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On strongly prime ideals, strongly multiplicative and insulating sets in rings

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1. Introduction

All rings in this paper are associative with identity element which should be preserved by ring homomorphisms and all modules are unitary.

The main object of this paper are strongly prime ideals, their characterization in terms of the strongly multiplicative sets and insulating systems. Recall that a ring R is called *strongly prime* if it is prime and its central closure $Q(R)$ is a simple ring. See [2] and [1] for definitions and basic properties of the central closure and the extended centroid of a semiprime ring. Particularly, Theorem 2.7 shows that strongly prime rings are very natural analogs of the commutative domains. An ideal of the ring is called strongly prime if corresponding factor ring is strongly prime. We characterize strongly prime rings as strongly prime modules over multiplication ring, also in terms of Procesi category and prove that for the strongly prime ring R its central closure is a flat epimorphism which defines a symmetric and perfect localization in the categories $Mod - R$ and $R - Mod$ - right and left R-modules, and that $Q(R)$ is canonically isomorphic to the quotient rings of R for these localizations. We also investigate relations of the ring and its multiplication ring. Proofs of some announced results will appear soon.

2. Terminology and basic results on strongly prime rings

By an ideal of the ring R we shall understand a two-sided ideal. We denote $\{a_1, ..., a_n\}$ set consisting of the elements $a_1, ..., a_n$, and by (a) an ideal of the ring generated by the element $a \in R$. $A \subset B$ means proper inclusion.

The subring of $End_Z R$, acting from the left on R , generated as a ring by all left and right mutiplications l_a and r_b, where $a, b \in R$, is called a *multiplication ring* of the ring R and will be denoted by $M(R)$. So each $\lambda \in M(R)$ is of the form $\lambda = \sum_k l_{a_k} r_{b_k}$ where $a_k, b_k \in R$ and can be represented as the sum $\sum_k a_k \otimes b_k^\circ$, where $b_k^\circ \in R^\circ$. Then $\lambda x = \sum_k a_k x b_k, x \in R$. It's clear that the canonical embedding $R \hookrightarrow M(R)$, sending $a \in R$ to l_a is onto if and only if R is commutative.

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Let M be an R-bimodule. Denote by $Z_M = Z_M(R) = \{\delta \in M \mid r\delta = \delta r, r \in R\}$ the set of R-centralizing elements of the M. A bimodule M is called *centred R-bimodule* if $M = RZ_M$.

Let $\phi : R \longrightarrow S$ be a ring homommrphism. We call ϕ *centred homomorphism* if S is centred R-bimodule. It's easy to see that centred extension of the ring R is a factor ring of a semigroup ring $R[G]$ where G is the free semigroup (with unit). Rings and their centred homomophisms form a category, which is called Procesi category.

$M \in R - Mod$ is called *strongly prime* if for any non-zero $x, y \in M$, there exits a finite set of elements $\{a_1, ..., a_n\} \subseteq R$, $n = n(x, y)$, such that $Ann_R\{a_1x, ..., a_nx\} \subseteq Ann_R\{y\}$. Taking $M = R$ notion of the one sided strongly prime ring is obtained.(See [3]). We look at the ring R as the R-bimodule taking into account left and right action of R on itself. This immediately leads to the notion of an ${}_{M(R)}R$-module.

We call an element $a \in R$ a *symmetric zero divisor* if tor any finite subset of elements $\{a_1, ..., a_n\} \subseteq (a)$, $Ann_{M(R)}\{a_1, ..., a_n\} \nsubseteq Ann_{M(R)}\{1_R\}$. Of course, when R is commutative, taking $n = 1, a_1 = a$, we obtain the usual definition of zero-divisors. Denote $zd(R)$ the set of zero divisors of the ring R.

Theorem 2.1. *For any nonzero ring R the following conditions are equivalent:*

(1) *R is a strongly prime ring;*
(2) $zd(R) = 0$;
(3) *R is a strongly prime module over its multiplication ring $M(R)$;*
(4) *for any nonzero $a, b \in R$ there are $\lambda_1, ..., \lambda_n \in M(R)$ such that $Ann_{M(R)}\{\lambda_1 a, ..., \lambda_n a\} \subseteq Ann_{M(R)}\{b\}$;*
(5) *for any nonzero $a \in R$ there are $\lambda_1, ..., \lambda_n \in M(R)$ such that $Ann_{M(R)}\{\lambda_1 a, ..., \lambda_n a\} \subseteq Ann_{M(R)}\{1_R\}$;*
(5') *for any nonzero $a \in R$ there exist $a_1, ..., a_n \in (a)$, such that*

$$\textstyle\sum_i x_i a_k y_i = 0, \textit{for all } 1 \leqslant k \leqslant n, \textit{ implies } \sum_i x_i y_i = 0;$$

(6) *there exists a centred monomorphism $\phi : R \longrightarrow K$ where K is a simple ring;*
(7) *There exists a centred monomorphism $\phi : R \longrightarrow S$, where the ring S has the following property: for each nonzero ideal $I \subseteq R$, its extension I^ε in S, $I^\varepsilon = SIS$, is equal to S.*

Proof. Equivalence of conditions (1), (3), (4), (5) is proved in [9] Thm. 35.6. Obviously, (3) $\Rightarrow$ (6) $\Rightarrow$ (7).

We prove (7) $\Rightarrow$ (5). Take nonzero $a \in R$. Then, by assumption, $(a)^\varepsilon = (a)Z_S = S$. This gives an expression $\sum_k a_k \delta_k = 1$ with some $a_1, ..., a_n \in (a)$, $\delta_k \in Z_S$. So we obtain that $Ann_{M(R)}\{a_1, ..., a_n\} \subseteq Ann_{M(R)}\{1_R\}$.

Equivalence (2) and (5) easily follows from the definition of the symmetric zero divisors.

(5') is exactly (5) written in the terms of elements of the ring R.

Particularly, by (2) of this theorem, each ring which is not strongly prime has nonzero symmetric zero-divisors. It is also clear that a strongly prime ring is left and right strongly prime over itself.

We note that for the strongly prime ring its central closure coinsides with right and left Martindale's quotient rings, and so with the symmetric ring of quotients. (See [6]).

Lemma 2.2. *Each centred homomorphism of rings $\phi : R \to S$ induces the canonical centred homomorphism $\phi' : M(R) \to M(S)$. If ϕ is a monomorphism, then also ϕ' is a monomorphism.*

For each prime ring R, we denote $F = Z(Q(R))$ the central closure of R which is a field.

Theorem 2.3. *A ring R is strongly prime if and only if its multiplication ring $M(R)$ is strongly prime. In this case their extended centroids are canonically isomorphic, and the central closure $Q(M(R)) \cong Q \otimes_F Q^\circ$, where $Q = Q(R)$.*

Theorem 2.4. *Let R be a strongly prime ring. If a ring S is Morita equivalent to the ring R, then S is strongly prime and their extended centroids are isomorphic.*

Let R be a ring. A finite set $A = \{a_1, ..., a_n\} \subseteq R$ is called an *insulator*, if

$$Ann_{M(R)}\{a_1, ..., a_n\} \subseteq Ann_{M(R)}\{1_R\};$$

i.e., if $\lambda a_1 = ... = \lambda a_n = 0$, implies $\lambda 1 = 0$.

In a semiprime ring R, insulators can be characterised in terms of the central closure $Q(R)$ and extended centroid $F(R)$ of the ring. Indeed, using Theorem 32.3 in [9], we obtain the following

Proposition 2.5. *In a semiprime ring R finite set $A = \{a_1, ..., a_n\}$ is an insulate if and only if $1 \in AF$, i.e. if*

$$a_1\varphi_1 + ... + a_n\varphi_n = 1$$

with suitable φ_k, $(1 \leqslant k \leqslant n)$ from the extended centroid F of the ring R.

Particularly this holds for the strongly prime rings.

Let R be a ring. Denote by $\mathcal{F}$ the set of the right ideals in R, containing an insulator. Analogously we define the set $\mathcal{F}'$ of the left ideals of R, containing an insulator.

Lemma 2.6. *Let R be a strongly prime ring. Then for every $q \in Q(R)$ there exist elements $i_1, ..., i_n \in R$ and $\psi_1, ..., \psi_n \in F$, such that $qi_k,\ i_kq \in R$, and $\sum_k i_k\psi_k = 1$.*

Proof. Let $q = r_1\varphi_1 + ... + r_m\varphi_m,\ r_k \in R,\ \varphi_k \in F$.

By the definition of the central closure of a prime ring, all φ_k can be represented as $M(R)$-homomorphisms $\varphi_k : I_k \to R$ where I_k and $I = \cap_k I_k$ are non zero idels in R, so I contains an insulator, Thus we have $1 = \sum_k i_k\psi_k$ for some $i_k \in I$ and $\psi_k \in F$. But $\varphi_k i = \varphi_k(i) \in R$ for all $i \in I$, so $qi_k,\ i_kq \in I,\ 1 \leqslant k \leqslant n$.

This lemma implies one of the equivalent conditions of the theorem Popescu-Spircu (see [8], Ch.XI, Proposition 3.4) and from it we option all the statements of the Theorem 2.7 except that Gabriel filters are symmetric. This fact can be proved directly. So we obtain the following crucial Theorem:

Theorem 2.7. *Let R be a strongly prime ring. Then $Q(R) \otimes_R Q(R) \cong Q(R)$, $Q(R)$ is left and right flat as the R-module. The sets $\mathcal{F}$ ang $\mathcal{F}'$ are symmetric Gabriel filters, the corresponding localizations are perfect, and the central closure $Q(R)$ is canonically isomorphic to the guotient ring of R with respect to $\mathcal{F}$ and $\mathcal{F}'$.*

3. Strongly prime ideals, strongly multiplicative and insulating sets

An ideal $\mathfrak{p} \subset R$ is called *strongly prime* if the factor ring $R/\mathfrak{p}$ is a strongly prime ring.

We can adapt the theorem 2.1 for equivalent characterizations of the strongly prime ideal. From the (5) of this theorem we obtain the following:

Proposition 3.1. *An ideal $\mathfrak{p} \subset R$ is strongly prime if and only if for each $a \notin \mathfrak{p}$, there exist elements $a_1, ..., a_n \in (a)$, $n = n(a)$, such that for each $\lambda \in M(R)$ with $\lambda 1 \notin \mathfrak{p}$, at least one of elements $\lambda a_k \notin \mathfrak{p}$.*

Clearly, maximal ideals of the ring are strongly prime. It is well known that in PI rings each prime ideal is strongly prime. Since not each prime ring has a simple central closure, prime ideals are not necessarily strongly prime. We easily obtain from Theorem 2.4, that strongly prime ideals are preserved under Morita equivalences. If $\phi : R \to S$ is a centred homomorphism of rings, and $\mathfrak{q} \subset S$ is a strongly prime ideal, we easily obtain from (6) of Theorem 2.1 that $\mathfrak{p} = \phi^{-1}\mathfrak{q}$ is the strongly prime ideal in R.

The intersection of all strongly prime ideals of the ring R we call a *strongly prime radical* and denote it by $sr(R)$. Let $R[X_1, ...X_n]$ be a polynomial ring over the ring R with commuting or noncommuting indeterminates.

Theorem 3.2. *$a \in sr(R)$ if and only if for each $n \in N$ and arbitrary elements $a_1, ..., a_n \in (a)$, the ideal in $R[X_1, ..., X_n]$, generated by polynomial $a_1X_1 + ... + a_nX_n - 1$ contains 1.*

Proof. If for some polynomial $a_1X_1 + ... + a_nX_n - 1$ generates a proper ideal in $R[X_1, ..., X_n]$, we can take a maximal ideal $\mathcal{M} \subset R[X_1, ..., X_n]$ containing this polynomial. Evidently $a \notin \mathcal{M}$. So we have the centred homomorphims $\phi : R \to R[X_1, ..., X_n]/\mathcal{M}$ with $\phi a \neq 0$ and $\phi^{-1}\mathcal{M}$ is the strongly prime ideal in R, not containing a.

If $a \notin sr(R)$, then $a \notin \mathfrak{p}$ for some strongly prime ideal $\mathfrak{p} \subset R$. So $(\bar{a})^{\varepsilon} = Q(R/\mathfrak{p})$ and we have an expression

$$\bar{a}_1 u_1 + ... + \bar{a}_n u_n = 1 \text{ in } Q(R/\mathfrak{p}) \text{ with } \bar{a}_1, ..., \bar{a}_n \in (\bar{a}),\ \ u_1, ..., u_n \in F(R/\mathfrak{p}).$$

So polynomial $a_1X_1 + \ldots + a_nX_n - 1$ is in the kernel of the homomorphism from $R[X_1, \ldots, X_n]$ to $Q(R/\mathfrak{p})$, which sends X_k to the u_k, $1 \leqslant k \leqslant n$. Thus the ideal, generated by this polynomial is proper.

Denote by $BMc(R)$ the Brown-McCoy radical of the ring R.

Corollary 3.3.

$$sr(R) = \cap_n BMc(R[X_1, \ldots, X_n]).$$

Recall that Levitzki radical of the ring is the biggest locally nilpotent ideal.

Theorem 3.4. *Strongly prime radical $sr(R)$ of the nonzero ring contains the Levitzki radical $L(R)$.*

It would be interesting to know if or under which conditions the upper nil-radical of the ring is contained in $sr(R)$.

Now we introduce the notion of a strongly multiplicative set of a ring and characterize strongly prime ideals in terms of these sets.

We call a subset $\mathcal{S} \subseteq R$ *strongly multiplicative*, or *sm-set*, if $1 \in \mathcal{S}$ and for any $a \in \mathcal{S}$ there exist elements $a_1, \ldots, a_n \in (a)$, $(n = n(a))$, such that for each $\lambda \in M(R)$ with $\lambda 1 \in \mathcal{S}$, we have $\lambda a_k \in \mathcal{S}$ for some $1 \leqslant k \leqslant n$.

Proposition 3.5. *If $\mathfrak{p} \subset R$ is a strongly prime ideal, its complement is a strongly multiplicative set.*

Indeed, this Proposition is just another form of Proposition 3.1. Example. Let $I \subset R$ be an ideal. The set of elements $\mathcal{S} = \{1 + i,\ i \in I\}$ is an sm-set. For a proof, take $n = 1$, $a_1 = a$ for each $a \in \mathcal{S}$.

Theorem 3.6. *Let $\mathcal{S} \subset R$, $0 \notin \mathcal{S}$ be a strongly multiplicative set. Each ideal $\mathfrak{p} \subset R$, maximal with respect to $\mathfrak{p} \cap \mathcal{S} = \emptyset$, is strongly prime. For each $a \in \mathcal{S}$, elements $a_1, \ldots, a_n \in (a)$ from the definition of the strongly multiplicative set are insulators in $R/\mathfrak{p}$.*

Proof. Let $x \notin \mathfrak{p}$. Then $p + \mu_0 x = a \in \mathcal{S}$, for some $p \in \mathfrak{p}$ and $\mu_0 \in M(R)$. Let $a_k = \lambda_k a = \lambda_k p + \lambda_k \mu_0 x \in (a)$, $1 \leqslant k \leqslant n$ be elements corresponding to a in the definition of the sm-set. Let $\lambda 1 \notin \mathfrak{p}$. Then $q + \nu_0 \lambda 1 = (l_q + \nu_0 \lambda)1 = \lambda' 1 \in \mathcal{S}$, where $l_q \in M(R)$ is the left multiplication by q. Then for some $\lambda' a_k \in \mathcal{S}$, for some k. thus not in $\mathfrak{p}$. We have

$$\lambda' a_k = (l_q + \nu_0 \lambda)(\lambda_k p + \lambda_k \mu_0 x) = q a_k + \nu_0 \lambda \lambda_k p + \nu_0 \lambda \lambda_k \mu_0 x \notin \mathfrak{p}.$$

But $q a_k$ and $\nu_0 \lambda \lambda_k p$ are in $\mathfrak{p}$, so $\lambda \lambda_k \mu_0 x \notin \mathfrak{p}$. Thus, for each $x \notin \mathfrak{p}$, there exist a finite set of elements $x_k = \lambda_k \mu_0 x \in (x)$, such that for each $\lambda \in M(R)$ with $\lambda 1 \notin \mathfrak{p}$, at least one of the elements $\lambda x_k \notin \mathfrak{p}$. By Proposition 3.1, ideal $\mathfrak{p}$ is strongly prime.

Let $S \subset R$ be a strongly multiplicative set. Similarly to the commutative case, we define the set $S' = \{u \in R \mid (u) \cap S \neq \emptyset\}$ and call it the *saturation* of S. Denote $H = \{\cup_\alpha \mathfrak{p}_\alpha \mid \mathfrak{p}_\alpha \cap S = \emptyset\}$, where $\mathfrak{p}_\alpha \subset R$ are strongly prime ideals.

Proposition 3.7. *Let S be a strongly multiplicative set. Then S' is also strongly multiplicative and $S' = R \setminus H$ - the complement to the union of all strongly prime ideals disjoint with S.*

Let $\mathcal{I}$ be a set which elements are finite subsets of the ring R. We call the set $\mathcal{I}$ an *insulating set* if for each $\{a_1, ..., a_n\} \in \mathcal{I}$ and each elements $\lambda_1, ..., \lambda_m \in M(R)$ such that $\{\lambda_1 1, ..., \lambda_m 1\} \in \mathcal{I}$, we have $\{\lambda_k a_l \in \mathcal{I}, 1 \leqslant k \leqslant m, 1 \leqslant l \leqslant n\} \in \mathcal{I}$. The set consisting of insulators of the ring is insulating.

Theorem 3.8. *Let $\mathcal{I}$ be an insulating set. Each ideal $\mathfrak{p} \subset R$, which does not contain subsets which belong to the insulating set and is maximal with respect to this property, is strongly prime. Elements from $\mathcal{I}$ are insulators in $R/\mathfrak{p}$.*

Metod for the proof can be extracted from the proof of the Theorem.

Corollary 3.9. *Each symmetric zero divisor of the ring is contained in some strongly prime ideal which does not contain an insulator.*

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Stipriai pirminiai idealai, stipriai pirminės ir izuoliuojančios sistemos žieduose

A. Kaučikas

Įrodyta, kad stipriai pirminių žiedų centrinis uždarymas yra vienpusės tobulos simetrinės lokalizacijos. Apibrėžtos stipriai multiplikatyvios ir izoliuojančios sistemos, kurių pagalba charakterizuoti stipriai pirminiai idealai. Rasti stipriai pirminio radikalo ryšiai su kitais žiedų radikalais.