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On analytic problems of combinatorial structures

Eugenijus MANSTAVIČIUS (VU)*
e-mail: eugenijus.manstavicius@maf.vu.lt

Assemblies, multisets, and selections represent the classes of combinatorial structures which were most intensively studied in recent years (see [1], [4], [10] and other). The neighbouring arithmetical semigroup theory was also being developed (see [5], [9], [11]). In [7], [8] the author proposed an analytic approach suitable to find asymptotic formulas for general multiplicative functions on combinatorial structures. The results were applied to the case of assemblies. We now extend the investigation [8] by some results valid for multisets, selections, and arithmetical semigroups.

For start, we point out some ambiguity in the use of terminology. A class $\mathcal{S}$ of combinatorial objects is said [3] to be *decomposable* over another class $\mathcal{P}$ if each element $\sigma \in \mathcal{S}$ may be uniquely decomposed into a finite multiset of elements $p \in \mathcal{P}$. The last ones are called *components* of σ and the very σ gains the term of *multiset*. Note that in contrast to the definition of *assembly* (see [1], [9]), we now do not involve any labeling in the structure p. If in the decomposition of σ, no repetition of elements p is allowed, such σ is called *selection*. We stress also that $\mathcal{P}$ may be identified with the subset of $\mathcal{S}$ comprised of one component and each multiset $\{p_i,\ i = 1,\dots,s\} \subset \mathcal{P}$ gives a unique $\sigma \in \mathcal{S}$. Therefore one could identify the decomposition of σ into this multiset with the formal product $\sigma = p_1 \cdots p_s$ and so define a commutatitive and associative multiplication in $\mathcal{S}$. Setting 1 for the empty product, we thus arrive to an arithmetic semigroup. Either of the concepts of multisets or arithmetic semigroups had been developed generalizing the set of monic polynomials over a Galois field, nevertheless, recent development took seemingly different directions. The papers [1], [4], [10] keep on the first concept based mainly on the terminology of [3] while the papers [5], [9], [12] follow the book [6]. This leads to some repetition of the results.

For a *size function* $\delta : \mathcal{S} :\longrightarrow \mathbb{N} \cup \{0\}$ we assume that $|\mathcal{S}_N| := |\{\sigma \in \mathcal{S} :\ \delta(\sigma) = N\}| =: S(N) < \infty$ for each $N \in \mathbb{N}$. Set $S(0) = 1$. Further, it is natural to require that the size of σ equals the sum of its components sizes and $\delta(p) \geqslant 1$ for each $\mathcal{P}$. The enumeration problem requires to derive an asymptotic formula for $p(N)$ from an information concerning $\pi(j) := |\{p \in \mathcal{P} :\ \delta(p) = j\}|$, $j \geqslant 1$. It is rather difficult to use the exact formula (see, for instance, [1], [10])

$$S(N) = \sum_{\substack{k_1,\dots,k_N \geqslant 0 \\ 1k_1+\cdots+Nk_N=N}} \binom{\pi(j)+k_j-1}{k_j}$$

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for large N.

Considering $\mathcal{S}$ an arithmetic semigroup, we define a multiplicative function f by

$$f(\sigma) = \prod_{p \in \mathcal{P}} f(p^{\alpha_p(\sigma)}), \tag{1}$$

where $\alpha_p(\sigma)$ denotes the multiplicity of p in the decomposition of σ and $f(1) = 1$. Now the asymptotic behaviour of the sum of values $f(\sigma)$ over $\sigma \in \mathcal{S}_N$ is of great interest.

To reckon the selections from an arithmetical semigroup, one can use the indicator $q(\sigma)$ of the set of square-free elements which is multiplicative function defined by (1) with $f(p^k) = q(p^k) := 1$ for $k = 1$ and zero otherwise. The direct calculations of the number of selections of size N which equals ([1], [10])

$$\widetilde{S}(N) = \sum_{\substack{k_1, \dots, k_N \geqslant 0 \\ 1k_1 + \cdots + Nk_N = N}} \binom{\pi(j)}{k_j}$$

is also rather complicated.

The purpose of the present remark is to indicate a new possibility to apply our analytic results [7], [8]. In particular, we will use the following corollary of Theorem 3 proved in [8]. It considers the asymptotic behaviour as $N \to \infty$ of the Taylor coefficients m_N of an analytic in $|z| < 1$ function having the expression

$$F(z) := \sum_{N \geqslant 0} m_N z^N =: H(z) \exp\{U(z)\} =: H(z) \exp\left\{ \sum_{j \geqslant 1} a_j z^j \right\} \tag{2}$$

where $a_j \in \mathbb{C}$. For a sequence d_j, set

$$D(z) := \sum_{N \geqslant 0} M_N z^N := \exp\left\{ \sum_{j \geqslant 1} d_j z^j \right\} =: \exp\{V(z)\}.$$

Theorem A ([8]). *Let $F(z)$ be given in* (2) *with an analytic in $|z| < 1$ and continuously differentiable on $|z| = 1$ function $H(z)$ and assume that $|a_j| \leqslant d_j$, $0 < \theta^- \leqslant jd_j \leqslant \theta^+ < \infty$, $j \geqslant 1$. If there exists $t_0 \in (-\pi, \pi]$ such that the series*

$$\sum_{j \geqslant 1} \left(d_j - a_j e^{-it_0 j}\right)$$

converges then

$$m_N = M_N \exp\left\{it_0 N + U(e^{-it_0}) - V(1)\right\}\left(H(e^{-it_0}) + o(1)\right).$$

Even used for enumeration problems, Theorem A yields extensions of the recent results which, typically, were proved under the conditions assuring analytic continuation of generating series outside the convergence disk, say, $|z| < r$. In many recent papers (see [1], [2], [4], [10] and the bibliography therein) the regions

$$\{z \in \mathbb{C}:\ |z| \leqslant r + \eta,\ |Arg(z - r)| \geqslant \eta_1\},\quad r > 0,\ \eta > 0,\ 0 < \eta_1 < \pi/2$$

are taken under consideration. In order to illustrate of our approach, we present a result on the so-called logarithmic multisets (arithmetic semigroups) and selections.

Theorem 1. *Let* $\pi(j) = \theta\rho^j/j + \rho^j r(j)$ *with some* $\theta > 0$ *and* $\rho > 1$ *so that*

$$\sum_{j=1}^{\infty} j|r(j)| < \infty. \tag{3}$$

Then

$$S(N) = \rho^N N^{\theta-1}\Gamma(\theta)^{-1}Q(1 + o(1))$$

and

$$\tilde{S}(N) = \rho^N N^{\theta-1}\Gamma(\theta)^{-1}\tilde{Q}(1 + o(1)).$$

Here Γ *denotes the Euler gamma function and the constants* Q, $\tilde{Q}$ *will be given in the proof.*

Proof. The generating function of the multiset equals by [1], formula (131),

$$\begin{aligned} Z(z) &= \sum_{N=0}^{\infty} S(N)z^N = \prod_{j=1}^{\infty}(1 - z^j)^{-\pi(j)} \\ &= \exp\left\{\sum_{k=1}^{\infty}\frac{\Pi(z^k)}{k}\right\} =: \exp\{\Pi(z)\}K(z), \end{aligned}$$

where

$$\Pi(z) = \sum_{j=1}^{\infty}\pi(j)z^j = -\theta\log(1 - \rho z) + \sum_{j=1}r(j)(\rho z)^j,\quad |z| < \rho^{-1}.$$

Hence

$$Z(z) = \frac{1}{(1 - \rho z)^\theta}\exp\left\{\sum_{k=1}^{\infty} r(j)(\rho z)^j\right\}K(z) =: \frac{H_0(z)}{(1 - \rho z)^\theta}.$$

We now apply Theorem A for $F(z) = Z(\rho^{-1}z)$, $a_j = d_j$, $U(z) = V(z) = -\theta\log(1-z)$. As it is known [2], in such a case $M_N \sim N^{\theta-1}/\Gamma(\theta)$ as $N \to \infty$. Condition (3) and the estimate

$$\sum_{k=2}^{\infty}\frac{1}{k}\sum_{j=1}^{\infty}\pi(j)|z|^{jk}jk \ll \sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\rho^{j(1-k)} \ll 1, \quad |z| \leqslant \rho^{-1},$$

assures the condition on the derivative of the function $H(z) = H_0(\rho^{-1}z)$. Thus we obtain from Theorem A

$$S(N) = \frac{\rho^N N^{\theta-1}}{\Gamma(\theta)}\exp\left\{\sum_{j=1}^{\infty}\frac{j\pi(j)\rho^{-j}-\theta}{j} + \sum_{k=2}^{\infty}\frac{\Pi(\rho^{-k})}{k}\right\}(1+o(1)).$$

In order to find an asymptotic formula for $\widetilde{S}(N)$, we start with the expression (see [1], formula (142))

$$\begin{aligned}\tilde{Z}(z) := \sum_{N=0}^{\infty}\widetilde{S}(N)z^N &= \prod_{j=1}^{\infty}(1+z^j)^{\pi(j)} \\ &= \exp\{\Pi(z)\}\exp\left\{\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\Pi(z^k)}{k}\right\}\end{aligned}$$

in the region $|z| < \rho^{-1}$ and repeat the previous argumentation. So we obtain the value

$$\tilde{Q} = \exp\left\{\sum_{j=1}^{\infty}\frac{j\pi(j)\rho^{-j}-\theta}{j} + \sum_{k=2}^{\infty}\frac{(-1)^{k-1}\Pi(\rho^{-k})}{k}\right\}.$$

Theorem 1 is proved.

Note that strengthening of condition (3) leads to estimates of the remainder terms in the formulas for $S(N)$ and $\widetilde{S}(N)$. Generalizing Theorem 1 we examine mean-values of multiplicative functions on $\mathcal{S}$.

Theorem 2. *Let f be a complex-valued multiplicative function defined by* (1) *on an arithmetical semigroup $\mathcal{S}$ with $\pi(j)$ satisfying condition* (3). *Suppose that*

$$|f(p)| \leqslant \tau \tag{4}$$

with some $\tau > 0$ such that the series

$$\sum_{p\in\mathcal{P}}\left(\tau - f(p)e^{-it_0\delta(p)}\right)\rho^{-\delta(p)} \tag{5}$$

converges for some $t_0 \in (-\pi, \pi]$ *and*

$$\sum_{p\in\mathcal{P}} \sum_{k=2}^{\infty} \frac{k\delta(p)|f(p^k)|}{\rho^{k\delta(p)}} < \infty. \tag{6}$$

Then there exist a constants $A_1, A_2 \in \mathbb{C}$ *such that*

$$M_N(f) := \sum_{\sigma\in\mathcal{S}_N} f(\sigma) = A_1\rho^N N^{\theta\tau-1} e^{it_0N}\big(A_2 + o(1)\big).$$

Proof. Consider the generating function

$$\begin{aligned} G(z) &= \sum_{\sigma\in\mathcal{S}} f(\sigma)z^{\delta(\sigma)} = \prod_{p\in\mathcal{P}} \left(1 + f(p)z^{\delta(p)} + f(p^2)z^{2\delta(p)} + \cdots\right) \\ &=: \prod_{p\in\mathcal{P}} \chi_p(z). \end{aligned}$$

By the conditions of Theorem 2, we can separate a finite subset $\mathcal{P}_1 \subset \mathcal{P}$ such that $|\chi_p(z) - 1| \leqslant 1/2$ for all $p \in \mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$ uniformly in $|z| \leqslant \rho^{-1}$. We have

$$\begin{aligned} G(z) &= \left[\prod_{p\in\mathcal{P}_1} \chi_p(z) \exp\left\{ - \sum_{p\in\mathcal{P}_1} f(p)z^{\delta(p)} + \sum_{p\in\mathcal{P}_2} \left(\log\chi_p(z) \right.\right.\right. \\ &\qquad \left.\left.\left. - f(p)z^{\delta(p)}\right)\right\}\right] \exp\left\{\sum_{p\in\mathcal{P}} f(p)z^{\delta(p)}\right\} \\ &=: K_1(z)\exp\left\{\sum_{p\in\mathcal{P}} f(p)z^{\delta(p)}\right\} =: K_1(z)\exp\{W(z)\}. \end{aligned}$$

Using the conditions one can verify that the function $K_1(z)$ analytic in $|z| < \rho^{-1}$ and continuously differentiable on $|z| = \rho^{-1}$. We now apply Theorem A with $F(z) = G(\rho^{-1}z)$, $H(z) = K_1(\rho^{-1}z)$, and $U(z) = W(\rho^{-1}z)$. The last function is now compared with $V(z) = \tau\Pi(\rho^{-1}z)$. To find M_N, which in this case are defined by

$$\sum_{N=0}^{\infty} M_N z^N = \exp\{\tau\Pi(\rho^{-1}z) = (1-z)^{-\theta\tau}\exp\left\{\tau\sum_{j=1}^{\infty} r(j)z^j\right\},$$

we again use Theorem A. It yields

$$M_N \sim \frac{N^{\theta\tau-1}}{\Gamma(\theta\tau)} \exp\left\{\tau\sum_{j=1}^{\infty} r(j)\right\} \tag{7}$$

as $N \to \infty$. Thus using the notation above from Theorem A we obtain

$$M_N(f) = \rho^N M_N \exp\{it_0N + W(\rho^{-1}e^{-it_0}) - \tau\Pi(\rho^{-1})\}\big(K_1(\rho^{-1}) + o(1)\big).$$

Inserting (7), we end the proof.

Observe that the constant A_1 in Theorem 2 also has the following expression

$$A_1 = \Gamma(\theta\tau)^{-1} \exp\left\{ \sum_{j=1}^{\infty} \left(\left(\sum_{\delta(p)=j} f(p) \right) \rho^{-j} e^{-it_0 j} - \frac{\theta\tau}{j} \right) \right\}.$$

A stronger version of Theorem A obtained in [8] yields asymptotic formulas for sums of multiplicative functions on S under all conditions of Theorem 2 but (5). Asymptotic formulas for sums of the values of multiplicative functions on the square-free semigroup elements (selections) follow from Theorem 2. To obtain them, it is enough to change $f(\sigma)$ by the product $f(\sigma)q(\sigma)$.

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Dėl analizinių kombinatorinių struktūrų problemų

E. Manstavičius

Darbe išplėstos autoriaus analizinio rezultato, įrodyto [8] straipsnyje, taikymo sritys. Apibendrintos kartotinių aibių bei atrankų skaičių asimptotinės formulės, aritmetiniuose pusgrupiuose gauta multiplikatyviųjų funkcijų reikšmių sumų išraiškų.