0^{x_s - y_s}\big[\partial_x g(t,y_t+u) - \partial_x g(s,y_t+u)\big]\mathrm{d}u\end{aligned}$$

$$+\int_0^{x_s-y_s}\left[\partial_x g(s,y_t+v)-\partial_x g(s,y_s+v)\right]\mathrm{d}v$$

for any $a \leqslant s < t \leqslant b$. Therefore

$$\begin{aligned}&\Big|\left[g(t,x_t)-g(t,y_t)\right]-\left[g(s,x_s)-g(s,y_s)\right]\Big|\\&\quad\leqslant \left|\partial_x g\right|_\infty\left|(x_t-y_t)-(x_s-y_s)\right|+\|g\|_T^{(\ell)}|x_s-y_s|\left[(t-s)^{\alpha/2}+|y_t-y_s|^\alpha\right].\end{aligned}$$

Now it is easy to finish the proof of the lemma.

Next we construct the Picard iteration

$$y_m(t)=\alpha+\int_{t_0}^t f(s,y_{m-1}(s))\,\mathrm{d}s+\int_{t_0}^t g(s,y_{m-1}(s))\,\mathrm{d}h_s,\quad m\geqslant 1$$

and $y_0(t)=\alpha$, for the equation (1).

Lemma 2. *Let* $2(1-1/p)<\alpha<1$ *and* $q=2/\alpha$. *If for some* $t>t_0$

$$V_p(h;[t_0,t])<\left(2\cdot C_{p,q}|\partial_x g|_{\infty,[t_0,T]}\right)^{-1} \tag{7}$$

then for each $m\geqslant 0$,

$$\begin{aligned}V_p\left(y_{m+1};[t_0,t]\right)<2\Big\{&|f|_{\infty,[t_0,t]}(t-t_0)+\left[C_{p,q}\|g\|_T^{(\ell)}(t-t_0)^{(1+\alpha)/2}\right.\\&\left.+|g|_{\infty,[t_0,t]}\right]V_p\left(h;[t_0,t]\right)\Big\}.\end{aligned}$$

Proof. For any $t_0\leqslant a<b\leqslant T$, by properties of the seminorm V_p and by inequality (6), we get

$$\begin{aligned}&V_p(y_{m+1};[a,b])\\&\leqslant V_1\left(\int_{t_0}^{\cdot} f(s,y_m(s))\,\mathrm{d}s;[a,b]\right)+V_p\left(\int_{t_0}^{\cdot} g(s,y_m(s))\,\mathrm{d}h_s;[a,b]\right)\\&\leqslant |f|_{\infty,[a,b]}(b-a)+\left[C_{p,q}V_q\left(g(\cdot,y_m);[a,b]\right)+|g|_{\infty,[a,b]}\right]V_p\left(h;[a,b]\right)\\&\leqslant |f|_{\infty,[a,b]}(b-a)+\Big\{C_{p,q}\Big[\|g\|_T^{(\ell)}(b-a)^{(1+\alpha)/2}\\&\quad+|\partial_x g|_{\infty,[a,b]}V_p\left(y_m;[a,b]\right)\Big]+|g|_\infty\Big\}V_p\left(h;[a,b]\right).\end{aligned} \tag{8}$$

Denote

$$A=|f|_{\infty,[t_0,t]}(t-t_0)+\left[C_{p,q}\|g\|_T^{(\ell)}(t-t_0)^{(1+\alpha)/2}+|g|_{\infty,[t_0,t]}\right]V_p\left(h;[t_0,t]\right).$$

By inequalities (7) and (8) we get

$$
\begin{aligned}
V_p\big(y_{m+1};[t_0,t]\big) &< \frac{1}{2}V_p\big(y_m;[t_0,t]\big)+A\\
&< \frac{1}{2^{m+1}}V_p\big(y_0;[t_0,t]\big)+\Big(1+\frac{1}{2}+\cdots+\frac{1}{2^m}\Big)A<2A
\end{aligned}
$$

since $V_p(y_0;[t_0,t])=0$. The proof of the lemma is complete.

As a consequence of Lemma 2 we get that $y_m\in \mathcal{W}_p([t_0,T])$ for each $m\geqslant 1$. Moreover $y_m\in C\mathcal{W}_p([t_0,T])$ since $h\in C\mathcal{W}_p([t_0,T])$.

Let $z_m(t):=y_m(t)-y_{m-1}(t)$, $m\geqslant 1$. Now we estimate $V_p(z_m;[t_0,t])$. Put $q=2/\alpha$. Similarly as in Lemma 2 we get

$$
\begin{aligned}
V_p(z_m;[t_0,t]) &\leqslant V_1\bigg(\int_{t_0}^{\cdot}\big[f\big(s,y_{m-1}(s)\big)-f\big(s,y_{m-2}(s)\big)\big]\mathrm{d}s;[t_0,t]\bigg)\\
&\quad+V_p\bigg(\int_{t_0}^{\cdot}\big[g\big(s,y_{m-1}(s)\big)-g\big(s,y_{m-2}(s)\big)\big]\mathrm{d}h_s;[t_0,t]\bigg)\\
&\leqslant |\partial_x f|_{\infty,[t_0,t]}(t-t_0)\big|y_{m-1}(s)-y_{m-2}(s)\big|_{\infty,[t_0,t]}\\
&\quad+\Big\{C_{p,q}V_q\big(g(\cdot,y_{m-1})-g(\cdot,y_{m-2});[t_0,t]\big)\\
&\quad+\big|g(\cdot,y_{m-1})-g(\cdot,y_{m-2})\big|_{\infty,[t_0,t]}\Big\}V_p\big(h;[t_0,t]\big).
\end{aligned}
$$

By Lemma 1 and inequality

$$
\sup_{t_0\leqslant s\leqslant t}\big|y_{m-1}(s)-y_{m-2}(s)\big|\leqslant V_p\big(z_{m-1};[t_0,t]\big)
$$

we get

$$
\begin{aligned}
V_p(z_m;[t_0,t]) &\leqslant \Big[|\partial_x f|_{\infty,[t_0,t]}+\big(C_{p,q}+1\big)\|g\|_T^{(\ell)}\\
&\quad+C_{p,q}\|g\|_T^{(\ell)}\big((t-t_0)^{\alpha/2}+V_p^{\alpha}\big(y_{m-2};[t_0,t]\big)\big)\Big]\\
&\quad\times V_p\big(z_{m-1};[t_0,t]\big)\cdot\max\Big\{(t-t_0),V_p\big(h;[t_0,t]\big)\Big\}.
\end{aligned}
$$

Denote

$$
\begin{aligned}
K_t &= |\partial_x f|_{\infty,[t_0,t]}+\big(C_{p,q}+1\big)\|g\|_T^{(\ell)}\\
&\quad+C_{p,q}\|g\|_T^{(\ell)}\Big\{(t-t_0)^{\alpha/2}+2^{\alpha}\big[|f|_{\infty,[t_0,t]}(t-t_0)\\
&\quad+\big(C_{p,q}\|g\|_T^{(\ell)}(t-t_0)^{(1+\alpha)/2}+|g|_{\infty,[t_0,t]}\big)V_p\big(h;[t_0,t]\big)\big]^{\alpha}\Big\}.
\end{aligned}
$$

Let $\tau > t_0$ be such that (7) holds. Then for $t \in [t_0, \tau]$ by Lemma 2 we get

$$V_p\big(z_m; [t_0, t]\big) < K_t V_p\big(z_{m-1}; [t_0, t]\big) \cdot \max\Big\{(t - t_0), V_p\big(h; [t_0, t]\big)\Big\}.$$

Now we take $t_1 \leqslant \tau$ such that

$$\max\Big\{(t_1 - t_0), V_p\big(h; [t_0, t_1]\big)\Big\} < \frac{1}{2} K_\tau^{-1}.$$

Then for $t \in [t_0, t_1]$

$$\begin{aligned} V_p\big(z_m; [t_0, t]\big) &< \frac{1}{2} V_p\big(z_{m-1}; [t_0, t]\big) < \frac{1}{2^{m-1}} V_p\big(z_1; [t_0, t]\big) \\ &= \frac{1}{2^{m-1}} V_p\big(y_1; [t_0, t]\big). \end{aligned}$$

Thus the series $\sum_{k=1}^{\infty} V_p(z_k; [t_0, t_1])$ converges. Now by the routine arguments one can prove that there is an element $\hat{y} \in C\mathcal{W}_p([t_0, t_1])$ such that $\hat{y}$ is the solution of the equation (1) in the interval $[t_0, t_1]$.

Further we can repeat the whole construction for the equation

$$\tilde{y}(t) = \hat{y}(t_1) + \int_{t_1}^{t} f(s, \tilde{y}(s))\, \mathrm{d}s + \int_{t_1}^{t} g(s, \tilde{y}(s))\, \mathrm{d}h_s, \quad t \in [t_1, T],$$

and prove the existence of the solution in some interval $[t_1, t_2]$. Then the function

$$\overline{y}(t) = \begin{cases} \hat{y}(t), & \text{for } t \in [t_0, t_1], \\ \tilde{y}(t), & \text{for } t \in [t_1, t_2] \end{cases}$$

is the solution of the equation (1) in $[t_0, t_2]$. After finitely number of steps we get the solution y on the whole interval $[t_0, T]$ and $y \in C\mathcal{W}_p([t_0, T])$.

Now we prove the uniqueness of the solution of equation (1). Let z be another solution of (1). Similarly to the estimate of $V_p(z_m; [t_0, t])$, we get

$$\begin{aligned} V_p\big(y - z; [t_0, t]\big) \leqslant \Big[&|\partial_x f|_{\infty,[t_0,t]} + \big(C_{p,p} + 1\big)\|g\|_T^{(\ell)} \\ &+ C_{p,q}\|g\|_T^{(\ell)}\big((t - t_0)^{\alpha/2} + V_p^\alpha\big(y; [t_0, t]\big)\big)\Big] \\ &\times V_p\big(y - z; [t_0, t]\big) \cdot \max\Big\{(t - t_0), V_{p,\infty}\big(h; [t_0, t]\big)\Big\}. \end{aligned}$$

Similarly to above we get

$$\begin{aligned} |y - z|_{\infty,[t_0,t]} \leqslant \Big[&|\partial_x f|_{\infty,[t_0,t]} + \big(C_{p,q} + 1\big)\|g\|_T^{(\ell)} + C_{p,q}\|g\|_T^{(\ell)}\big((t - t_0)^{\alpha/2} \\ &+ V_p^\alpha\big(y; [t_0, t]\big)\big)\Big] V_p\big(y - z; [t_0, t]\big) \cdot \max\Big\{(t - t_0), V_{p,\infty}\big(h; [t_0, t]\big)\Big\}. \end{aligned}$$

Therefore one can find a $\delta > 0$ such that

$$V_{p,\infty}(y - z; [t_0, t]) < \frac{1}{2} V_{p,\infty}(y - z; [t_0, t])$$

for $t \in [t_0, t_0 + \delta]$. From the last inequality we get $V_{p,\infty}(y - z; [t_0, t_0 + \delta]) = 0$. Since $V_{p,\infty}$ is the norm then $y(t) - z(t) = 0$ on the interval $[t_0, t_0 + \delta]$. After finitely many steps we get $y(t) = z(t)$ on $[t_0, T]$. The proof of the Theorem is complete.

References

[1] R.M. Dudley, R. Norvaiša, Differentiability of six operators on nonsmooth functions and p-variation. *Lecture Notes Math.*, Springer, vol. 1703 (1999).

[2] R.M. Dudley, Picard iteration and p-variation: the work of Lyons (1994), *Mini-proceedings: Workshop on Product Integrals and Pathwise Integration*, MaPhySto (1999).

[3] T. Lyons, Differential equations driven by rough signals (I): An extension of an inequality of L.C. Young, *Mathematical Research Letters*, **1**, 451–464 (1994)

[4] L.C. Young, An inequality of the Hölder type, connected with Stieltjes integration, *Acta Math.*, **67**, 251–282 (1936).

Integralinės lygties generuotos funkcijos, turinčios baigtinę p-variaciją, sprendinio egzistavimas ir vienatis

K. Kubilius

Tarkime, kad funkcija $f(t, x)$ apibrėžta aibėje $[t_0, T] \times \mathbf{R}$ yra aprėžta ir turi aprėžtą dalinę išvestinę x atžvilgiu, o $g \in H^{\ell/2,\ell}([t_0, T] \times \mathbf{R})$, $p < \alpha/2 + 1 < 2$. Įrodyta, kad lygtis

$$x_t = \alpha + \int_{t_0}^{t} f(s, x_s)\, ds + \int_{t_0}^{t} g(s, x_s)\, dh_s, \qquad t_0 \leqslant t \leqslant T,$$

čia h yra tolydi funkcija turinti baigtinę p-variaciją tam tikram p, $1 < p < 2$, turi vienintelį sprendinį tolydžių baigtinės p-variacijos funkcijų klasėje.