LMD mokslo darbai, III tomas, 100–103

The generalized numbers and modified L-functions

Eugenijus STANKUS (VU)
e-mail:eugenijus.stankus@maf.vu.lt

The real numbers $1 < \beta_1 \leqslant \beta_2 \leqslant \cdots$ for which $\lim_{n\to\infty} \beta_n = \infty$ are called the generalized primes (g-primes). Then the elements ν_i $(i = 1, 2, \cdots)$, $\nu_0 = 1$ of the multiplicative semigroup generated by g-primes are the generalized integers called g-integers (Beurling, 1937 [1]).

In this paper we consider the problems associated with the distribution of g-integers in arithmetical progressions. The application of analytical methods reduces these problems to consideration of corresponding modified L-functions. For example in order to establish the asymptotic behaviour of counting function

$$P(x; k, l) = \sum_{\substack{\varphi(n)\leqslant x \\ n\equiv l (\mathrm{mod}\, k)}} 1$$

for Euler totient function $\varphi(n)$ it is necessary to examine the modified L-functions

$$\begin{aligned}\Phi(s, \chi) &= \sum_{n=1}^{\infty} \chi(n)\varphi(n)^{-s} \\ &= \prod_{p} \left(1 + \chi(p)(p-1)^{-s}\left(1 - \chi(p)p^{-s}\right)^{-1}\right), \quad s = \sigma + it,\ \sigma > 1.\end{aligned}$$

Here and further p denotes the rational prime numbers. In [4] the analyticity of the functions $\Phi(s, \chi)$ in the half-plane $\sigma > 0$ for $\chi \neq \chi_0$ was proved. Let us take notice that the values of Euler function are closely related to g-integers generated by the sequence of g-primes $\{\beta_i\}$, $\beta_i = p_{i+1} - 1$, $i = 1, 2, \cdots$, where p_n is the nth prime number.

The values of divisor sum function $\sigma(n)$ are associated with the g-integers generated by g-primes $\{\beta_i\}$, $\beta_i = p_i + 1$, $i = 1, 2, \cdots$. The corresponding modified L-function

$$\begin{aligned}G(s, \chi) &= \sum_{n=1}^{\infty} \chi(n)\sigma(n)^{-s} \\ &= \prod_{p} \left(1 + \sum_{j=1}^{\infty} \chi^j(p)(p^j + p^{j-1} + \cdots + p + 1)^{-s}\right), \quad \sigma > 1\end{aligned}$$

is analytically continued to the half-plane $\sigma > 0$ for $\chi \neq \chi_0$ as well [5].

A bit different problem is to investigate the number of g-integers ν_i not exceeding x in arithmetical progression, i.e. $\nu_i \equiv l(\bmod k)$, $\nu_i \leqslant x$, $(l,k) = 1$. Denote this number by $N(x;k,l)$. This problem for any sequences of g-primes is rather complicated, but it is solvable for special cases. For example in the case of g-primes $\beta_i = vp_i$, $(i = 1, 2, \cdots)$, v is integer, $v > 1$ (see [2]) the asymptotic formula for $N(x;k,l)$ was obtained in [3]:

$$N(x;k,l) = ax(\log x)^{\frac{1}{v}-1} + O(xe^{-c\sqrt{\log x}}),$$

where c and a are positive constants depending on k and v.

The applied method of the proof needs the analytical properties of the modified L-function

$$L(s;v,\chi) = \sum_{i=0}^{\infty} k_i\chi(\nu_i)\nu_i^{-s} = \prod_p \left(1 - \chi(vp)(vp)^{-s}\right)^{-1}, \quad \sigma > 1.$$

The g-integer ν_i arise as distinct products of g-primes; k_i denote the number of such products $k_0 = 1$. The analytical continuation of the function $L(s;v,\chi)$ to the left of the line $\sigma = 1$ is more complicated than in the cases above. The domain of analyticity of $L(s;v,\chi)$ is $D_\chi = \{s : \sigma > 0\} \setminus U_\chi$, where

$$U_\chi = \bigcup_n \bigcup_\rho \{s : s = (x\beta + i\gamma)/n,\ 0 < x < 1\} \cup (0;1];$$

the union is taken over all positive integers n and over all non-real zeros $\rho = \beta + i\gamma$ of Dirichlet L-functions $L(s,\chi^n)$.

The purpose of this paper is the analytical continuation of modified L-function associated with the g-primes $\beta_i = p_i + r$; r is the arbitrary positive integer. A similar problem was considered in [6].

Let χ is a non-principal Dirichlet character mod$q$, where q is a rational prime number. Now we consider the modified L-function

$$L(s;\chi,r) = \sum_{i=0}^{\infty} k_i\chi(\nu_i)\nu_i^{-s} = \prod_p \left(1 - \chi(p+r)(p+r)^{-s}\right)^{-1}, \quad \sigma > 1,$$

which may be used to investigate the number of g-integers not exceeding x in arithmetical progression.

Theorem 1. *The modified L-function $L(s;\chi,r)$ for non-principal character with prime modulus q is analytic in the domain $\sigma > 0.9$.*

Proof. Let $L(s,\chi)$ be the Dirichlet L-function with character $\chi(\bmod q)$. Then we have

$$\begin{aligned} L(s;\chi,r) &= L(s,\chi)\cdot\prod_p \left(1 + \chi(p+r)(p+r)^{-s} - \chi(p)p^{-s} + O(p^{-2\sigma})\right) \\ &= L(s,\chi)\cdot G(s,\chi)\cdot W(s,\chi), \end{aligned}$$

where $G(s,\chi)$ is analytic in the half-plane $\sigma > \frac{1}{2}$ and

$$W(s,\chi) = \prod_p \left(1 + \chi(p+r)(p+r)^{-s} - \chi(p)p^{-s}\right).$$

Further we write

$$W(s,\chi) = \prod_p \left(1 + \chi(p+r)((p+r)^{-s} - p^{-s}) + p^{-s}(\chi(p+r) - \chi(p))\right).$$

The difference $(p+r)^{-s} - p^{-s}$ can be easily estimated:

$$|(p+r)^{-s} - p^{-s}| = \left| - s \int_p^{p+r} u^{-s-1} du \right| \leqslant r|s|p^{-\sigma-1}.$$

So that we have

$$W(s,\chi) = H(s,\chi) \cdot V(s,\chi),$$

where the function $H(s,\chi)$ is analytic in the half-plane $\sigma > 0$ and

$$\begin{aligned} V(s,\chi) &= \prod_p \left(1 + p^{-s}(\chi(p+r) - \chi(p))\right) \\ &= U(s,\chi) \cdot \exp\left\{ \sum_p p^{-s}(\chi(p+r) - \chi(p)) \right\}; \end{aligned}$$

$U(s,\chi)$ is analytic function in the half-plane $\sigma > \frac{1}{2}$.

In order to evaluate the last series over primes we consider the sum

$$S(y) = \sum_{p \leqslant y} p^{-s}(\chi(p+r) - \chi(p)).$$

After the partial summation we apply the following deep result of I.M. Vinogradov [7]:

$$\sum_{p \leqslant y} \chi(p+r) << y^{1+\varepsilon}\left(q^{\frac{1}{4}} y^{-\frac{1}{3}} + y^{-\frac{1}{10}}\right).$$

Thus $S(y) << 1$ for $\sigma \geqslant 0.9 + \varepsilon$ and the theorem is proved.

References

[1] A. Beurling, Analyse de la loi asymptotic de la distribution des nombres premièrs généralisés I, *Acta Math.*, **68**, 225–291 (1937).
[2] C. Ryavec, The analytic continuation of Euler products with applicatios to asymptotic formulae, *Illinois J. Math.*, **17**, 608–618 (1973).
[3] E. Stankus, Analytic continuation of modified L-function, *Lith. Math. J.*, **24**, 176–181 (1984).
[4] E. Stankus, On the Euler function values in arithmetical progressions, Preprintas 97-26, Vilniaus universiteto matematikos fakultetas (1997).
[5] E. Stankus, On the divisor sums in arithmetical progressions, in: *LMD XXXIX konf. darbai*, spec. „Liet. matem. rink." priedas, Vilnius, Technika (1998), pp. 76–79.
[6] N. Prusakova, E. Stankus, Modifikuotosios L-funkcijos, in: *LMD XXXVI konf. pranešimų tezės*, Vilnius (1995), p. 27.
[7] И.М. Виноградов, Улучшение оценки для суммы значений $\chi(p+k)$, *Изв. АН СССР, Сер. матем.* **17**, 285–290 (1953).

Apibendrintieji skaičiai ir modifikuotosios L-funkcijos

E. Stankus

Nagrinėjami įvairių apibendrintųjų skaičių sistemų pasiskirstymų aritmetinėse progresijose klausimai. Įrodomas modifikuotosios L-funkcijos, susijusios su apibendrintųjų pirminių skaičių seka $\{p + r\}$, analiziškumas pusplokštumėje $\sigma > 0,9$.